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Boolean Hypercubes, Classification of Natural Numbers, and the Collatz Conjecture

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Abstract

Using simple arguments derived from the Boolean hypercube configuration, the structure of natural spaces, and the recursive exponential generation of the set of natural numbers, a linear classification of the natural numbers is presented. The definition of a pseudolinear Collatz operator, the description of the set of powers of 2, and the construction of the natural numbers via this power set might heuristically prove the Collatz conjecture from an empirical point of view.

1. Introduction

The Collatz conjecture, associated with a simple computational algorithm over the natural set $\mathbb{N}$, remains unsolved as far as the authors know, despite having attracted the interest of many researchers; see reference [1] for literature on the subject and references [2]–[21] for recent studies and more literature. Recently, a preprint claimed to have reached the solution, but the file still is not posted [22]. More recently, a paper [23] indicates the intractable nature of the conjecture.

1.1. Preliminary considerations

Reference [1] reported that a good deal of computations was performed on $\mathbb{N}$ subsets with the assistance of a Python 3.0 program. Some calculations were related to large Mersenne numbers, and others with numerical series of similar structures of natural numbers. Despite the significant quantity of tested numbers in reference [1], no natural number studied was found *Collatz **not** compliant*. Any tested natural number submitted to the Collatz algorithm, see the **Appendix** for more details, seems to yield unity.

The largest Mersenne number[1] tested recently in this laboratory is number 39 on the list of Mersenne primes; see the web page in reference [24]: $\mu_2(13466917) \approx 3 \cdot 10^{21}$, which has 4053946 digits and has taken 51440634 steps of the Collatz algorithmic path to yield 1. Such a calculation has expended a considerable time, even running on a ten-core i9 CPU.

One can be confident in the previous computational experience, see reference [1]. Also, while writing the present paper, we performed a large set of additional tests. In this way, we observed more information about the behavior of the Collatz algorithm. All tests keep yielding no natural number Collatz compliant exception.

Considering this, the present authors have carried on a previous discussion, connecting the quantum mechanical harmonic oscillator with the Collatz conjecture [19]. The present study corresponds to another point of view of the problem based on numerical and algebraic empirical considerations.

[1]A Mersenne number can be described as: $\mu_2(N) = 2^N - 1$.

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1.2. Empirical computational proof of Collatz conjecture frontiers

In the opinion of the present authors, outside of the strictly computational area, some action has to be done to prove empirically that all the elements of the set of natural numbers $\mathbb{N}$ are Collatz compliant. Such an aim is based, apart from the previously mentioned computations, on two points consisting in that:

First, Gödel-like reasoning can be easily applied [25] to the solution of the Collatz conjecture by computational means. As in this case will be no a priori limit to testing a natural number if, for instance, it is chosen as a Mersenne number $\mu_2(N)$, with N growing indefinitely. But also, one can consider that at some step S of the Collatz path, see the **Appendix** for more information, which can be represented as: $C_S[\mu_2(N)] = M$, a resultant natural number M might grow larger than $\mu_2(N)$, making any attempt to use an induction reasoning useless.

Second, the application of the Erdös discrepancy conjecture [26] permits us to admit that there will be, in any dimension, some Boolean vector $\langle \mathbf{h}_M|$ present, which might be the binary representation of a more significant decimal number. Note that the index M means that it can be any binary representation of the natural numbers in the interval: $\left\{2^N;\mu_2[M]\right\}$

Due to all of these previous ideas, continuing to try to understand the Collatz problem by following a different way than the previous experience, the present paper seeks to empirically demonstrate that the set of natural numbers $\mathbb{N}$ is heuristically Collatz compliant.

1.3. Structure of this study on Collatz conjecture

To achieve this objective, the current analysis will be developed as follows.

Initially, we set as a starting point the description of how the Collatz algorithm and the definition of a Collatz operator work. Essentially, this corresponds to describing the operator action on the set of powers of 2 and the possibility of considering the Collatz operator (pseudo-)distributive concerning the sum of two natural numbers.

Next appears the discussion of an essential part of this work: the structure of Boolean hypercubes and the possibility of describing a recursive building of the natural number set. Such construct permits to devise of an empirical-heuristic demonstration of the Collatz algorithm convergence for all elements of the natural number set.

The following section studies the formalism of natural and Collatz vector spaces, shown to correspond to an extension of the application of the Collatz algorithm to N-tuples constructed with natural numbers.

After these preliminaries, the discussion studies the expression of any natural number, taking the set of natural powers of 2: $2^{\mathbb{N}}$, as a basis. That opens the door to analyzing the action of the Collatz operator, defined in reference [1] over *any* natural number.

Afterward, the line of work directs to the description of the classification of natural numbers using what can be named (one)classes.

Finally, such a previous step allows empirically studying the Collatz compliance of both natural numbers and natural classes. Such a view of the natural numbers proves $\mathbb{N}$ is heuristically Collatz compliant.

2. Collatz Algorithm and Collatz Operator

2.1. Introduction: Collatz algorithm and operator, Collatz algorithm path and steps, and Collatz compliance

Although an algorithm variant leads to a shorter number of Collatz algorithm steps, namely the Syracuse algorithm, the original Collatz algorithm is easily described for the present paper using pseudocode, presented in the **Appendix**. As already explained in reference [1], we performed many computations within both algorithmic paths, the original and the Syracuse. The previous results yield no exception of Collatz compliance in any of the substantial natural number set tested.

Also, the Collatz algorithm can be formally defined using:

$$\forall n \neq m \wedge S \in \mathbb{N} : C_S[n] = m \in \mathbb{N}, \tag{2.1}$$

where the subindex S means the Collatz heuristic operator $C[\circ]$ has been applied S-th times over an initial natural number n yielding the natural number m.

