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Research Article

Estimation of entropies on time scales by Lidstone's interpolation using Csiszár-type functional

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Abstract

The inequality containing Csiszár divergence on time scales is generalized for $2n$-convex functions by using Lidstone interpolating polnomial. As an application, new entropic bounds on time scales are also computed. Several inequalities in quantum calculus and h-discrete calculus are also established. The relationship between Shannon entropy, Kullback-Leibler divergence and Jeffreys distance with Zipf-Mandelbrot entropy are also established.

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1. Introduction

The concept of time scales has attracted many mathematicians for the past quarter-century. Theory of time scales plays an important part in mathematical analysis. Some of the most well-known examples of calculus on time scales are quantum calculus, difference calculus and differential calculus. The books of Bohner and Peterson [19, 20] cover many of the essential aspects of time scales. In recent years, various researchers did a lot of work on time scale calculus and got fantastic results (see [2, 4, 14–16, 22, 61, 62] and the references cited therein). In addition to mathematics, Dynamic inequalities and equations have a wide range of applications. For example; finance problems, quantum mechanics, physical problems, optical problems, wave equations, population dynamics and heat transfer [18, 40, 67].

Quantum calculus is the contemporary name for a kind of calculus that works without the notion of limits. It was initially based on the notion of finite difference re-scaling and is also known as q-calculus. In the 1740s, Euler proposed the theory of partitions, usually called

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analytic number theory, which gave rise to the concept of q-calculus. In 1910, Jackson [38] established the notion of q-definite integrals and generalized the theory of q-calculus. The q-calculus has developed a bridge between mathematics and physics due to substantial role of mathematics related to modeling of quantum computing. Kac and Cheung's book [40] explains various important concept of quantum calculus. There has already been significant advancement in q-calculus over the last few decades, see [21,37,48,51,71–74,77] and references therein.

Tariboon *et al.* [68,69] established the notion of q-derivatives over the finite intervals and studied a number of quantum counterparts of classical mathematical inequalities. Sudsutad *et al.* [63] established several Hermite-Hadamard type quantum integral inequalities for convex functions. Chen and Yang in [30] and Liu and Yang in [44] utilized quantum integrals and established numerous Chebyshev and Grüss type inequalities on finite intervals, respectively. In [33], Erden *et al.* established a number of quantum integral inequalities for convex functions. A generalized q-integral identity containing q-differentiable function is established by Awan *et al.* [13]. Various quantum bounds by considering the class of preinvex functions are also determined. The Green function approach was used by Khan *et al.* [42] to establish quantum Hermite-Hadamard inequality. In [43], Kunt *et al.* determined new version of the celebrated Montgomery identity via quantum integral operators. The obtained result is used to established some quantum integral inequalities of Ostrowski type. In [17], Ali *et al.* proved some new Ostrowski-type integral inequalities for q-differentiable bounded functions. In [46], Li *et al.* obtained various new estimates of Hermite-Hadamard type quantum integral inequalities. In [29], Ben *et al.* established q-fractional integral inequalities of Henry-Gronwall type.

Despite its resemblance to q-calculus, h-calculus is quite different. It is, in fact the calculus of finite differences, but a more precise similarity with classical calculus makes it clear. For example, Newton's interpolation formula is similar to h-Taylor formula, and Abel transform is similar to h-integration by parts. The definite h-integral is same as Riemann sum, consequently, the fundamental theorem of h-calculus permits one to estimate finite sums.

The theory of convexity plays a significant role in the development of inequalities. In spite of that the importance of inequalities containing convex functions is magnificent as it tackles numerous problems in various fields of mathematics at a substantial rate. Consequently, the study of these inequalities has gained a tons of attention (see [8,9,36,54] and the references cited therein). Over the recent years, the inequalities for n-convex functions are generalized by numerous researchers. Lidstone polynomials are helpful in literature to generalize numerous well-known inequalities. Jensen's inequality and its converses generalized by Gazić *et al.* [3] for $2n$-convex functions by utilizing Lidstone's interpolating polynomials. In [55], Pečarić *et al.* introduced a new class of n-convex functions. They proposed an interesting theory to evaluate linear operator inequalities utilizing n-convex functions. This approach guides to various impressive and insightful results with a number of developments in statistics and operator theory. In [5], Adil *et al.* used Abel-Gontscharoff formula along with Green function and established generalized majorization theorem for higher order convex functions. Further in [23], Butt *et al.* utilized Abel-Gontscharoff interpolation and generalized Popoviciu inequality for n-convex functions. In [56], Pečarić *et al.* obtained generalizations of Steffensen's inequality using Abel-Gontscharoff formula.

In [24], Butt *et al.* obtained useful identities via Taylor polynomial and generalized Popoviciu inequality for n-convex functions. Sherman's inequality is generalized by Agarwal *et al.* [6] for $2n$-convex functions by using Lidstone's interpolating polynomial. Jensen's and Jensen-Steffensen's inequalities and their converses generalized by Vukelic [70] *et al.* in both the integral and the discrete case by using Lidstone's interpolating polynomials and majorization theorems. The generalizations of majorization inequalities is obtained by Adil *et al.* [7] by using conditions on Green's functions and Lidstone interpolation. In [25], Bibi *et al.* used Lidstone's interpolating polynomial and generalized Jensen's inequality for diamond integrals on time scales for $2n$-convex functions. Popoviciu's inequality is generalized by Butt *et al.* [26] for n-convex functions by employing Fink's identity in combination with new Green's function. In [34], Fahad *et al.* proved a new generalization of Steffense's inequality by using Green's function, Lidstone interpolation and Montogomery's identity. Using Taylor's polynomial along with new Green functions, Latif *et al.* in [45] obtained generalized results concerning majorization inequality. In [52], Nosheen *et al.* used Taylor's formula with Green function and obtained some improvements of Jensen-Steffensen inequality for diamond integrals. In [41], Khan *et al.* generalized new inequalities of Rényi Shannon entropies and refinement of Jensen's inequality for n-convex functions by utilizing Montgomery identity.

In [57], Pečarić and Praljak obtained Popoviciu type inequalities for higher order convex functions using Lidstone's interpolating polynomial. In [53], Niaz *et al.* estimated various entropies by utilizing the Jensen's type functionals. In addition, the authors generalized new inequalities for higher order convex functions employing Taylor's formula. Fahad and Pečarić [35] gave Abel-Gontscharoff interpolation of composition functions and proved generalized Steffensen-type inequalities. Further in [27], the authors used Taylor's formula and obtained various extensions of Jensen type inequalities for k-convex functions. Levinson's inequality has been generalized for 3-convex function utilizing Green functions by Adeel *et al.* [10]. In addition, the obtained results are used in information theory via Shannon entropy, f-divergence and Rényi divergence. Further in [12], the authors used Taylor's polynomial and generalized Levinson type inequalities for the class of m-convex functions. The obtained results are applied in information theory. In [28], Butt *et al.* used Abel-Gontscharoff formula and new Green functions and extended the continuous and discrete cyclic refinements of Jensen's inequality for higher order convex function. As an application, they developed a link among novel entropic bounds for Relative, Mandelbrot and Shannon entropies.

