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π-BAER $*$-RINGS

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Abstract. A $*$-ring R is called a π-Baer $*$-ring, if for any projection invariant left ideal Y of R, the right annihilator of Y is generated, as a right ideal, by a projection. In this note, we study some properties of such $*$-rings. We indicate interrelationships between the π-Baer $*$-rings and related classes of rings such as π-Baer rings, Baer $*$-rings, and quasi-Baer $*$-rings. We announce several results on π-Baer $*$-rings. We show that this notion is well-behaved with respect to polynomial extensions and full matrix rings. Examples are provided to explain and delimit our results.

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1. Introduction

Throughout this paper R denotes an associative ring with unity. Let us recall that a *$*$-ring* (or an *involutive ring*) R is a ring with an operation $* : R \to R$, called *involution*, such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, and $(x^*)^* = x$, for all $x, y \in R$. An idempotent p of a $*$-ring R is called a *projection* if p is self-adjoint ($p^* = p$). An idempotent $e \in R$ is called *right* (resp., *left*) *semicentral* if $ex = exe$ (resp., $xe = exe$), for all $x \in R$ [11]. We denote by $\mathbf{S}_r(R)$ (resp., $\mathbf{S}_\ell(R)$) the set of all right (resp., left) semicentral idempotents of R. If X is a nonempty subset of R, then $r_R(X)$ (resp., $\ell_R(X)$) is used for the right (resp., left) annihilator of X over R. We use $\mathrm{M}_n(R)$, $R[x]$, and $R[[x]]$ for the n by n full matrix ring over R, the ring of polynomials, and the ring of formal power series, respectively. The ring of integers and the ring of integers modulo n are denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively.

Recall from [17], a ring R is called a *Baer ring* if the right annihilator of every nonempty subset of R is generated, as a right ideal, by an idempotent. If R is a $*$-ring, then R is called a *Baer $*$-ring* if the right annihilator of every nonempty subset is generated, as a right ideal, by a projection. Kaplansky shows in [17] that the definitions of a Baer ring and a Baer $*$-ring are left-right symmetric. The subject of Baer $*$-rings is essentially pure algebra, with historic roots in operator algebras and lattice theory. Baer $*$-rings are a common generalization of AW*-algebras and complete $*$-regular rings. The AW*-algebras are precisely the Baer $*$-rings that happen to be C*-algebras, the complete $*$-regular rings are the Baer $*$-rings that happen to be regular in the sense of von Neumann.

Various weaker versions of Baer and Baer $*$-rings have been studied. From [15], a ring R is *quasi-Baer* if the right annihilator of every right ideal is generated, as a right ideal, by an idempotent. This is a nontrivial generalization of the class of Baer rings. For example, prime rings with nonzero right singular ideal are quasi-Baer and not Baer, since Baer rings are nonsingular. The quasi-Baer ring property is left-right symmetric.

In [10], Birkenmeier and Park introduced a *quasi-Baer $*$-ring* as a $*$-ring R in which the right annihilator of every ideal is generated by a projection. As in the case of Baer $*$-rings, the involution can be used to show that this notion is left-right symmetric. If R is a commutative non-Prüfer domain then $\mathrm{M}_n(R)$, with the transpose involution, is a quasi-Baer $*$-ring which is not a Baer $*$-ring.

In [6], Birkenmeier et al. introduced another generalization of Baer rings. Recall that a ring R is said to be a *π-Baer ring* if the right annihilator of every projection invariant left ideal Y (i.e., $Ye \subseteq Y$ for all $e = e^2 \in R$) is generated by an idempotent. Like the Baer and quasi-Baer properties, the π-Baer property is left-right symmetric. The π-Baer condition is strictly between the Baer and quasi-Baer conditions.

To transfer the quasi-Baer $*$-condition from a $*$-ring R to various extensions (e.g., full matrix rings over R or $R[x]$ or $R[[x]]$) one needs no additional conditions which is certainly not the case for the Baer $*$-condition (see [8, Example 2.3 and Theorem 2.5]). So, it is natural to ask: is there a condition strictly between the Baer $*$ and quasi-Baer $*$-conditions, which is able to combine some of the notable features of the Baer $*$ and quasi-Baer $*$-conditions?