Such an operator, described in the equation (2.1) applied over a natural number, corresponds to arriving, after a sequence of S steps, at one location on the Collatz n-path as defined by the **Appendix** algorithm, while the resultant natural number m might appear greater than 1.

In the case that the Collatz algorithm applied to the number n reaches in his n-path the ending number 1 after S steps, that is:

$$n \in \mathbb{N} : C_S[n] = 1, \tag{2.2}$$

we can denominate the natural number n as Collatz compliant.

Nonetheless, the number of times the operator is applied to a given natural number in the path to reach the number 1, is relatively irrelevant when one tries only to see if a given natural number is Collatz compliant. If this is the case, we can simplify the notation and write the end of the Collatz algorithm applied on any number using the operator $C[\circ]$ without a subindex, like:

$$n \in N : C[n] = 1, \tag{2.3}$$

or with some explicit subindex symbol, like in the equation (2.2), meaning that for the natural number n, the Collatz algorithm has ended. Thus, indicating that n is Collatz compliant following a corresponding n-path. That is: a sequence of natural numbers starting at n and ending at 1 after following several n-path steps S.

It is interesting to note that, according to the Collatz algorithm shown in the **Appendix**, the equalities: $C[0] = 1$ and $C[1] = 1$ hold in any case.

When the Collatz algorithm path ends with unit 1, the Collatz operator, if the Collatz conjecture is true, can be considered as a transformation of each element of the whole natural number set $\mathbb{N}$ into 1. That is, one can suppose the following equality involving all the natural number set:

$$C[\mathbb{N}] = (1, 1, 1, \ldots 1, \ldots) = \langle \mathbf{1}|$$

will hold, and thus symbolically represents the so-called Collatz conjecture.

2.2. The behavior of the Collatz operator and stopping Collatz algorithm before the end

A remark about the application of the Collatz operator (2.1) is due here. Any Collatz n-path sequence of a given number n, if converging to 1, implies that at some n-path step, one shall necessarily arrive at a number that has to be *lesser than the initial tested number*.

It must be kept in mind now that there is no problem in supposing that, when applying the Collatz operator to a number n, all the previous numbers $m < n$ tested are Collatz compliant: $C[m] = 1$.

That is, supposing that the application of the operator depicted in the equation (2.1) to a number n is performed after using, previously and systematically, the Collatz operator on an increasing sequence of natural numbers like:

$$\mathbb{M}_n = \{0, 1, 2, \ldots, m, \ldots, (n-1)\} \tag{2.4}$$

and we can hypothesize that we previously found it Collatz compliant fulfilling:

$$C[\mathbb{M}_n] = \langle \mathbf{1}_n| \, . \tag{2.5}$$

If at some step S of the n-path the Collatz algorithm arrives at some number $m \in \mathbb{M}_n$, being $m < n$, then at this moment the Collatz algorithm does not need to continue, whenever one can suppose that the m-path is known and converges:

$$if \quad C_S[n] = m \wedge m < n \wedge C[m] = 1 \Rightarrow C[n] = 1.$$

Such a possibility might shorten the number of steps dramatically in the Collatz algorithm in some cases, at the expense of compulsorily implementing a comparison between the initial number and the number resulting at each Collatz algorithmic step. Nevertheless, such a modification could slow the Collatz algorithm application computational speed in cases where the lesser number lies far from the algorithmic beginning within many steps of the Collatz path.

Looking for this feature, we tested many Collatz compliant numbers.

2.2.1. Testing Collatz compliance of even natural numbers

In this sense, *even* natural numbers present a Collatz compliant condition at the first iteration. The result of the Collatz algorithm will transform the number into half of the initial value. Therefore, the set of even natural numbers can be considered Collatz compliant, a consequence already described in the reference [1].

One can easily prove even natural numbers Collatz compliance. Just consider the sequence of the equation (2.4) submitted to the property of being Collatz compliant with the equation (2.5). We can write the even numbers associated with the sequence (2.4) as:

$$\mathbb{E}_n = 2 \otimes \mathbb{M}_n = \{0, 2, 4, \ldots, 2m, \ldots, 2(n-1)\},$$

where the symbol $2\otimes$ means that every element of the set $\mathbb{M}_n$ is multiplied by 2. Then, the application of the Collatz algorithm first step to the even set $\mathbb{E}_n$ can be symbolized by:

$$C_{\langle \mathbf{1}|}[\mathbb{E}_n] = \mathbb{M}_n$$

thus, in the following steps, one can suppose that the Collatz compliance expressed in the equation (2.5) holds for the set $\mathbb{M}_n$; therefore, this is the same to say $\mathbb{E}_n$ being Collatz compliant, that is:

$$C_{\langle \mathbf{S}|}[\mathbb{E}_n] = C_{(\langle \mathbf{S}| - \langle \mathbf{1}|)}[\mathbb{M}_n] = \langle \mathbf{1}| \, .$$

Empirically, odd natural numbers possess a similar property to even numbers at other positions of the Collatz path.

2.3. Third step concerning Collatz compliance

For several odd numbers, it has been computationally found within this research that the Collatz path for them provides lesser natural elements than the initial tested number, just at the algorithm's *third* step.

For the set of Mersenne twins, defined as $\nu_2(N) = \mu_2(N) + 2 = 2^N + 1$, we already found such an occurrence, see reference [1]. Whatever the attached Boolean hypercube dimension N, one obtains: $C_3[\nu_2(N)] < \nu_2(N)$, meaning that the Mersenne twins are heuristically Collatz compliant.

On the contrary, the Collatz operator applied on Mersenne numbers yields, after some indefinite number of steps S, a value less than the starting Mersenne number: $C_S[\mu_2(N)] = m < \mu_2(N)$.