Cyclic refinements of Jensen's inequality is generalized by Mehmood *et al.* [50] by utilizing Lidstone's polynomial and Green functions. Further obtained new entropic bounds and established the link among Relative and Shannon entropy with Zipf-Mandelbrot entropy. Levinson type inequalities is generalized by Adeel *et al.* [11] for $(2\text{p} + 1)$-convex functions by using Lidstone interpolating polynomial and Green functions. They also established some inequalities for Shannon entropies. In [58], Pečarić *et al.* obtained new generalizations of Steffensen's inequality for $2n$-convex and $(2n + 1)$-convex functions utilizing Lidstone's polynomial. In [64], Siddique *et al.* used Fink's identity and Green functions and obtained generalized results related to majorization-type inequalities. They also gave a generalized majorization theorem for higher order convex functions. The obtained results are applied with regard to Kullback-Leibler divergence and Shannon entropy. In [59], Ramzan *et al.* utilized extended Montgomery identity and Jensen's inequality for diamond integrals and generalized it for n-convex functions.

Motivated by above discussion, we generalize an inequality containing Csiszár divergence on time scales for $2n$-convex functions by utilizing Lidstone's interpolating polynomial, as unification of both discrete and integral cases. In addition, we estimate bounds of different divergence measures, in particular, Kullback-Leibler divergence, differential entropy, Shannon entropy, Jeffreys distance and triangular discrimination on time scales, q-calculus and h-discrete calculus. Some estimates for Zipf-Mandelbrot entropy are also given.

2. Preliminaries

Let us take a quick look at time scales, as well as the essential definitions and notations. The details can be followed from [19]:

For $\varsigma \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as

$$\sigma(\varsigma) := \inf\{\upsilon \in \mathbb{T} : \upsilon > \varsigma\}.$$

Assume that $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$ is a function. Then $\mathfrak{f}$ is rd-continuous, if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limit is finite at left-dense points of $\mathbb{T}$. The set of rd-continuous functions $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$ will be denoted in this paper by C_{rd}.
The derived set $\mathbb{T}^k$ is defined as follows: Given $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; else, $\mathbb{T}^k = \mathbb{T}$.
Let $\mathfrak{f} : \mathbb{T} \to \mathbb{R}$ and $\varsigma \in \mathbb{T}^k$. Then $\mathfrak{f}^\Delta(\varsigma)$ is defined to be the number (if it exists) with the property that for any $\epsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of ς such that

$$|\mathfrak{f}(\sigma(\varsigma)) - \mathfrak{f}(\upsilon) - \mathfrak{f}^\Delta(\varsigma)(\sigma(\varsigma) - \upsilon)| \leq \epsilon|\sigma(\varsigma) - \upsilon|, \quad \forall\, \upsilon \in \mathfrak{U}.$$

Then $\mathfrak{f}$ is known as delta differentiable at ς.
If $\mathbb{T} = \mathbb{R}$, then $\mathfrak{f}^\Delta$ reduces to usual derivative $\mathfrak{f}'$, and $\mathfrak{f}^\Delta$ becomes forward difference operator $\Delta\mathfrak{f}(\varsigma) = \mathfrak{f}(\varsigma+1) - \mathfrak{f}(\varsigma)$ for $\mathbb{T} = \mathbb{Z}$. If $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, q-difference operator ($q > 1$) is given by

$$\mathfrak{f}^\Delta(\varsigma) = \frac{\mathfrak{f}(q\varsigma) - \mathfrak{f}(\varsigma)}{(q-1)\varsigma}, \quad \mathfrak{f}^\Delta(0) = \lim_{\upsilon \to 0} \frac{\mathfrak{f}(\upsilon) - \mathfrak{f}(0)}{\upsilon}.$$

Theorem A. Every $\mathfrak{f} \in C_{rd}$ has an antiderivative. For $\varsigma_0 \in \mathbb{T}$, $\mathfrak{F}$ is given as

$$\mathfrak{F}(\varsigma) := \int_{\varsigma_0}^{\varsigma} \mathfrak{f}(x)\Delta x \qquad for \quad \varsigma \in \mathbb{T}^k,$$

is an antiderivative of $\mathfrak{f}$.

If $\mathbb{T} = \mathbb{R}$, then $\int_a^b \mathfrak{f}(\varsigma)\Delta\varsigma = \int_a^b \mathfrak{f}(\varsigma)d\varsigma$, and $\int_a^b \mathfrak{f}(\varsigma)\Delta\varsigma = \sum\limits_{\varsigma=a}^{b-1} \mathfrak{f}(\varsigma)$ for $\mathbb{T} = \mathbb{N}$, where $a, b \in \mathbb{T}$ with $a \leq b$.
Assume the following set of all probability densities on time scale $\mathbb{T}$ to be

$$\Omega := \left\{ p \middle| p : \mathbb{T} \to [0, \infty), \int_a^b p(\varsigma)\Delta\varsigma = 1 \right\}.$$

The following inequality is proved by Ansari *et al.* in [15]:
Theorem B. Suppose that $\Psi : [0, \infty) \to \mathbb{R}$ is a convex and continuous function on the interval $[\gamma_1, \gamma_2] \subset [0, \infty)$ and $\gamma_1 \leq 1 \leq \gamma_2$. If $p_1, p_2 \in \Omega$ with $\gamma_1 \leq \frac{p_1(\varsigma)}{p_2(\varsigma)} \leq \gamma_2$ for all $\varsigma \in \mathbb{T}$, then

$$\int_a^b p_2(\varsigma)\Psi\left(\frac{p_1(\varsigma)}{p_2(\varsigma)}\right)\Delta\varsigma \leq \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}\Psi(\gamma_1) + \frac{1 - \gamma_1}{\gamma_2 - \gamma_1}\Psi(\gamma_2). \tag{2.1}$$

Assume the hypothesis of Theorem B, we define the following Csiszár-type linear functional:

$$\mathbf{J}(\Psi) := \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}\Psi(\gamma_1) + \frac{1 - \gamma_1}{\gamma_2 - \gamma_1}\Psi(\gamma_2) - \int_a^b p_2(\varsigma)\Psi\left(\frac{p_1(\varsigma)}{p_2(\varsigma)}\right)\Delta\varsigma. \tag{2.2}$$

Remark 2.1. Under the conditions of Theorem B, (2.2) will be non negative.