On the other hand, in the presence of an involution, the projections are vastly easier to work with than idempotents. In this paper, we introduce a *π-Baer $*$-ring* as a $*$-ring R in which the right annihilator of every projection invariant left ideal of R is generated by a projection. These $*$-rings are generalizations of Baer $*$-rings, and there are examples distinguishing these classes.

The organization of our paper is as follows. In Section 2, we introduce the notion of π-Baer $*$-rings, and we study its properties and relations with other Baer-type rings such as π-Baer rings, Baer $*$-rings, and quasi-Baer $*$-rings.

Section 3 is devoted to the study of extensions of π-Baer $*$-rings. We prove that the n by n full matrix rings over π-Baer $*$-rings are π-Baer $*$-rings. It is shown that being a π-Baer $*$-ring is preserved by polynomial extensions. Also, it is proved that the essential overrings of π-Baer $*$-rings are π-Baer $*$-rings.

2. Basic results

In this section, the π-Baer $*$-rings and and their basic properties are introduced. Furthermore, the relations between the notion of a π-Baer $*$-ring and other Baer-type notions are verified.

Definition 2.1. Let R be a $*$-ring. We say that R is a π-Baer $*$-ring if for any projection invariant left ideal Y of R, the right annihilator of Y is generated, as a right ideal, by a projection; i.e., there is a projection $p \in R$ such that $r_R(Y) = pR$.

Remark 2.2. Let R be a $*$-ring.

(i) If R is a π-Baer $*$-ring then R has a unity. This follows from Definition 2.1 by taking $Y = 0$.

(ii) The definition of a π-Baer $*$-ring is left-right symmetric. For this, let R be a π-Baer $*$-ring and let Y be a projection invariant right ideal of R. It is not hard to see that Y^* is a projection invariant left ideal of R. Then $r_R(Y^*) = pR$, for some projection $p \in R$. Hence $\ell_R(Y) = (r_R(Y^*))^* = Rp$.

In the following example we see that there is a $*$-ring which is a π-Baer ring, but it is not a π-Baer $*$-ring.

Example 2.3. Let R be the free ring $\mathbb{Z}\langle x, y\rangle$. Then by [6, Example 2.1] R is π-Baer and the only idempotents of R are 0 and 1. Take $T = R \oplus R^{op}$, where R^{op} denotes the opposite ring of R. Let $* : T \to T$ be the exchange involution (i.e., $(a, b)^* = (b, a)$). By [6, Proposition 2.10] the $*$-ring T is π-Baer. We show that T is not a π-Baer $*$-ring. Note that the only projections of T are $(0, 0)$ and $(1, 1)$. One can show that $r_T(R) = (0, 1)T$. So that $r_T(R)$ does not contain a nonzero projection of T. Hence T is not a π-Baer $*$-ring.

Lemma 2.4. *Let R be a $*$-ring.*

(i) *If $p \in R$ is a projection and pR is a projection invariant right ideal, then p is central.*

(ii) *If $p \in R$ is a projection and Rp is a projection invariant left ideal, then p is central.*

Proof. We prove only Part (i). Note that [6, Lemma 2.1] implies that $p \in \mathbf{S}_\ell(R)$. Now, by [1, Lemma 2.3(i)] p is central. □

The following result will be used many times in the sequel.