2.3.1. Example about a Collatz algorithm third step concerning a lesser number

Another interesting example of the Collatz algorithm yielding a lesser number at the third step corresponds to the sequence of pairs of the powers of odd natural numbers, using prime numbers in both the basis B and the power P:

$$\left\{Z(B,P) = B^P \pm (B-1)\,|B;P = 3,5,7,...\right\} \Rightarrow C_3[Z] < Z; \tag{2.6}$$

which corresponds to a sequence associated with some generalization of Mersenne numbers and his twins. For example, the pair:

$$Z(11,7) = 11^7 \pm 10 \equiv \{19487161;19487181\},$$

submitted to the Collatz Algorithm produces at the third step, two numbers less than the $Z(11,7)$ pair:

$$C_3[Z] = \{14615371;14615386\}.$$

Finally, meaning that these numbers might be empirically considered Collatz compliant or that the following property holds for the whole set $Z(B,P)$ defined in the equation (2.6):

$$\forall Z : C_3[Z] = 1$$

Noting that using composite numbers as a basis B, the finding of a lesser number is apparent in the Z-path, but the path position of this occurrence fluctuates. However, some cases keep the 3-step trend in one of the pairs, for example:

$$\left(6^7 - 5\right) > C_{13}\left[6^7 - 5\right] \wedge \left(6^7 + 5\right) > C_3\left[6^7 + 5\right].$$

The third step rule, though, does not disappear whenever the composite number used as a basis is a product of prime numbers. Also, if the basis is a composite of products of primes, we can choose the power as an even number without losing the third step property somehow. For example:

$$\left(20^6 - 19\right) > C_3\left[20^6 - 19\right] \wedge \left(20^6 + 19\right) > C_6\left[20^6 + 19\right],$$

those results indicate that a systematic search of this kind of extended numbers could be interesting from the point of view of the Collatz algorithm properties.

The whole situation is interesting because it opens a heuristic Collatz compliance landscape to a large set of natural numbers represented not only by the Mersenne twins but by the extended $Z(B,P)$ set.

2.4. Powers of two and the Collatz operator action on them

When applied over the natural number set, the tree's central trunk generated by the Collatz algorithm or operator appears as the set of the powers of 2.

First of all, considering this, we can write the set of all powers of two as:

$$\mathbf{2}^{\mathbb{N}} = \left\{2^0, 2^1, 2^2, ..., 2^N, ...\right\} = \left\{1, 2, 4, ..., 2^N, ...\right\} \equiv \left\langle \mathbf{2}^{\mathbb{N}} \right|, \tag{2.7}$$

one can see that it possesses as many elements as the natural number set:

$$\mathbb{N} = \{0, 1, 2, ..., N, ...\} \equiv \langle \mathbb{N}|, \tag{2.8}$$

yet one has to write:

$$2^{\mathbb{N}} \subset \mathbb{N}$$

Anyway, when applying the Collatz operator over any power of 2 using the number of Collatz operator 2^N- path steps explicitly, one can describe that the following relation holds:

$$\forall N \in \mathbb{N} : C_N\left[2^N\right] = 1 \equiv C\left[2^N\right] = 1, \tag{2.9}$$

according to equations (2.7) and (2.8), a general application result of the Collatz operator over the whole set of powers of 2 is easy to write as:

$$C_{\langle N|}\left[2^N\right] = \langle \mathbf{1}| \tag{2.10}$$

It must be repeated that, in general, the Collatz operator might be applied to any natural number for a certain number of steps until it yields 1. Then this number could be considered Collatz compliant. In this sense, all the elements of the even number set $2^{\mathbb{N}}$ are Collatz compliant with certainty.

Compared with the set $2^{\mathbb{N}}$, the numbers, $Z(B,P)$ defined in the equation (2.6) possess another kind of Collatz compliance, empirically obtained.

2.5. When loops might be present in the Collatz algorithm

Now we must say that in the case at some step S in a Collatz path, the following circumstance is found:

$$C_S[n] = n, \tag{2.11}$$

then an infinite loop will be present in the Collatz algorithm application. Therefore, if the whole set of natural numbers has to be Collatz compliant, the equality in the equation (2.11) *cannot* be present in any natural number Collatz path.

3. The Collatz Operator over a Sum of Natural Numbers

3.1. Introduction: On the possibility of defining a pseudolinear Collatz operator

Let us denote by $C_{S_a}[a]$ the symbolic implementation of the full Collatz operation-algorithm acting on the natural number a and ending on 1, with S_a being the number of Collatz algorithm steps associated with a. Moreover, $C_{S_b}[b]$ is the symbolic implementation of the full Collatz operation-algorithm acting on the natural number b and ending on 1, S_b meaning the number of Collatz algorithm steps associated with it.

Then $C_{S_{(a+b)}}[a+b]$ is the symbolic implementation of the full Collatz algorithm action on the sum $a+b$ and ending on 1, with $S_{(a+b)}$ being the number of steps associated with $a+b$, and which *do not* need to be equal to $S_a + S_b$.

With these definitions, from now on, the present work is based on one, and only one heuristic assumption:

If, and only if, the Collatz symbolic operation satisfies the following relation for any pair of natural numbers $\{a,b\}$:

$$C_{S_{(a+b)}}[a+b] \equiv C_{S_a}[a] + C_{S_b}[b], \tag{3.1}$$

we will be able to show that the Collatz conjecture is true, at least in an empirical-heuristic manner.

In other words, the Collatz conjecture can be recast in the pseudolinear or pseudo distributive forms of the Collatz operator displayed by the equation (3.1).

Strictly speaking, the equation (3.1) is **not** linear because when $a \neq b$ the three symbolic operations: $C_{S_a}[a]$, $C_{S_b}[b]$, and $C_{S_{(a+b)}}[a+b]$ represent three **different** Collatz operator path results.