In [75], Widder established the following result:
Lemma A. If $\Psi \in C^\infty[0, 1]$,

$$\Psi(t) = \sum_{v=0}^{m-1} \left[\Psi^{(2v)}(0)\Phi_v(1-t) + \Psi^{(2v)}(0)\Phi_v(t)\right] + \int_0^1 G_m(t, s)\Psi^{(2m)}(t)dt,$$

where Φ_v is a polynomial of degree $2v + 1$ defined by the relations

$$\Phi_0(t) = t, \quad \Phi_m''(t) = \Phi_{m-1}(t), \quad \Phi_m(0) = \Phi_m(1) = 0, \quad m \geq 1$$

and

$$G_1(t, z) = G(t, z) = \begin{cases} (t-1)z, & z \leq t; \\ (z-1)t, & t \leq z, \end{cases} \tag{2.3}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{dz^2}$ on $[0,1]$, and with the successive iterates of $G(t,z)$

$$G_m(t,z) = \int_0^1 G_1(t,y)G_{m-1}(y,z)dy \quad m \geq 2. \tag{2.4}$$

The Lidstone polynomial may be represent in connection with $G_m(t,z)$ as

$$\Phi_m(t) = \int_0^1 G_m(t,z)zdz. \tag{2.5}$$

Lidstone series representation of $\Psi \in C^{2m}[\zeta_1, \zeta_2]$ given in [1] as follows:

$$\begin{aligned}\Psi(x) \quad = \quad & \sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\Psi^{(2v)}(\zeta_1)\Phi_v\Big(\frac{\zeta_2-x}{\zeta_2-\zeta_1}\Big) + \sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\Psi^{(2v)}(\zeta_2)\times \\ & \Phi_v\Big(\frac{x-\zeta_1}{\zeta_2-\zeta_1}\Big) + (\zeta_2-\zeta_1)^{2m-1}\int_{\zeta_1}^{\zeta_2} G_m\Big(\frac{x-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)\Psi^{(2m)}(t)dt.\end{aligned} \tag{2.6}$$

3. Generalization of the Csiszár type linear functional

Let us begin with the following result in which we construct the generalized identity involving Csiszár divergence on time scales using Lidstone interpolating polynomial:

Theorem 3.1. *Assume the conditions of Theorem B with $\Psi \in C^{2m}[\zeta_1, \zeta_2]$ and G_m be the same as given in (2.4). Then*

$$\begin{aligned}& I_\Psi(p_1,p_2) = \int_a^b p_2(\varsigma)\Psi\Big(\frac{p_1(\varsigma)}{p_2(\varsigma)}\Big)\Delta\varsigma \\ & = \frac{\gamma_2-1}{\gamma_2-\gamma_1}\Psi(\gamma_1) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\Psi(\gamma_2) - \sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\Psi^{(2v)}(\zeta_1)\mathbf{J}\Big(\Phi_v\Big(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\Big)\Big) \\ & - \sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\Psi^{(2v)}(\zeta_2)\mathbf{J}\Big(\Phi_v\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big) - (\zeta_2-\zeta_1)^{2m-1}\times \\ & \int_{\zeta_1}^{\zeta_2}\mathbf{J}\Big(G_m\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}, \frac{z-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big)\Psi^{(2m)}(z)dz,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\mathbf{J}\Big(\Phi_v\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big) \quad = \quad & \frac{\gamma_2-1}{\gamma_2-\gamma_1}\Phi_v\Big(\frac{\gamma_1-\zeta_1}{\zeta_2-\zeta_1}\Big) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\Phi_v\Big(\frac{\gamma_2-\zeta_1}{\zeta_2-\zeta_1}\Big) - \\ & \int_a^b \Phi_v\Big(\frac{p_1(\varsigma)-\zeta_1 p_2(\varsigma)}{\zeta_2-\zeta_1}\Big)\Delta\varsigma\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}& \mathbf{J}\Big(G_m\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}, \frac{z-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big) = \frac{\gamma_2-1}{\gamma_2-\gamma_1}G_m\Big(\frac{\gamma_1-\zeta_1}{\zeta_2-\zeta_1}, \frac{z-\zeta_1}{\zeta_2-\zeta_1}\Big) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\times \\ & G_m\Big(\frac{\gamma_2-\zeta_1}{\zeta_2-\zeta_1}, \frac{z-\zeta_1}{\zeta_2-\zeta_1}\Big) - \int_a^b p_2(\varsigma)G_m\Big(\frac{\frac{p_1(\varsigma)}{p_2(\varsigma)}-\zeta_1}{\zeta_2-\zeta_1}, \frac{z-\zeta_1}{\zeta_2-\zeta_1}\Big)\Delta\varsigma.\end{aligned} \tag{3.3}$$

Proof. Use (2.6) in (2.2) and the linearity of $\mathbf{J}(\cdot)$ to obtain (3.1). □

The following result is related to the generalization of Csiszár type linear functional for $2m$-convex function.

Theorem 3.2. *Consider $\Psi \in C^{2m}[\zeta_1, \zeta_2]$ be such that Ψ is $2m$-convex function together with the conditions of Theorem 3.1. If*

$$\mathbf{J}\left(G_m\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}, \frac{z-\zeta_1}{\zeta_2-\zeta_1}\right)\right) \geq 0, \tag{3.4}$$

then

$$\begin{aligned} &\frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}\Psi(\gamma_1) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\Psi(\gamma_2) - \int_a^b p_2(\varsigma)\Psi\left(\frac{p_1(\varsigma)}{p_2(\varsigma)}\right)\Delta\varsigma \\ \geq\ &\sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\left[\Psi^{(2v)}(\zeta_1)\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right) + \Psi^{(2v)}(\zeta_2)\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right] \end{aligned} \tag{3.5}$$

Proof. Since $\Psi \in C^{2m}[\zeta_1, \zeta_2]$ and Ψ is $2m$-convex function, therefore $\Psi^{(2m)}(\cdot) \geq 0$ (see [54, p. 16]). Apply Theorem 3.1 and utilizing the assumption (3.4) to get (3.5). □

Theorem 3.3. *Assume the conditions of Theorem 3.1. Consider Ψ is $2m$-convex function and $p \in C([a,b]\mathbb{T}, \mathbb{R})$ be positive such that $\int_a^b p(\varsigma)\Delta\varsigma = 1$.*