Proposition 2.5. *Let R be a $*$-ring. Then the following are equivalent.*

(i) *R is a π-Baer $*$-ring;*
(ii) *R is a π-Baer ring in which every left (right) semicentral idempotent is a central projection;*
(iii) *For each projection invariant left (right) ideal Y, there exists a central projection $p \in R$ such that $r_R(Y) = pR$ ($\ell_R(Y) = Rp$).*

Proof. (i)⇒(ii) Let R be a π-Baer $*$-ring. Then R is π-Baer. Let $e \in \mathrm{S}_\ell(R)$. Since $(1-e)Re = 0$, we get $eR \subseteq r_R(R(1-e)R)$. But $r_R(R(1-e)R) \subseteq r_R(1-e) = eR$. Thus $r_R(R(1-e)R) = eR$. Since R is a π-Baer $*$-ring, there exists a projection $p \in R$ such that $eR = r_R(R(1-e)R) = pR$. Note that p is a central projection since pR is an ideal and this implies that p is a left semicentral projection and so it is central by [1, Lemma 2.3(i)]. Thus $e = pe = ep = p$ and e is a central projection.

(ii)⇒(iii) Let Y be a projection invariant left ideal of R. Then there is an idempotent $e \in R$ such that $r_R(Y) = eR$. By [6, Lemma 2.1(i)] eR is a projection invariant right ideal. Hence Lemma 2.4(i) implies that e is central.

(iii)⇒(i) It is obvious. □

Proposition 2.6. *Let R be a $*$-ring. Then the following are equivalent.*

(i) *R is a π-Baer $*$-ring.*
(ii) *For any nonempty subset S of R, if $Se \subseteq S$ for each idempotent $e \in R$, then there exists a central projection $p \in R$ such that $r_R(S) = pR$.*
(iii) *Every projection invariant right annihilator is generated, as a right ideal, by a central projection of R.*

Proof. (i)⇒(ii) Let R be a π-Baer $*$-ring and $\emptyset \neq S \subseteq R$ such that $Se \subseteq S$ for each idempotent $e \in R$. Then RS is a left ideal of R. Moreover, RS is a projection invariant left ideal of R since $(RS)e = R(Se) \subseteq RS$, for each idempotent $e \in R$. Thus, there exists a central projection $p \in R$ such that $r_R(RS) = pR$ by Proposition 2.5. Hence, $r_R(S) = r_R(RS) = pR$.

(ii)⇒(iii) Let $Y = r_R(S)$, for some $\emptyset \neq S \subseteq R$, be a projection invariant right annihilator. Then for each idempotent $e \in R$, $(\ell_R(Y)e)Y = \ell_R(Y)(eY) \subseteq \ell_R(Y)Y =$

0. Thus, $X = \ell_R(Y)$ is a projection invariant left ideal of R. Hence there exists a central projection $p \in R$ such that $r_R(X) = pR$ by Condition (ii). Then $r_R(S) = r_R(\ell_R(r_R(S))) = r_R(X) = pR$.

(iii)$\Rightarrow$(i) Let Y be a projection invariant left ideal of R. By [6, Lemma 2.1(i)] $r_R(Y)$ is a projection invariant right annihilator. From Condition (iii), $r_R(Y) = pR$ for some central projection $p \in R$. Hence, R is a π-Baer $*$-ring. □

Proposition 2.7. *Let R be a $*$-ring. Consider the following conditions.*

(i) *R is a Baer $*$-ring;*

(ii) *R is a π-Baer $*$-ring;*

(iii) *R is a quasi-Baer $*$-ring.*

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

Proof. The fact that every projection invariant left ideal is a subset yields the implication (i)$\Rightarrow$(ii). The implication (ii) $\Rightarrow$(iii) follows from the fact that every two-sided ideal is a projection invariant left ideal. □

We remark that when R is commutative, conditions (i), (ii), and (iii) of Proposition 2.7 are equivalent. The next example shows that the converse of each of the implications in Proposition 2.7 does not hold.

Example 2.8. (i) Let $R = \mathrm{M}_n(\mathbb{Z}[x])$. Consider R with the $*$-transpose involution. By [19, Theorem 7.7], the ring $\mathbb{Z}[x]$ is not a Prüfer domain. Thus [16, Theorem 2.3] implies that R is not a Baer $*$-ring. On the other hand, by [6, Theorem 4.1], $\mathbb{Z}[x]$ is a π-Baer ring. Then $\mathbb{Z}[x]$, with the identity map as the involution, is a π-Baer $*$-ring. Now Proposition 3.4 below implies that R is a π-Baer $*$-ring.