To obtain this new perspective, and without considering relevant the number of steps needed to arrive at the Collatz algorithm completion, but the completion of the algorithm by itself, one can heuristically write for the sum of two Collatz compliant natural numbers:

$$\begin{aligned} &if \quad \{a,b\} \in \mathbb{N} \wedge \{C[a] = 1 \wedge C[b] = 1\} \\ &\qquad \to C^{[2]}[a+b] = C[C[a]+C[b]] = C[1+1] = C[2] = 1. \end{aligned} \tag{3.2}$$

Note that the operator $C^{[2]}[+]$ has to be *strictly* applied to the sum of two Collatz compliant numbers. It is equivalent to using the Collatz algorithm in two steps: first, as a pseudolinear operator over the two terms of the sum, and second, as the Collatz operator over the sum of the two final units yielded by the previous application.

3.2. Some examples

To have a particular view of what represents the equation (3.2) above, one could talk about a Collatz n-path for all the steps obtained when applying the Collatz algorithm to a natural number n.

For instance, the Collatz 3-path possesses 7 elements or steps:

$$3:10:5:16:8:4:2:1$$

while the Collatz 7-path possesses 16 steps:

$$7:22:11:34:17:52:26:13:40:20: \mathit{10: 5: 16: 8: 4: 2: 1.}$$

As a curiosity, the Collatz 3-path coincides with the last 7 steps of the Collatz 7-path. This fact is signaled by writing the coincident steps in *cursive*. Moreover, it seems that this is a property of the Mersenne numbers, which have in common these 7 last steps, at least for several of the continuing elements of the Mersenne sequence, for instance: 15, 31, and 63.

The significant situation in 3- and 7-path cases is that both are Collatz compliant; thus, we can write: $C[3] = 1$ and $C[7] = 1$. Therefore, according to the previous considerations, we can also write:

$$C[10] \equiv C^{[2]}[3+7] = C[C[3]+C[7]] = C[2] = 1,$$

noting the equivalence but not the equality between $C[10]$ and $C^{[2]}[3+7]$.

To stress the need for the twofold application of the Collatz operator, we can alternatively write more compactly:

$$C^{[2]}[3+7] = C[C[3+7]] = C[2] = 1.$$

That proves that the number 10 is Collatz compliant whenever the numbers 3 and 7 are Collatz compliant. It is a prominent property of the number 10, as this number, as commented, appears in both the Collatz 3-path and Collatz 7-path and the three successive Mersenne numbers Collatz paths.

Also, one must consider that number 10 is even, and thus in the first iteration yielding 5, a number less than 10. As commented before, this will be sufficient to consider the number 10 as Collatz compliant.

3.3. Collatz algorithm on particular sums of natural numbers

These last considerations allow us to observe that if a natural number $n \in \mathbb{N}$ is Collatz compliant: $C[n] = 1$, then according to the definition of the Collatz operator acting on a sum of natural numbers, the following equalities for even natural numbers hold:

$$\begin{aligned} C^{[2]}[n+n] &= C[C[n]+C[n]] = C[2] = 1 \\ &\rightarrow C^{[2]}[n+n] = 1 \rightarrow C[2n] = 1. \end{aligned} \tag{3.3}$$

This last result constitutes a coherent outcome concerning the previously discussed behavior of even natural numbers.

The addition of the number 1 to any Mersenne number $\mu(N)$ provides the corresponding power of 2: $\mu(N)+1 = 2^N$, therefore one can write:

$$C[\mu(N)+1] = C\left[2^N\right] = 1,$$

but according to the previous considerations about the pseudolinearity of the Collatz operator, one can also write:

$$C^{[2]}[\mu(N)+1] = C[C[\mu(N)]+C[1]] = C[2] = 1.$$

Also, one must be aware that the odd numbers, derived from natural numbers which are Collatz compliant, can be considered as being Collatz compliant too because one can write:

$$\begin{aligned} &C[n] = 1 \wedge C[2n] = 1 \Rightarrow \\ &\quad C^{[2]}[2n+1] = C[C[2n]+C[1]] = C[2] = 1 \rightarrow C^{[2]}[2n+1] = 1. \end{aligned} \tag{3.4}$$

Alternatively, this can also be written in an extended way, involving the sum of three, instead of two, natural numbers:

$$\begin{aligned} &C[n] = 1 \Rightarrow \\ &\quad C^{[2]}[2n+1] = C[C[n]+C[n]+C[1]] = C[3] = 1 \\ &\quad \rightarrow C^{[2]}[2n+1] = 1. \end{aligned} \tag{3.5}$$

Therefore, one can deduce that if any natural number $n \in \mathbb{N}$ is Collatz compliant, that is: $C[n] = 1$, then the derived even $2n$ and odd $2n+1$ numbers are also Collatz compliant.

Such a property involving both even, and odd natural numbers, might be considered sufficient for a simple initial empirical proof, showing all natural numbers are Collatz compliant.

However, one can obtain better, refined, and general heuristic proofs of the Collatz conjecture, as described below in the following paragraphs.

3.4. Collatz compliance of a sum of a Collatz compliant natural number set

The previous results about sums of two natural numbers, three in the case of the equation (3.5), and their Collatz compliance can be easily generalized.