(i) *The inequality (3.5) is valid for odd m.*
(ii) *Let the inequality (3.5) be satisfied and*

$$\sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\left[\Psi^{(2v)}(\zeta_1)\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right) + \Psi^{(2v)}(\zeta_2)\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right] \geq 0, \tag{3.6}$$

then

$$\frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}\Psi(\gamma_1) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\Psi(\gamma_2) - \int_a^b p_2(\varsigma)\Psi\left(\frac{p_1(\varsigma)}{p_2(\varsigma)}\right)\Delta\varsigma \geq 0. \tag{3.7}$$

Proof. As G_1 is convex and $G_m\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}, \frac{z-\zeta_1}{\zeta_2-\zeta_1}\right) \geq 0$ for odd m, therefore (3.4) holds. Moreover, Ψ is $2m$-convex function, thus by utilizing Remark 2.1 and Theorem 3.2 to get (3.7). □

Remark 3.4. If $\mathbb{T} = \mathbb{R}$ and $m = 1$, i.e. Ψ is convex, then (3.7) becomes [31, (2.1)].

Remark 3.5. It is also possible to compute Grüss, Cebyšev and Ostrowski-type bounds corresponding to the identity (3.1) related to the generalization of an inequality containing Csiszár divergence on time scales.

4. Bounds of divergence measures

Shannon entropy is the fundamental term in information theory and is often dealt with measure of uncertainty. The random variable, entropy, is characterized regarding its probability distribution and it can appear as a better measure of uncertainty or predictability. Shannon entropy allows the estimation of the normal least number of bits essential to encode a string of symbols based on alphabet size and frequency of symbols.
Let X be a continuous random variable and p is positive density function on time scale $\mathbb{T}$ to X such that $\int_a^b p(\varsigma)\Delta\varsigma = 1$, if the integral exists.
On time scales, Ansari *et al.* [14] introduced the differential entropy which is given as

$$h_{\bar{b}}(X) := \int_a^b p(\varsigma)\log\frac{1}{p(\varsigma)}\Delta\varsigma, \tag{4.1}$$

where $\bar{b} > 1$ is base of log.

Theorem 4.1. *Let X be a continuous random variable and assume the conditions of Theorem 3.1 with Ψ is $2m$-convex function. If m is odd, then*

$$\begin{aligned}
&h_{\bar{b}}(X) \leq \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}\log(\gamma_1) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\log(\gamma_2) - \int_a^b p_1(\varsigma)\log(p_2(\varsigma))\Delta\varsigma - \\
&\sum_{v=0}^{m-1}(2v-1)!(\zeta_2-\zeta_1)^{2v}\Big[\frac{1}{(\zeta_1)^{2v}}\mathbf{J}\Big(\Phi_v\Big(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\Big)\Big) \\
&+\frac{1}{(\zeta_2)^{2v}}\mathbf{J}\Big(\Phi_v\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big)\Big]
\end{aligned} \tag{4.2}$$

where $\mathbf{J}\Big(\Phi_v\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big)$ and $h_{\bar{b}}(X)$ are given in (3.2) and (4.1) respectively.

Proof. Use $\Psi = -\log\varsigma$ in Theorem 3.2 to get (4.2). □

Kullback-Leibler divergence is one of the best known among information divergences. The well-known divergence measure is used in information theory, mathematical statistics and signal processing (see [76]). Ansari *et al.* [15] defined the Kullback-Leibler divergence on time scales by

$$D(p_1,p_2) = \int_a^b p_1(\varsigma)\ln\left[\frac{p_1(\varsigma)}{p_2(\varsigma)}\right]\Delta\varsigma. \tag{4.3}$$

Theorem 4.2. *Let X be a continuous random variable and assume the conditions of Theorem 3.1 with Ψ is $2m$-convex function. If m is odd, then*

$$\begin{aligned}
&D(p_1,p_2) \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\gamma_1\ln(\gamma_1) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\gamma_2\ln(\gamma_2) - \sum_{v=0}^{m-1}(2v-2)!(\zeta_2-\zeta_1)^{2v}\times \\
&\Big[\frac{1}{(\zeta_1)^{2v-1}}\mathbf{J}\Big(\Phi_v\Big(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\Big)\Big) + \frac{1}{(\zeta_2)^{2v-1}}\mathbf{J}\Big(\Phi_v\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big)\Big]
\end{aligned} \tag{4.4}$$

where $\mathbf{J}\Big(\Phi_v\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big)$ and $D(p_1,p_2)$ are given in (3.2) and (4.3) respectively.

Proof. Use $\Psi = \varsigma\ln\varsigma$ in Theorem 3.2 to get (4.4). □

Jeffreys distance have many applications in statistics and pattern recognition (see [39, 65]). Ansari *et al.* [15] defined the Jeffreys distance on time scales by

$$D_J(p_1,p_2) := \int_a^b (p_1(\varsigma)-p_2(\varsigma))\ln\left[\frac{p_1(\varsigma)}{p_2(\varsigma)}\right]\Delta\varsigma. \tag{4.5}$$

Theorem 4.3. *Let X be a continuous random variable and assume the conditions of Theorem 3.1 with Ψ is $2m$-convex function. If m is odd, then*

$$\begin{aligned}
D_J(p_1,p_2) \quad \leq \quad & \frac{\gamma_2-1}{\gamma_2-\gamma_1}(\gamma_1-1)\ln(\gamma_1) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}(\gamma_2-1)\ln(\gamma_2) - \sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\times \\
&\Big[\Big(\frac{(2v-1)!}{(\zeta_1)^{2v}} + \frac{(2v-2)!}{(\zeta_1)^{2v-1}}\Big)\mathbf{J}\Big(\Phi_v\Big(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\Big)\Big) + \Big(\frac{(2v-1)!}{(\zeta_2)^{2v}} + \\
&\frac{(2v-2)!}{(\zeta_2)^{2v-1}}\Big)\mathbf{J}\Big(\Phi_v\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big)\Big]
\end{aligned} \tag{4.6}$$

where $\mathbf{J}\Big(\Phi_v\Big(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big)$ and $D_J(p_1,p_2)$ are given in (3.2) and (4.5) respectively.