(ii) Let R be a prime ring with trivial idempotents which is not domain. For example, let $R = KG$, where K is a field of characteristic $p > 0$, and $G = C_p \wr A$ be the restricted wreath product of C_p, the cyclic group of order p, and an infinite elementary abelian p-group (see [14, Example 3.4]). Let $*$ be the involution on the group ring R defined by $(\sum a_g g)^* = \sum a_g^* g^{-1}$. Then R is a quasi-Baer $*$-ring that is not a π-Baer $*$-ring because R is not nonsingular.

Let R be a ring with an involution $*$. Recall that, $*$ is called a *proper* involution if for every $x \in R$, $xx^* = 0$ implies $x = 0$ [4]. Also, $*$ is called a *semiproper* involution if $xRx^* = 0$ implies $x = 0$ [12].

Proposition 2.9. *Let R be a π-Baer $*$-ring. Then $*$ is a semiproper involution, so R is semiprime.*

Proof. The proof follows from Proposition 2.7 and [12, Proposition 4.3] or [13, Lemma 10.2.13]. □

Corollary 2.10. *A $*$-ring R is a π-Baer $*$-ring if and only if R is a semiprime π-Baer ring in which every central idempotent is a projection.*

Proof. This result follows from Propositions 2.7 and 2.9 and the fact that in any semiprime ring all semicentral idempotents are central [13, Proposition 1.2.6]. □

The following example illustrates that there is an involutive semiprime π-Baer ring which is not a π-Baer $*$-ring.

Example 2.11. Let S be a prime ring and take $R = \mathrm{M}_n(S) \oplus \mathrm{M}_n(S)$. From [6, Corollary 2.2(iii), and Proposition 2.10] R is a π-Baer ring. Define $* : R \to R$ by $(A, B)^* = (B^*, A^*)$ for all $A, B \in \mathrm{M}_n(S)$, where A^* (resp., B^*) is the $*$-transpose of A (resp., B). Then $(1, 0)$ is a central idempotent, but it is not a projection. Thus R is not a π-Baer $*$-ring by Corollary 2.10.

Proposition 2.12. *Let R be a $*$-ring. Then the following are equivalent.*

(i) *R is a π-Baer $*$-ring.*

(ii) *For each projection invariant left (right) ideal Y of R, there exists a central projection $p \in R$ such that $Y \subseteq Rp$ and $r_R(Y) \cap Rp = 0$ ($\ell_R(Y) \cap pR = 0$).*

Proof. (i)⇒(ii) This implication follows from [6, Proposition 2.3], [13, Proposition 1.2.6], and Proposition 2.9.

(ii)⇒(i) Let Y be a projection invariant left ideal of R. Choose a central projection $p \in R$ such that $Y \subseteq Rp$ and $r_R(Y) \cap Rp = 0$. Then $(1-p)R = r_R(Rp) \subseteq r_R(Y)$. Let $a \in r_R(Y)$, then $a = ap + a(1-p)$. Since $ap \in r_R(Y) \cap Rp$, $ap = 0$. Thus $a = a(1-p) = (1-p)a \in (1-p)R$. Hence, $r_R(Y) \subseteq (1-p)R$. Therefore, R is a π-Baer $*$-ring. □

Let M_R be a right R-module. A submodule N_R of M_R is called *essential* in M_R if for any $x \in M \setminus \{0\}$, there exists $r \in R$ such that $0 \neq xr \in N$. Also recall a *right essential overring* T of R is an overring of R such that R_R is essential in T_R.

Recall that for a ring R, the *left socle* of R, $\mathrm{Soc}({}_RR)$, is defined as the sum of all minimal left ideals of R. Equivalently, $\mathrm{Soc}({}_RR)$ is the intersection of all essential left ideals of R (see [18, Exercise 6.12]). The *right socle*, $\mathrm{Soc}(R_R)$, is defined similarly. It is easy to check that both socles are ideals of R.