For this purpose, we can now suppose known a set of different natural numbers:

$$\mathbb{A} = \{a_I \,|\, I = 1,N\} \subset \mathbb{N},$$

such that their elements are Collatz compliant, that is[2]:

$$\begin{aligned}&\forall I=1,N: C\left[a_I\right]=1 \Rightarrow \\ &C\left[\langle A|\right]=\langle \mathbf{1}_N|=(1,1,1,...,1) \Leftarrow \langle A|=\left(a_1,a_2,a_3,...,a_N\right).\end{aligned}$$

The setup shown above can be used as a first stage to prove that the sum of the elements of a Collatz compliant set $\mathbb{A}$, is Collatz compliant[3]:

$$\begin{aligned}\langle\langle\mathbb{A}|\rangle=\sum_{I=1}^{N}a_I \rightarrow C\left[\langle\langle\mathbb{A}|\rangle\right]=\sum_{I=1}^{N}C\left[a_I\right]=\sum_{I=1}^{N}1=N \\ \rightarrow C^{[2]}\left[\langle\langle\mathbb{A}|\rangle\right]=C\left[C\left[\langle\langle\mathbb{A}|\rangle\right]\right]=C\left[N\right]=1.\end{aligned} \tag{3.6}$$

This result implies that a sum of Collatz compliant numbers is Collatz compliant whenever the total number of elements of the sum is Collatz compliant. That is, if the equality: $C\left[N\right]=1$, also holds.

Considering that the sum of N different natural numbers greater than 1 is always higher than the number of elements of the sum itself, or: $\langle\langle\mathbb{A}|\rangle > N$.

Thus, one can suppose that to be Collatz compliant, a given natural number must rely on that all the lesser natural numbers have been previously found Collatz compliant.

The important thing to underline here is that a sum of N Collatz compliant natural numbers can be Collatz compliant whenever the *number of N terms of the sum* is Collatz compliant. In other words: if the cardinality of a natural number set is Collatz compliant, the whole elements' sum of the set is Collatz compliant.

One can be aware that we reach the *minimum* value of the sum of N ordered different natural numbers when all the sum terms are equal to the set $\mathbb{T}_N=\{0,1,2,\ldots,N-1\}$.

However, as the number of elements of $\mathbb{T}_N$ is N, if all of them are Collatz compliant, then N is Collatz compliant. That is so because if: $C\left[N-1\right]=1$ as $N=(N-1)+1 \rightarrow C\left[N\right]=1$ using equations (3.4) and (3.5).

Therefore, when summing up the elements of a Collatz compliant natural number set having a cardinality N, it is only necessary to know the Collatz compliance of the number N of terms to be summed up to deduce the Collatz compliance of the resultant sum.

4. Natural and Collatz Vector Spaces

We can also define an N-dimensional natural (row) vector (*semi*)space[4] $\mathbb{V}_N\left(\mathbb{N}\right)$ as the Cartesian power $\mathbb{N}^N$ of the row orderings of N natural numbers, that is:

$$\forall\langle\mathbf{a}|\in\mathbb{V}_N\left(\mathbb{N}\right)\Rightarrow\langle\mathbf{a}|=\left(a_1,a_2,a_3,\ldots,a_N\right)\wedge\left\{a_I\left|I=1,N\right.\right\}\subset\mathbb{N}.$$

One might call a natural vector space a Collatz vector space when the dimension of the natural vector space is Collatz compliant; that is when: $C\left[N\right]=1$ holds.

In a Collatz space, a vector $\langle\mathbf{a}|$ will be called Collatz compliant when the whole set of its elements $\left\{a_I\left|I=1,N\right.\right\}$ is Collatz compliant. That is, one can write:

$$\begin{aligned}&\forall I=1,N : C\left[a_I\right]=1 \Rightarrow \\ &\qquad C\left[\langle\mathbf{a}|\right]=\left(C\left[a_1\right],C\left[a_2\right],C\left[a_3\right],\ldots,C\left[a_N\right],\right)=(1,1,1,...,1)=\langle\mathbf{1}_N|\end{aligned}$$

for all Collatz compliant vectors.

In this case, if the Collatz conjecture holds, one can write a Collatz conjecture for natural spaces. We can write it as:

$$C\left[\mathbb{N}\right]=\langle\mathbf{1}|\Rightarrow\forall\langle\mathbf{a}|\in V_N\left(N\right): C\left[\langle\mathbf{a}|\right]=\langle\mathbf{1}_N|\wedge C\left[N\right]=1.$$

and that corresponds to a general conjecture related to ordered sets of natural numbers, including matrices and hypermatrices or tensors. It is well-known that one can reorder them as row vectors of the adequate dimension.

Moreover, for more information, any natural vector space (see references [27]–[29]) might be considered a Banach space, where one can define some vector norms. Among them, the simplest is the Minkowski norm, defined as:

$$\forall\langle\mathbf{a}|\in\mathbb{V}_N\left(\mathbb{N}\right): M\left[\langle\mathbf{a}|\right]=\langle\langle\mathbf{a}|\rangle=\sum_{I=1}^{N}a_I\rightarrow M\left[\langle\mathbf{a}|\right]\in\mathbb{N},$$

[2] An ordered set of natural numbers can be also represented for the purposes of the present work as a row vector belonging to a row vector space, defined over the natural numbers, and using a Dirac bra notation to describe it: $A=\left\{a_I\left|I=1,N\right.\right\}\subset\mathbb{N}\Leftrightarrow\langle A|=\left(a_1,a_2,a_3,...,a_N\right)\in V_N\left(\mathbb{N}\right)$.

[3] The bracket symbol $\langle\quad\rangle$ is used as a symbolic algorithm to compute the complete sum over the subindices of the elements of a vector $\langle\mathbf{a}|=\left(a_1,a_2,...,a_N\right)$: $\langle\langle\mathbf{a}|\rangle=\sum_{I=1}^{N}a_I$.

[4] The name (*semi*)space is used to stress the fact that no other addition operation is allowed than these associated to an addition semigroup. See references [28]–[30] for more details. The particle (*semi*) will be omitted from now on. Such spaces can be also called *orthants*. In this precise context are also related to lattices.

defined in the natural number context as the sum of the vector elements, which is of use in this paper. We can admit this, because the Minkowski norm of the unity vector yields the dimension of the natural space, or:

$$M\left[\langle \mathbf{1}_N|\right] = N.$$

Therefore, in Collatz vector spaces, one can write:

$$C\left[M\left\langle\langle \mathbf{1}_N|\right\rangle\right] = C[N] = 1.$$

Now one must stress that this result, applied to Collatz natural vectors, looks similar to the descriptions and properties of the previous paragraphs. Similarly, as in the equations (3.2) and (3.6), a Collatz compliant vector defined in a Collatz vector space can be associated with the sum of the elements of the vector.