Proof. Use $\Psi = (\varsigma-1)\ln\varsigma$ in Theorem 3.2 to get (4.6). □

Triangular discrimination have many applications in statistics and information theory (see [39, 66]). Ansari *et al.* [15] defined the triangular discrimination on time scale by

$$D_{\Delta}(p_1,p_2)=\int_a^b \frac{[p_2(\varsigma)-p_1(\varsigma)]^2}{p_2(\varsigma)+p_1(\varsigma)}\Delta\varsigma. \tag{4.7}$$

Theorem 4.4. *Let X be a continuous random variable and assume the conditions of Theorem 3.1 with Ψ is $2m$-convex function. If m is odd, then*

$$\begin{aligned}D_{\Delta}(p_1,p_2) &\le \frac{\gamma_2-1}{\gamma_2-\gamma_1}\frac{(\gamma_1-1)^2}{\gamma_1+1}+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\frac{(\gamma_2-1)^2}{\gamma_2+1}-\sum_{v=0}^{m-1}4(2v)!\times\\ &(\zeta_2-\zeta_1)^{2v}\left[\frac{1}{(\zeta_1+1)^{2v+1}}\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\frac{1}{(\zeta_2+1)^{2v+1}}\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right]\end{aligned} \tag{4.8}$$

where $\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)$ and $D_{\Delta}(p_1,p_2)$ are given in (3.2) and (4.7) respectively.

Proof. Use $\Psi=\dfrac{(\varsigma-1)^2}{\varsigma+1}$ in Theorem 3.2 to get (4.8). □

4.1. Inequalities in classical calculus (continuous case)

In this section, new bounds of Csiszár divergence, differential entropy, Kullback-Leibler divergence, Jeffrey distance and triangular discrimination are given, respectively:
If $\mathbb{T}=\mathbb{R}$ in Theorem 3.2, the inequality (3.5) have the following form and gives new bound for Csiszár divergence:

$$\begin{aligned}&\int_a^b p_2(\varsigma)\Psi\left(\frac{p_1(\varsigma)}{p_2(\varsigma)}\right)d\varsigma \le \frac{\gamma_2-1}{\gamma_2-\gamma_1}\Psi(\gamma_1)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\Psi(\gamma_2)-\\ &\sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\left[\Psi^{(2v)}(\zeta_1)\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\Psi^{(2v)}(\zeta_2)\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right],\end{aligned}$$

where

$$\begin{aligned}\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right) &= \frac{\gamma_2-1}{\gamma_2-\gamma_1}\Phi_v\left(\frac{\gamma_1-\zeta_1}{\zeta_2-\zeta_1}\right)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\Phi_v\left(\frac{\gamma_2-\zeta_1}{\zeta_2-\zeta_1}\right)-\\ &\int_a^b \Phi_v\left(\frac{p_1(\varsigma)-\zeta_1 p_2(\varsigma)}{\zeta_2-\zeta_1}\right)d\varsigma.\end{aligned} \tag{4.9}$$

If $\mathbb{T}=\mathbb{R}$ in Theorem 4.1 - Theorem 4.4, inequalities (4.2), (4.4), (4.6) and (4.8) take the following new form, respectively:

$$\begin{aligned}&\int_a^b p_2(\varsigma)\log\frac{1}{p_2(\varsigma)}d\varsigma \le \frac{\gamma_2-1}{\gamma_2-\gamma_1}\log(\gamma_1)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\log(\gamma_2)-\int_a^b p_1(\varsigma)\log(p_2(\varsigma))d\varsigma-\\ &\sum_{v=0}^{m-1}(2v-1)!(\zeta_2-\zeta_1)^{2v}\left[\frac{1}{(\zeta_1)^{2v}}\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\frac{1}{(\zeta_2)^{2v}}\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right],\end{aligned}$$

$$\begin{aligned}&\int_a^b p_1(\varsigma)\ln\frac{p_1(\varsigma)}{p_2(\varsigma)}d\varsigma \le \frac{\gamma_2-1}{\gamma_2-\gamma_1}\gamma_1\ln(\gamma_1)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\gamma_2\ln(\gamma_2)-\sum_{v=0}^{m-1}(2v-2)!\times\\ &(\zeta_2-\zeta_1)^{2v}\left[\frac{1}{(\zeta_1)^{2v-1}}\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\frac{1}{(\zeta_2)^{2v-1}}\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right],\end{aligned}$$

$$\int_a^b [p_1(\varsigma) - p_2(\varsigma)] \ln \frac{p_1(\varsigma)}{p_2(\varsigma)} d\varsigma \leq \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}(\gamma_1 - 1)\ln(\gamma_1) + \frac{1-\gamma_1}{\gamma_2 - \gamma_1}(\gamma_2 - 1)\ln(\gamma_2)$$
$$- \sum_{v=0}^{m-1} (\zeta_2 - \zeta_1)^{2v} \left[\left(\frac{(2v-1)!}{(\zeta_1)^{2v}} + \frac{(2v-2)!}{(\zeta_1)^{2v-1}} \right) \mathbf{J}\left(\Phi_v\left(\frac{\zeta_2 - t}{\zeta_2 - \zeta_1} \right) \right) + \right.$$
$$\left. \left(\frac{(2v-1)!}{(\zeta_2)^{2v}} + \frac{(2v-2)!}{(\zeta_2)^{2v-1}} \right) \mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2 - \zeta_1} \right) \right) \right],$$

$$\int_a^b \frac{[p_2(\varsigma) - p_1(\varsigma)]^2}{p_1(\varsigma) + p_2(\varsigma)} d\varsigma \leq \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} \frac{(\gamma_1 - 1)^2}{\gamma_1 + 1} + \frac{1 - \gamma_1}{\gamma_2 - \gamma_1} \frac{(\gamma_2 - 1)^2}{\gamma_2 + 1} - \sum_{v=0}^{m-1} 4(2v)! \times$$
$$(\zeta_2 - \zeta_1)^{2v} \left[\frac{1}{(\zeta_1 + 1)^{2v+1}} \mathbf{J}\left(\Phi_v\left(\frac{\zeta_2 - t}{\zeta_2 - \zeta_1} \right) \right) + \frac{1}{(\zeta_2 + 1)^{2v+1}} \mathbf{J}\left(\Phi_v\left(\frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) \right) \right].$$

where $\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1} \right) \right)$ is given in (4.9).

4.2. Inequalities in h-discrete calculus

The following inequalities give new bound of Csiszár divergence, Shannon entropy, Kullback-Leibler divergence, Jeffrey distance and triangular discrimination in h-discrete calculus respectively. In this section, discrete case of these divergence measures are also given.