Recall from [9], that a ring R is called a *right FI-extending ring* if every ideal is right essential in an idempotent generated right ideal of R. A left FI-extending ring is defined similarly. A ring is called *FI-extending* if it is both right and left FI-extending.

Proposition 2.13. *If R is a π-Baer $*$-ring, then R is right and left FI-extending.*

Proof. This result follows from Propositions 2.7 and 2.9 and [9, Theorem 4.7] or [13, Theorem 3.2.37]. □

Corollary 2.14. *Let R be a π-Baer $*$-ring and X denotes the right socle of R, the prime radical of R, the Jacobson radical of R, or the right (or left) singular ideal of R. Then $R = A \oplus B$ (ring direct sum), where X is right and left essential in A and B has zero right socle, zero prime radical, zero Jacobson radical, or zero right (or left) singular ideal, respectively.*

Proof. This corollary is an immediate consequence of Proposition 2.13. □

Note that the rings A and B in Corollary 2.14 are π-Baer $*$-rings from Proposition 2.17 below. Thus, the class of π-Baer $*$-rings can be split into the subclasses of π-Baer $*$-rings with essential right socle and those with zero right socle, or those with essential prime radical and those with zero prime radical, or those with essential Jacobson radical and those with zero Jacobson radical, or those with essential right (left) singular ideal and those with zero right (left) singular ideal.

Proposition 2.15. *Let R be a $*$-ring and e a central projection of R. If R is a π-Baer $*$-ring, then eRe is also a π-Baer $*$-ring.*

Proof. The proof is straightforward. □

Proposition 2.16. *The center of a π-Baer $*$-ring is a Baer $*$-ring* (*and hence π-Baer $*$-ring*).

Proof. The proof follows from [13, Proposition 10.2.14] and Proposition 2.7. □

Proposition 2.17. *Let Λ be a nonempty set and let R_λ be a $*$-ring for each $\lambda \in \Lambda$. Then $R = \prod_{\lambda\in\Lambda} R_\lambda$ is a π-Baer $*$-ring if and only if R_λ is a π-Baer $*$-ring for each $\lambda \in \Lambda$.*

Proof. Assume that R is a π-Baer $*$-ring and $\lambda \in \Lambda$. Then Proposition 2.15 implies that R_λ is a π-Baer $*$-ring.

Conversely, assume that R_λ is a π-Baer $*$-ring, for each $\lambda \in \Lambda$. Let Y be a projection invariant left ideal of R. It is easy to see that $Y = \prod_{\lambda\in\Lambda} Y_\lambda$ for some projection invariant left ideals Y_λ of R_λ. As R_λ is a π-Baer $*$-ring, $r_{R_\lambda}(Y_\lambda) =$

$e_\lambda R_\lambda$, for some central projections $e_\lambda \in R_\lambda$. Then $r_R(Y) = \prod_{\lambda\in\Lambda} e_\lambda R_\lambda$. Put $e = (e_1, e_2, \ldots) \in R$, it is not hard to see that e is a projection of R. Hence $r_R(Y) = eR$ and that R is a π-Baer $*$-ring. □

Let R be a $*$-ring. An ideal I of R is said to be *$*$-essential* in R if $I \neq 0$ and $I \cap J \neq 0$ for any nonzero self-adjoint ideal J of R. An ideal P of R is said to be a *$*$-prime* ideal of R if $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$, where I and J are self-adjoint ideals of R (see [5]).

Proposition 2.18. *Every $*$-prime (prime) ideal of a π-Baer $*$-ring R is either generated by a central projection or it is a $*$-essential (essential) ideal.*

Proof. We prove the case of $*$-prime ideal, the other one can be proved similarly. Let P be a $*$-prime ideal of R. If P is not $*$-essential in R, then there is a nonzero self-adjoint ideal I of R such that $P \cap I = 0$. Since R is a π-Baer $*$-ring, there exists a central projection $e \in R$ such that $r_R(I) = eR$. One can show that $P \subseteq r_R(I) = eR$. On the other hand, I and eR are self-adjoint and $I(eR) = 0$. Then $I \subseteq P$ or $eR \subseteq P$, since P is $*$-prime. If $I \subseteq P$ then $I \cap P = I = 0$ which leads a contradiction. Hence $e \in P$ and that $P = eR$. □