That is, in a Collatz space involving a vector, constructed with Collatz compliant natural numbers, one can write:

$$C[\mathbb{N}] = \langle \mathbf{1}| \Rightarrow \forall \langle \mathbf{a}| \in \mathbb{V}_N(\mathbb{N}) : C[\langle \mathbf{a}|] = \langle \mathbf{1}_N| \wedge C\left[\left\langle\langle \mathbf{a}|\right\rangle\right] = 1$$

which one can write compactly, using the previous paragraphs notation:

$$\forall \langle \mathbf{a}| \in \mathbb{V}_N(\mathbb{N}) : C^{[2]}[\langle \mathbf{a}|] = 1,$$

which one can interpret as:

$$C^{[2]}[\langle \mathbf{a}|] = C\left[C\left[\left\langle\langle \mathbf{a}|\right\rangle\right]\right] = C\left[\left\langle C[\langle \mathbf{a}|]\right\rangle\right] = C[N] = 1.$$

Now we can propose a Collatz extended conjecture from what one has commented on before and compactly write it as:

$$C^{[2]}[\mathbb{V}_N(\mathbb{N})] = \langle \mathbf{1}|.$$

5. N-dimensional Boolean Hypercubes and Binary Expression of any Natural Number

One can construct any N-dimensional Boolean hypercube $\mathbf{H}_N$, see references [27]–[30] for more information and applications, as a set of 2^N vertices. We can consider every vertex formed as a string of N bits. That is, with elements made of the two binary digits: $\mathbb{B} = \{0, 1\}$.

One can name the vertices of such a construct as the set:

$$\mathbf{H}_N = \left\{\langle \mathbf{h}_0|;\langle \mathbf{h}_1|;\langle \mathbf{h}_2|;\ldots\left\langle \mathbf{h}_{\mu(N)}\right|\right\},$$

where the last subindex $\mu_2(N) = 2^N - 1$ corresponds to the already encountered Mersenne number, associated in turn with the unity (or Mersenne) vertex in *any* Boolean Hypercube dimension:

$$\forall N \in \mathbb{N} : \left\langle \mathbf{h}_{\mu(N)}\right| = \langle \mathbf{1}_N| = (1, 1, 1, \ldots, 1).$$

Another characteristic vertex, which is well-structured in any dimension Boolean Hypercube, agrees with the zero vertex:

$$\forall N \in \mathbb{N} : \langle \mathbf{h}_0| = \langle \mathbf{0}_N| = (0, 0, 0, \ldots, 0).$$

The remaining vertices of $\mathbf{H}_N$ are the remnant combinations of the binary set $\mathbb{B}$ taken by N by N.

However, one can interpret these vertices in various manners. The following section will deal with this.

5.1. Interpretations of the Boolean hypercube vertices

One alternative is considering the two bits of $\mathbb{B}$ as natural numbers. Then the Boolean hypercube transforms into a *natural* hypercube of the same dimension. One can also consider the natural hypercube vertices made by strings of the most straightforward natural set: $\mathbb{S}_2 = \{0, 1\} \subset \mathbb{N}$, constituting the decimal translation of the vertices of the monodimensional Boolean hypercube $\mathbf{H}_1$.

Another possibility is to consider them as a set of logical Kronecker deltas, see references [31]–[33] for more information and applications, which, taking L as any logical expression, can be defined as:

$$\delta(L = .False.) = 0 \wedge \delta(L = .True.) = 1.$$

For example:

$$(1, 0, 0, 1) \Rightarrow (\delta(L_3); \delta(L_2); \delta(L_1); \delta(L_0)),$$

where we also define a logical vector with the following values:

$$\langle \mathbf{L}| = (L_3; L_2; L_1; L_0) \equiv (.True.; .False.; .False.; .True.).$$

5.2. Binary expression of any natural number

Now, any natural number can be expressed as the complete sum of an inward (or Hadamard, or diagonal, ...) product of two vectors [28],[29],[34],[35], using as coordinates of one vector a vertex of an appropriate N-dimensional Boolean hypercube, and the other vector constructed by the powers of 2 ordered in the same way.

That is, we can construct the reference binary basis as a vector made of the convenient powers of 2:

$$\left\langle \mathbf{2}^N \right| = \left(2^{N-1};2^{N-2};...;2^2;2^1;2^0\right)$$

Then, constructing the elements of an appropriate Boolean hypercube vertex according to the rightmost bit being the less significative:

$$\langle \mathbf{h}_n| = \left(h_{(N-1)n}, h_{(N-2)n}, h_{(N-3)n}, ..., h_{1n}, h_{0n}\right),$$

the 2^N numbers in the interval:

$$[0,\mu(N)] \to \mathbb{S}_N = \{0,1,2,\ldots,n,\ldots,\mu(N)\} \tag{5.1}$$

can be generated with the following algorithm[5]:

$$\forall n \in \mathbb{S}_N : n = \left\langle \langle \mathbf{h}_n| * \left\langle 2^N \right| \right\rangle = \sum_{I=0}^{N-1} h_{In} 2^I, \tag{5.2}$$

considering the bits of the vertex $\langle \mathbf{h}_n|$ equivalent to some elements of the natural set $\mathbb{S}_2$.

Alternatively, one can use a logical Kronecker's delta expression:

$$\forall n \in \mathbb{S}_N : n = \left\langle \langle \mathbf{h}_n| * \left\langle \mathbf{2}^N \right| \right\rangle = \sum_{I=0}^{N-1} \delta(h_{In} = 1) 2^I,$$

which is valid for both Boolean and natural options to construct the hypercube vertices.