Put $\mathbb{T} = h\mathbb{Z}$ ($h > 0$) in Theorem 3.2, the inequality (3.5) have the following form

$$\sum_{v=\frac{a}{h}}^{\frac{b}{h}-1} p_2(vh) h \Psi\left(\frac{p_1(vh)}{p_2(vh)} \right) \leq \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} \Psi(\gamma_1) + \frac{1 - \gamma_1}{\gamma_2 - \gamma_1} \Psi(\gamma_2) - \sum_{v=0}^{m-1} (\zeta_2 - \zeta_1)^{2v} \times$$
$$\left[\Psi^{(2v)}(\zeta_1) \mathbf{J}\left(\Phi_v\left(\frac{\zeta_2 - t}{\zeta_2 - \zeta_1} \right) \right) + \Psi^{(2v)}(\zeta_2) \mathbf{J}\left(\Phi_v\left(\frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) \right) \right], \tag{4.10}$$

where

$$\mathbf{J}\left(\Phi_v\left(\frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) \right) = \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} \Phi_v\left(\frac{\gamma_1 - \zeta_1}{\zeta_2 - \zeta_1} \right) + \frac{1 - \gamma_1}{\gamma_2 - \gamma_1} \Phi_v\left(\frac{\gamma_2 - \zeta_1}{\zeta_2 - \zeta_1} \right) -$$
$$\sum_{v=\frac{a}{h}}^{\frac{b}{h}-1} \Phi_v\left(\frac{p_1(vh)h - \zeta_1 p_2(vh)h}{\zeta_2 - \zeta_1} \right). \tag{4.11}$$

Put $\mathbb{T} = h\mathbb{Z}$ ($h > 0$) in Theorem 4.1 - Theorem 4.4, the inequalities (4.2), (4.4), (4.6) and (4.8) takes the following new forms in h-discrete calculus, respectively:

$$\sum_{v=\frac{a}{h}}^{\frac{b}{h}-1} p_2(vh)h \ \log \frac{1}{p_2(vh)h} \leq \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} \log(\gamma_1) + \frac{1 - \gamma_1}{\gamma_2 - \gamma_1} \log(\gamma_2) - \sum_{v=\frac{a}{h}}^{\frac{b}{h}-1} p_2(vh)h \ \times$$
$$\log[p_1(vh)h] - \sum_{v=0}^{m-1} (2v-1)! (\zeta_2 - \zeta_1)^{2v} \left[\frac{1}{(\zeta_1)^{2v}} \mathbf{J}\left(\Phi_v\left(\frac{\zeta_2 - t}{\zeta_2 - \zeta_1} \right) \right) + \frac{1}{(\zeta_2)^{2v}} \right.$$
$$\left. \mathbf{J}\left(\Phi_v\left(\frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) \right) \right], \tag{4.12}$$

$$\sum_{v=\frac{a}{h}}^{\frac{b}{h}-1} p_1(vh)h\ln\left[\frac{p_1(vh)}{p_2(vh)}\right] \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\gamma_1\ln(\gamma_1)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\gamma_2\ln(\gamma_2)-\sum_{v=0}^{m-1}(2v-2)!\times$$
$$(\zeta_2-\zeta_1)^{2v}\left[\frac{1}{(\zeta_1)^{2v-1}}\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\frac{1}{(\zeta_2)^{2v-1}}\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right], \tag{4.13}$$

$$\sum_{v=\frac{a}{h}}^{\frac{b}{h}-1}(p_1-p_2)(vh)h\ln\frac{p_1(vh)}{p_2(vh)} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}(\gamma_1-1)\ln(\gamma_1)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}(\gamma_2-1)\ln(\gamma_2)$$
$$-\sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\left[\left(\frac{(2v-1)!}{(\zeta_1)^{2v}}+\frac{(2v-2)!}{(\zeta_1)^{2v-1}}\right)\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\left(\frac{(2v-1)!}{(\zeta_2)^{2v}}+\right.\right.$$
$$\left.\left.\frac{(2v-2)!}{(\zeta_2)^{2v-1}}\right)\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right], \tag{4.14}$$

$$\sum_{v=\frac{a}{h}}^{\frac{b}{h}-1} h\frac{[p_2(vh)-p_1(vh)]^2}{p_1(vh)+p_2(vh)} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\frac{(\gamma_1-1)^2}{\gamma_1+1}+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\frac{(\gamma_2-1)^2}{\gamma_2+1}-$$
$$\sum_{v=0}^{m-1}4(2v)!(\zeta_2-\zeta_1)^{2v}\left[\frac{1}{(\zeta_1+1)^{2v+1}}\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\frac{1}{(\zeta_2+1)^{2v+1}}\right.$$
$$\left.\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right]. \tag{4.15}$$

where $\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)$ is given in (4.11).

Remark 4.5. If $h=1, a=0, b=m, p_1(v)=(p_1)_v$ and $p_2(v)=(p_2)_v$, the inequality (4.10) takes the following new form and gives new bound for discrete Csiszár divergence:

$$\sum_{v=1}^{m}(p_2)_v\Psi\left(\frac{(p_1)_v}{(p_2)_v}\right) \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\Psi(\gamma_1)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\Psi(\gamma_2)-\sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\times$$
$$\left[\Psi^{(2v)}(\zeta_1)\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\Psi^{(2v)}(\zeta_2)\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right],$$

where

$$\begin{aligned}\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right) &= \frac{\gamma_2-1}{\gamma_2-\gamma_1}\Phi_v\left(\frac{\gamma_1-\zeta_1}{\zeta_2-\zeta_1}\right)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\Phi_v\left(\frac{\gamma_2-\zeta_1}{\zeta_2-\zeta_1}\right)- \\ &\quad \sum_{v=1}^{n}\Phi_v\left(\frac{(p_1)_v-\zeta_1(p_2)_v}{\zeta_2-\zeta_1}\right).\end{aligned} \tag{4.16}$$

Remark 4.6. Put $h=1, a=0, b=m, p_1(v)=(p_1)_v$ and $p_2(v)=(p_2)_v$, the inequality (4.12) takes the following form and gives new bound for discrete Shannon entropy:

$$S=\sum_{v=1}^{m}(p_2)_v\ \log\frac{1}{(p_2)_v} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\log(\gamma_1)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\log(\gamma_2)-\sum_{v=1}^{m}(p_2)_v\ \log(p_1)_v-$$
$$\sum_{v=0}^{m-1}(2v-1)!(\zeta_2-\zeta_1)^{2v}\left[\frac{1}{(\zeta_1)^{2v}}\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\frac{1}{(\zeta_2)^{2v}}\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right], \tag{4.17}$$

where $\mathbf{J}\bigg(\Phi_v\bigg(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)\bigg)$ is given in (4.16).