Proposition 2.19. *Let R be a π-Baer $*$-ring. Then there is no nonzero ideal I of R such that $r_R(I)$ is $*$-essential in R.*

Proof. Let I be a nonzero ideal of R such that $r_R(I)$ is $*$-essential in R. Since R is a π-Baer $*$-ring, there exists a central projection $e \in R$ such that $r_R(I) = eR$. As $I \neq 0$, $e \neq 1$. Then $(1-e)R$ is a nonzero self-adjoint ideal and $r_R(I) \cap (1-e)R = eR \cap (1-e)R = 0$, a contradiction since $r_R(I)$ is $*$-essential in R. Hence, $I = 0$. □

Recall from [3] that, a ring R is said to satisfy the *IFP (insertion of factors property)* if $r_R(x)$ is an ideal of R for all $x \in R$. A ring R is called *abelian* if every idempotent in it is central. It is evident that any reduced ring satisfies IFP and any ring with IFP is abelian.

Proposition 2.20. *Let R be a $*$-ring. Then the following conditions are equivalent.*

(i) *R is an Abelian π-Baer $*$-ring;*
(ii) *R is an Abelian Baer $*$-ring;*
(iii) *R is a reduced quasi-Baer $*$-ring;*
(iv) *R is quasi-Baer $*$-ring with IFP;*
(v) *R is a reduced Baer $*$-ring;*
(vi) *R is a Baer $*$-ring with IFP;*

(vii) *R is a reduced π-Baer $*$-ring;*

(viii) *R is a π-Baer $*$-ring with IFP.*

Proof. (i)⇒(ii) This implication follows from [6, Lemma 2.3].

(ii)⇒(iii) This implication follows from [7, proposition 1.5] and the fact that every Baer $*$-ring is a quasi-Baer $*$-ring.

(iii)⇒(iv) It is trivial.

(iv)⇒(v) If quasi-Baer $*$-ring R satisfies IFP, then it is immediate that R is a Baer $*$-ring. Since every ring with IFP is abelian, the implication (ii)⇒(iii) implies that R is reduced.

(v)⇒(vi) It is trivial.

(vi)⇒(vii) By Proposition 2.7, R is a π-Baer $*$-ring. Since every ring with IFP is abelian, the implication (ii)⇒(iii) yields that R is reduced.

(vii)⇒(viii) It is trivial.

(viii)⇒(i) It is trivial. □

3. Extensions

Theorem 3.1. *Let R be a π-Baer $*$-ring, T a right (or left) essential overring of R, and the involution of R extends to T. Then T is a π-Baer $*$-ring and R contains all central projections of T.*

Proof. Let Y be a projection invariant left ideal of T and $X = Y \cap R$. It is easily seen that X is a projection invariant left ideal of R. So there is a central projection $p \in R$ such that $r_R(X) = pT$. We claim that $r_T(Y) = pT$. Let $a \in r_T(Y)$ and assume that $0 \neq (1-p)a$. Since $R_R \leq^{ess} T_R$, there exists $r \in R$ such that $0 \neq (1-p)ar \in R$. Then $0 \neq (1-p)ar \in r_R(Y) \subseteq r_R(X)$, a contradiction. Thus, $r_T(Y) \subseteq pT$. Now, suppose $pT \nsubseteq r_T(Y)$. Then there is $y \in Y$ such that $0 \neq yp$. Since ${}_RR \leq^{ess} {}_RT$ by [1, Lemma 2.26], there is $s \in R$ such that $0 \neq syp \in R$. Hence $syp \in Y \cap R = X$. Then $syp = (syp)p \in Xp = 0$, a contradiction. Therefore, $r_T(Y) = pT$, and consequently T is a π-Baer $*$-ring.