5.3. Collatz operator acting on the recursive generation of natural numbers associated with a Boolean hypercube

The equation (5.1) has been used in reference [1] to design a manner to construct the natural number set recursively. That has been so because when describing the set defined in the equation (5.1), the attached Boolean hypercube dimension augments in one unit. One can write the resultant natural number set:

$$[0,\mu(N+1)] \to \mathbb{S}_{N+1} = \{0,1,2,\ldots,n,\ldots,\mu(N+1)\},$$

as the union of the initial set $\mathbb{S}_N$ with a new set, $\mathbb{A}_N$ say, which one can write in terms of $\mathbb{S}_N$, using the algorithm:

$$\begin{aligned} \mathbb{A}_N &= 2^N \oplus \mathbb{S}_N \\ &= \left\{2^N; \left(2^N+1\right); \left(2^N+2\right); \ldots; \left(2^N+\mu(N)\right) = \mu(N+1)\right\} \\ &\Rightarrow \mathbb{S}_{N+1} = \mathbb{S}_N \cup \mathbb{A}_N; \end{aligned}$$

where the symbol $2^N \oplus$ means that the power 2^N is summed to every element of the set $\mathbb{S}_N$.

Then, the application of the Collatz operator over the set $\mathbb{S}_N$, if previously found Collatz compliant:

$$C[\mathbb{S}_N] = \langle \mathbf{1}_{2^N}|,$$

as well as the already discussed property of powers of 2:

$$\forall N \in \mathbb{N} : C_N\left[2^N\right] = 1,$$

implies that, when used over the new natural set $\mathbb{A}_N$, one can write:

$$C[\mathbb{A}_N] = C\left[2^N \oplus \mathbb{S}_N\right] = C\left[2^N\right] \oplus C[\mathbb{S}_N] = 2\langle \mathbf{1}_{2^N}|,$$

implying that the new set acts over the Collatz operator as:

$$C[\mathbb{S}_{N+1}] = \langle \mathbf{1}_{2^{N+1}}|,$$

and therefore, $\mathbb{S}_{N+1}$ might be considered Collatz compliant.

We can also see such reasoning as an inductive way to prove the Collatz compliance of the natural number set heuristically.

[5] The scalar product of two vectors can be expressed as the complete sum of the inward product of two vectors:

$$\langle \mathbf{p}| = \langle \mathbf{a}| * \langle \mathbf{b}| = (a_1b_1, a_2b_2, ..., a_Nb_N) \to \langle\langle \mathbf{a}| * \langle \mathbf{b}|\rangle = \langle \mathbf{a} \mid \mathbf{b}\rangle = \sum_{I=1}^{N} a_I b_I.$$

6. Application of the Collatz Algorithm or Operator to any Natural Number

Besides the results of the previous section, here is the possibility to obtain alternative heuristical proof that any natural number is Collatz compliant. Considering the Collatz operator applied up to completion over a natural number yielding the unit, one can apply it to the equation (5.2), for instance:

$$C[n] = \left\langle \langle \mathbf{h}_n | * C\left[\left\langle \mathbf{2}^N \right|\right] \right\rangle = \langle \langle \mathbf{h}_n | * \langle \mathbf{1}_{N+1} | \rangle = \langle \langle \mathbf{h}_n | \rangle, \tag{6.1}$$

and one might see the hypercube vertices as zero or unit weights or coordinates of the powers of 2 sum. As a definition of the operator action in similar cases, in the development of equations (5.2) and (6.1), one must consider that, it has been used both the pseudolinearity of the Collatz operator and the application of such an operator over the natural vector: $\left\langle \mathbf{2}^N \right|$ only, leaving the coefficients of the Boolean Hypercube vertex vector $\langle \mathbf{h}_n|$ intact.

In the case of the equation (6.1), the operator does not even need to be considered linear. We must accept that when applying the Collatz operator to any vector made by natural numbers, the result is another vector with the Collatz algorithm results of each natural number element of the vector, as discussed in paragraph **4**.

In this case, the result is the unity vector $\langle \mathbf{1}|$ of the appropriate dimension because every initial element is a power of 2.

The result of the sum (6.1), whenever $\forall n > 0$, might be considered from two points of view. Considering the vertex $\langle \mathbf{h}_n|$ as:

A. Binary, then: $\langle \langle \mathbf{h}_n | \rangle = 1$,
B. Natural, then: $\langle \langle \mathbf{h}_n | \rangle = \sigma_1 \left(\langle \mathbf{h}_n | \right)$,

where $\sigma_1 \left(\langle \mathbf{h}_n | \right)$ is the number of ones contained in the hypercube vertex: $\langle \mathbf{h}_n|$.

6.1. The definition of natural number (one)classes and Collatz compliance

One might consider the function $\sigma_1 \left(\langle \mathbf{h} | \right)$ a simple tool to classify the vertices of the N-dimensional Boolean hypercube in $N+1$ (one)classes, as the vertices might possess zero, one, two, ... up to N unit numbers, 1, which one can find within their elements.

Consequently, one can also classify the natural numbers in this way, and accordingly, in any natural subset $\mathbb{S}_N$ of cardinality 2^N, there are present $N+1$ (one)classes.

For example, Mersenne numbers $\mu(N)$ are associated with the unit vector $\langle \mathbf{1}_N|$, attached to N ones. Thus, such numbers belong to the $N+1-th$ (one)class holding N ones, the unique occurrence of one vertex in this kind of $N-$ dimensional hypercubes.