Remark 4.7. Consider $h = 1, a = 0, b = m, p_1(v) = (p_1)_v$ and $p_2(v) = (p_2)_v$, the inequality (4.13) takes the following form and gives new bound for discrete Kullback-Leibler divergence:

$$KL(p_1,p_2) = \sum_{j=1}^{n}(p_1)_v \ln\frac{(p_1)_v}{(p_2)_v} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\gamma_1\ln(\gamma_1) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\gamma_2\ln(\gamma_2) -$$
$$\sum_{v=0}^{m-1}(2v-2)!(\zeta_2-\zeta_1)^{2v}\bigg[\frac{1}{(\zeta_1)^{2v-1}}\mathbf{J}\bigg(\Phi_v\bigg(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\bigg)\bigg) + \frac{1}{(\zeta_2)^{2v-1}}\mathbf{J}\bigg(\Phi_v\bigg(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)\bigg)\bigg], \tag{4.18}$$

where $\mathbf{J}\bigg(\Phi_v\bigg(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)\bigg)$ is given in (4.16).

Remark 4.8. Put $h = 1, a = 0, b = m, p_1(v) = (p_1)_v$ and $p_2(v) = (p_2)_v$, the inequality (4.14) takes the following new form and gives new bound for discrete Jeffreys distance:

$$J_a(p_1,p_2) = \sum_{v=1}^{m}(p_1-p_2)_j \ln\frac{(p_1)_v}{(p_2)_v} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}(\gamma_1-1)\ln(\gamma_1) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}(\gamma_2-1)\times$$
$$\ln(\gamma_2) - \sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\bigg[\bigg(\frac{(2v-1)!}{(\zeta_1)^{2v}} + \frac{(2v-2)!}{(\zeta_1)^{2v-1}}\bigg)\mathbf{J}\bigg(\Phi_v\bigg(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\bigg)\bigg)$$
$$+\bigg(\frac{(2v-1)!}{(\zeta_2)^{2v}} + \frac{(2v-2)!}{(\zeta_2)^{2v-1}}\bigg)\mathbf{J}\bigg(\Phi_v\bigg(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)\bigg)\bigg], \tag{4.19}$$

where $\mathbf{J}\bigg(\Phi_v\bigg(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)\bigg)$ is given in (4.16).

Remark 4.9. Take $h = 1, a = 0, b = m, p_1(v) = (p_1)_v$ and $p_2(v) = (p_2)_v$, the inequality (4.15) takes the following new form and gives new bound for discrete triangular discrimination:

$$\sum_{v=1}^{m}\frac{[(p_2)_v-(p_1)_v]^2}{(p_1)_v+(p_2)_v} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\frac{(\gamma_1-1)^2}{\gamma_1+1} + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\frac{(\gamma_2-1)^2}{\gamma_2+1} - \sum_{v=0}^{m-1}4(2v)!\times$$
$$(\zeta_2-\zeta_1)^{2v}\bigg[\frac{1}{(\zeta_1+1)^{2v+1}}\mathbf{J}\bigg(\Phi_v\bigg(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\bigg)\bigg) + \frac{1}{(\zeta_2+1)^{2v+1}}\mathbf{J}\bigg(\Phi_v\bigg(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)\bigg)\bigg],$$

where $\mathbf{J}\bigg(\Phi_v\bigg(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)\bigg)$ is given in (4.16).

4.3. Inequalities in q-calculus

The following inequalities give new bound of Csiszár divergence, Shannon entropy, Kullback-Leibler divergence, Jeffrey distance and triangular discrimination in q-calculus respectively.
Let $\mathbb{T} = q^{\mathbb{N}_0}, q > 1, a = q^k$ and $b = q^m$ with $k < m$, the inequality (3.5) takes the following form in q-calculus:

$$\sum_{v=k}^{m-1}q^{v+1}p_2(q^v)\Psi\bigg(\frac{p_1(q^v)}{p_2(q^v)}\bigg) \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\Psi(\gamma_1) + \frac{1-\gamma_1}{\gamma_2-\gamma_1}\Psi(\gamma_2) - \sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\times$$
$$\bigg[\Psi^{(2v)}(\zeta_1)\mathbf{J}\bigg(\Phi_v\bigg(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\bigg)\bigg) + \Psi^{(2v)}(\zeta_2)\mathbf{J}\bigg(\Phi_v\bigg(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)\bigg)\bigg],$$

where

$$\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right) = \frac{\gamma_2-1}{\gamma_2-\gamma_1}\Phi_v\left(\frac{\gamma_1-\zeta_1}{\zeta_2-\zeta_1}\right)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\Phi_v\left(\frac{\gamma_2-\zeta_1}{\zeta_2-\zeta_1}\right)- \sum_{v=k}^{m-1} q^{v+1}\Phi_v\left(\frac{p_1(q^v)-\zeta_1 p_2(q^v)}{\zeta_2-\zeta_1}\right). \tag{4.20}$$

Use $\mathbb{T}=q^{\mathbb{N}_0}, q>1, a=q^k$ and $b=q^m$ with $k<m$, in Theorem 4.1 - Theorem 4.4, inequalities (4.2), (4.4), (4.6) and (4.8) take the following new form in q-calculus, respectively:

$$\sum_{v=k}^{m-1} q^{v+1}p_2(q^v)\ \log\frac{1}{p_2(q^v)} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\log(\gamma_1)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\log(\gamma_2)-\sum_{v=k}^{m-1} q^{v+1}p_2(q^v)\times \log[p_1(q^v)]-\sum_{v=0}^{m-1}(2v-1)!(\zeta_2-\zeta_1)^{2v}\left[\frac{1}{(\zeta_1)^{2v}}\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\frac{1}{(\zeta_2)^{2v}}\times \mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right],$$

$$\sum_{v=k}^{m-1} q^{v+1}p_1(q^v)\ \ln\frac{p_1(q^v)}{p_2(q^v)} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\gamma_1\ln(\gamma_1)+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\gamma_2\ln(\gamma_2)-\sum_{v=0}^{m-1}(2v-2)!\times (\zeta_2-\zeta_1)^{2v}\left[\frac{1}{(\zeta_1)^{2v-1}}\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\frac{1}{(\zeta_2)^{2v-1}}\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right],$$

$$\sum_{v=k}^{m-1} q^{v+1}[p_1(q^v)-p_2(q^v)]\ \ln\frac{p_1(q^v)}{p_2(q^v)} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}(\gamma_1-1)\ln(\gamma_1) +\frac{1-\gamma_1}{\gamma_2-\gamma_1}(\gamma_2-1)\ln(\gamma_2)-\sum_{v=0}^{m-1}(\zeta_2-\zeta_1)^{2v}\left[\left(\frac{(2v-1)!}{(\zeta_1)^{2v}}+\frac{(2v-2)!}{(\zeta_1)^{2v-1}}\right)\times \mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\left(\frac{(2v-1)!}{(\zeta_2)^{2v}}+\frac{(2v-2)!}{(\zeta_2)^{2v-1}}\right)\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right],$$