To prove the last part of the statement, let $p \in T$ be a central projection. Set $Y = T(1-p)$ and $X = Y \cap R$. Then there exists a central projection $q \in R$ such that $r_R(X) = qR$. It follows that $r_T(Y) = qT$. On the other hand, $r_T(Y) = pT$. So $pT = qT$ and $p = q$. Hence R contains all central projections of T. □

We need the following Lemma will be useful.

Lemma 3.2 ([11], Theorem 2.3)**.** *For a ring R, let T be $R[X]$, or $R[[X]]$, where X is a nonempty set of not necessarily commuting indeterminates. If $e(x) \in \mathrm{S}_\ell(T)$, then $e_0 \in \mathrm{S}_\ell(R)$, where e_0 is the constant term of $e(x)$. Moreover, $e(x)T = e_0T$.*

It was shown in [2] that a reduced ring R is a Baer ring if and only if $R[x]$ is a Baer ring. In the next theorem, we show that being a π-Baer $*$-ring is preserved by various polynomial extensions. Note that the involution of a $*$-ring R can be naturally extended to an involution on $R[x]$, and $R[[x]]$.

Theorem 3.3. *Let R be a $*$-ring. Then the following conditions are equivalent:*

(i) *R is a π-Baer $*$-ring;*
(ii) *$R[x]$ is a π-Baer $*$-ring;*
(iii) *$R[[x]]$ is a π-Baer $*$-ring.*

Proof. We will prove the equivalence (i)$\Leftrightarrow$(iii). The other one can be proved similarly. Assume that R is a π-Baer $*$-ring. Let Y be a projection invariant right ideal of $T := R[[x]]$. By [6, Theorem 4.1], T is a π-Baer ring. Thus, $\ell_T(Y) = Te(x)$ for some idempotent $e(x) \in \mathrm{S}_\ell(T)$. By Lemma 3.2, $e_0 \in \mathrm{S}_\ell(R)$ where e_0 is the constant term of $e(x)$. By Proposition 2.5, e_0 is a central projection, since R is a π-Baer $*$-ring. By Lemma 3.2, $e(x)T = e_0T$. Then $T(e(x))^* = Te_0^* = e_0T = e(x)T$, and hence $e(x) = e(x)(e(x))^* = (e(x))^*$. So $e(x)$ is a projection of T. Thus T is a π-Baer $*$-ring.

Conversely, suppose that T is a π-Baer $*$-ring. Let Y be a projection invariant right ideal of R. By [6, Lemma 4.1(iv)], $Y[[x]]$ is a projection invariant right ideal of T. Then $\ell_T(Y[[x]]) = Te(x)$ for some central projection $e(x)$ of T. Since $e(x) \in T$ is a projection, it follows that e_0 is a projection of R. We claim that $\ell_R(Y) = Re_0$. As $e(x)Y = 0$, then $c_0Y = 0$. So $Re_0 \subseteq \ell_R(Y)$. Let $a \in \ell_R(Y)$. Then $a \in \ell_T(Y[[x]])$. Hence $a = ae(x) = ae_0 + ae_1x + \cdots$. Thus, $ae_0 = a$ and $ae_1 = ae_2 = \cdots = 0$. Therefore, $\ell_R(Y) = Re_0$, and consequently R is a π-Baer $*$-ring. □

Proposition 3.4. *Let R be a π-Baer $*$-ring. Then $\mathrm{M}_n(R)$, with the $*$-transpose involution, is a π-Baer $*$-ring for each positive integer n.*

Proof. Let R be a π-Baer $*$-ring. Then by Proposition 2.7, R is quasi-Baer $*$-ring. Now [8, Proposition 2.6] implies that $\mathrm{M}_n(R)$ is a quasi-Baer $*$-ring. Since $\mathrm{M}_n(R)$ is generated by its idempotents, every projection invariant one-sided ideal of $\mathrm{M}_n(R)$ is an ideal of $\mathrm{M}_n(R)$ by [6, Corollary 2.2(iii)]. Hence, $\mathrm{M}_n(R)$ is a π-Baer $*$-ring. Moreover, this assertion follows from [6, Proposition 3.2] and Proposition 2.5. □

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