One can construct the hypercube vertices associated with the powers of 2, which with the algorithm:

$$\begin{aligned} \forall P = 0, N-1 : \langle \mathbf{h}_{2^P} | = \left\{ h_{I;2^P} = \delta\left(I = P\right) | I = 0, N-1 \right\} \\ \rightarrow \sigma_1 \left(\langle \mathbf{h}_{2^P} | \right) = \langle \langle \mathbf{h}_{2^P} | \rangle = \kappa_{2^P} = 1, \end{aligned}$$

are elements of the (one)class holding just one 1.

While the complementary vertices to the collection $\{\langle \mathbf{h}_{2^P}|\}$ represent the numbers:

$$\forall P = 0, N-1 : \chi(P) = \mu_2(N) - 2^P = 2^P(2^{N-P} - 1) - 1 = 2^P \mu_2(N-P) - 1,$$

which we can write as:

$$\begin{aligned} \forall P = 0, N-1 : \left\langle \mathbf{h}_{\chi(P)} \right| = \left\{ h_{I;\chi(P)} = \delta\left(I \neq P\right) | I = 0, N-1 \right\} \\ \rightarrow \sigma_1 \left(\left\langle \mathbf{h}_{\chi(P)} \right| \right) = \left\langle \left\langle \mathbf{h}_{\chi(P)} \right| \right\rangle = \kappa_{\chi(P)} = N-1 \end{aligned}$$

that is, the vector set $\left\{ \left\langle \mathbf{h}_{\chi(P)} \right| \right\}$ belongs to the $N-th$ (one)class holding $N-1$ ones.

One must realize that the set cardinality duplicates from the set $\mathbb{S}_N$ to the set $\mathbb{S}_{N+1}$. Similarly, the number of vertices duplicates from the N-dimensional Boolean hypercube $\mathbf{H}_N$ to $\mathbf{H}_{N+1}$; the number of associated (one)classes augments in one unit.

That is: defining the cardinality of (one)classes in a natural subset $\mathbb{S}_N$ as $K(\mathbb{S}_N) = N+1$, then it can be written: $K(\mathbb{S}_{N+1}) = N+2$. However, if we write the cardinality of both sets as $L(\mathbb{S}_N) = 2^N$, then $L(\mathbb{S}_{N+1}) = 2^{N+1}$.

6.2. Collatz compliance in natural number and natural number classes

As a consequence of the above-obtained properties, to heuristically prove that a natural number is Collatz compliant, one has just to apply the Collatz algorithm to the natural number classes.

To apply the Collatz operator to the resultant (one)class κ_n of a natural number $n \in \mathbb{N}$, first, just consider that in any case:

$$C[n] = \langle\langle\mathbf{h}_n|\rangle = \sigma_1(\langle\mathbf{h}_n|) = \kappa_n \rightarrow 0 \leq \kappa_n \leq N.$$

Now, suppose that all the $N+1$ class numbers κ_n of the subset S_N are Collatz compliant, or what is the same[6]:

$$\forall I = 0, N : C[I] = 1.$$

Implying an induction reasoning might heuristically prove that (one)class numbers associated with the set $\mathbb{S}_N$ are Collatz compliant. When used in the set $\mathbb{S}_{N+1}$, one can consider that the following (one)class number $N+1$ is also Collatz compliant. Because using similar reasoning as in the equations (3.4) and (3.5), and the pseudolinearity of the Collatz operator, one can write:

$$C[N+1] = C[N] + 1 = 2 \wedge C[2] = 1 \rightarrow C^{[2]}[N+1] = 1.$$

That is, in general, for any natural number n, the following sequence holds:

$$\begin{aligned} \forall n \in N : C[n] = \langle\langle\mathbf{h}_n|\rangle = \sigma_1(\langle\mathbf{h}_n|) = \kappa_n \wedge C[\kappa_n] = 1 \\ \rightarrow C^{[2]}[n] = C[\langle\langle\mathbf{h}_n|\rangle] = C[\kappa_n] = 1, \end{aligned}$$

whenever the implied (one)class number κ_n is Collatz compliant.

We have to consider here the squared Collatz operator, as it has been pointed out reiteratively before, as the subsequent application of the Collatz operator over the result of some previous application of the Collatz algorithm n-path complete steps.

Note also that the number of ones in the Boolean hypercube vertex vector representing the number n, in any case, fulfills the relation: $\kappa_n < n$, and as n grows larger: $\kappa_n \ll n$.

7. Conclusion

The results found all along this paper lead to the empirical proof, such that one can write:

$$C^{[2]}[\mathbb{N}] = \langle\mathbf{1}|. \tag{7.1}$$

Then Collatz conjecture appears to be heuristically true: any natural number is *Collatz compliant*. But also considering this statement can be extended to N-tuple orderings of natural numbers in the way:

$$\forall N \in \mathbb{N} \wedge C[N] = 1 : C^{[2]}\left[\mathbb{N}^N\right] = \langle\mathbf{1}|. \tag{7.2}$$

The reported discussion and results are of empirical and heuristic nature.

Perhaps there are no other means to prove the Collatz conjecture. Suppose this statement is true, then equations (7.1) and (7.2) constitute an important landmark in studying the Collatz conjecture structure.
On the contrary, whenever a complete description of a mathematical proof might appear in the future, this paper's results will still constitute a reliable, first-step heuristic source of the natural number set Collatz conjecture compliance.

8. Appendix: Collatz Algorithm

Pseudocode depicting the original Collatz algorithm.

Algorithm: Collatz or $(3x+1)$ **procedure**

$$\begin{aligned} & Input : n \in \mathbb{N}; \\ & I = 0; \\ & Define\ C_I[n]; \\ & \textbf{while}\ n > 1; \\ & \quad I = I+1; c \leftarrow n/2; \\ & \quad \textbf{if}\ 2*c \neq n : n \leftarrow 3*n+1;\ \textbf{else}: n \leftarrow c; \end{aligned}$$

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[6] As it was already defined in reference [1], the result: $C[0] = 1$ is also adopted here.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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