$$\sum_{v=k}^{m-1} q^{v+1}\frac{[p_2(q^v)-p_1(q^v)]^2}{p_1(q^v)+p_2(q^v)} \leq \frac{\gamma_2-1}{\gamma_2-\gamma_1}\frac{(\gamma_1-1)^2}{\gamma_1+1}+\frac{1-\gamma_1}{\gamma_2-\gamma_1}\frac{(\gamma_2-1)^2}{\gamma_2+1}- \sum_{v=0}^{m-1}4(2v)!(\zeta_2-\zeta_1)^{2v}\left[\frac{1}{(\zeta_1+1)^{2v+1}}\mathbf{J}\left(\Phi_v\left(\frac{\zeta_2-t}{\zeta_2-\zeta_1}\right)\right)+\frac{1}{(\zeta_2+1)^{2v+1}}\times \mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right].$$

where $\mathbf{J}\left(\Phi_v\left(\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\right)$ is given in (4.20).

5. Zipf-Mandelbrot law

In the field of information sciences, Zipf's law is used for indexing [32,60], in ecological field studies [49] and it plays an important role in art for identifying the aesthetics criteria in music [47].

For $m \in \{1, 2, \dots\}, r \geq 0$ and $l > 0$ the Zipf-Mandelbrot law (probability mass function) is defined as

$$f(v; m, r, l) = \frac{1}{(v + r)^l H_{m,r,l}}, \quad v = 1, \dots, m, \tag{5.1}$$

where

$$H_{m,r,l} = \sum_{u=1}^{m} \frac{1}{(u + r)^l} \tag{5.2}$$

is a generalization of a harmonic number.
Let $m \in \{1, 2, \dots\}, r \geq 0$ and $l > 0$, then Zipf-Mandelbrot entropy may be defined as

$$Z(H; r, l) = \frac{l}{H_{m,r,l}} \sum_{v=1}^{m} \frac{\ln(v + r)}{(v + r)^l} + \ln(H_{m,r,l}). \tag{5.3}$$

Assume

$$q_v = f(v; m, r, l) = \frac{1}{(v + r)^l H_{m,r,l}}. \tag{5.4}$$

Use $(p_2)_v = \frac{1}{(v+r)^l H_{m,r,l}}$ in (4.17) to get the following result which establishes the link of Mandelbrot entropy (5.3) with discrete Shannon entropy:

$$Z(H; r, l) \leq \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} \log(\gamma_1) + \frac{1 - \gamma_1}{\gamma_2 - \gamma_1} \log(\gamma_2) - \sum_{v=1}^{m} \left(\frac{1}{(v + r)^l H_{m,r,l}} \right) \log(p_1)_v -$$
$$\sum_{v=0}^{m-1} (2v - 1)!(\zeta_2 - \zeta_1)^{2v} \left[\frac{1}{(\zeta_1)^{2v}} \mathbf{J}\left(\Phi_v\left(\frac{\zeta_2 - t}{\zeta_2 - \zeta_1} \right) \right) + \frac{1}{(\zeta_2)^{2v}} \mathbf{J}\left(\Phi_v\left(\frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) \right) \right],$$

where

$$\begin{aligned} \mathbf{J}\left(\Phi_v\left(\frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) \right) &= \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} \Phi_v\left(\frac{\gamma_1 - \zeta_1}{\zeta_2 - \zeta_1} \right) + \frac{1 - \gamma_1}{\gamma_2 - \gamma_1} \Phi_v\left(\frac{\gamma_2 - \zeta_1}{\zeta_2 - \zeta_1} \right) - \\ & \sum_{v=1}^{m} \Phi_v\left(\frac{(p_1)_v - \frac{\zeta_1}{(v+r)^l H_{m,r,l}}}{\zeta_2 - \zeta_1} \right). \end{aligned}$$

Use $(p_1)_v = \frac{1}{(v+r_1)^{l_1} H_{m,r_1,l_1}}$ and $(p_2)_v = \frac{1}{(v+r_2)^{l_2} H_{m,r_2,l_2}}$ in (4.18) to get following result which establishes the link of Mandelbrot entropy (5.3) with Kullback-Leibler divergence:

$$Z(H; r_1, l_1) \geq \frac{l_2}{H_{m,r_1,l_1}} \sum_{v=1}^{m} \frac{\ln(v + r_2)}{(v + r_1)^{l_1}} + \ln(H_{n,r_2,l_2}) - \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} \gamma_1 \ln(\gamma_1)$$
$$- \frac{1 - \gamma_1}{\gamma_2 - \gamma_1} \gamma_2 \ln(\gamma_2) + \sum_{v=0}^{m-1} (2v - 2)!(\zeta_2 - \zeta_1)^{2v} \left[\frac{1}{(\zeta_1)^{2v-1}} \mathbf{J}\left(\Phi_v\left(\frac{\zeta_2 - t}{\zeta_2 - \zeta_1} \right) \right) \right.$$
$$\left. + \frac{1}{(\zeta_2)^{2v-1}} \mathbf{J}\left(\Phi_v\left(\frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) \right) \right],$$

where $H_{m,r_1,l_1} = \frac{1}{(v+r_1)^{l_1}}, H_{m,r_2,l_2} = \frac{1}{(v+r_2)^{l_2}}$ and

$$\begin{aligned} \mathbf{J}\left(\Phi_v\left(\frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) \right) &= \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} \Phi_v\left(\frac{\gamma_1 - \zeta_1}{\zeta_2 - \zeta_1} \right) + \frac{1 - \gamma_1}{\gamma_2 - \gamma_1} \Phi_v\left(\frac{\gamma_2 - \zeta_1}{\zeta_2 - \zeta_1} \right) - \\ & \sum_{v=1}^{m} \Phi_v\left(\frac{\frac{1}{(v+r_1)^{l_1} H_{m,r_1,l_1}} - \frac{\zeta_1}{(v+r_2)^{l_2} H_{m,r_2,l_2}}}{\zeta_2 - \zeta_1} \right). \end{aligned}$$

.

Remark 5.1. Similarly, use $(p_1)_v = \frac{1}{(v+r_1)^{l_1} H_{m,r_1,l_1}}$ and $(p_2)_v = \frac{1}{(v+r_2)^{l_2} H_{m,r_2,l_2}}$ in (4.19) to find the relationship of Jeffreys distance $J(p_1, p_2)$ with Mandelbrot entropy (5.3).

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