e-ISSN: 2564-7954 CUJSE 19(01): 013-021 (2022) Research Article

Çankaya University

Journal of Science and Engineering

https://dergipark.org.tr/cankujse

CUJSE

Graph Translation Surface in the Lorentz-Heisenberg 3-space with constant curvatures

Brahim Medjahdi *[1] and Zoubir Hanifi [2]

[1,2]Department of la Formation préparatoire en sciences et technologies, ENP Oran Maurice Audin (Ex ENSET), Algeria.

Keywords

Lorentz-Heisenberg 3-space, Graph Translation Surface, Mean curvature, Gaussian curvature.

Abstract

In this paper, we study graph translation surfaces in a 3-dimensional Lorentz-Heisenberg 3-space $\mathbb{H}_3$. The classification theorems of the considered surfaces with zero and nonzero mean and Gaussian curvatures are given. Contrary to the Euclidean case, there is evidence that , translation surfaces with constant Gaussian curvature K that are not cylindrical surfaces, with constant mean curvature $H \neq 0$ which are not settled.

1. Introduction

In classical differential geometry, the problem of obtaining the mean curvature H and Gaussian curvature K of a surface in the three dimensional Euclidian space $\mathbb{E}^3$ and in other spaces is one of the most important problems.

In particular, for the immersed graph z into $\mathbb{E}^3$, such a problem is reduced to solve the Monge-Ampère equation given by ([1], [2])

$$\det\left(\frac{\partial z}{\partial x \partial y}\right) = K(1+|\nabla z|^2)^2,$$

and the equation of mean curvature type in divergence form

$$div\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}}\right) = H,$$

where ∇ denotes the gradient of E^2 ([3], [4], [5]).

An interesting class of surfaces in $\mathbb{E}^3$ is that of the graph tanslation surfaces, which can be locally parametrized as

$$r(s,t) = (s,t,u(s)+v(t)),$$

where u and v are smooth functions of a single variable.

Such a surfaces has been invertigated from various points of view by many geometers. One of the famous examples of minimal surfaces in $\mathbb{E}^3$ is a Scherk's minimal graph translation surfaces. In fact, in [6], Sherk showed that exept for the planes, the only minimal graph translation surfaces are the surfaces given by

$$z(x,y) = \frac{1}{a}\log\left|\frac{\cos(ax)}{\cos(ay)}\right|, \tag{1}$$

*Corresponding author: medjahdib41@gmail.com
Received: July 15, 2021, Accepted: January 26, 2022

where a is a nonzero constant.

On the other hand, in [7], H. Liu has presented a classification of translation surfaces with a constant mean curvature or constant Gaussian curvature in the three dimensional Euclidian space $\mathbb{E}^3$ and the three dimensional Minkowski space $\mathbb{E}_1^3$. In [8], L. Verstraelen, J.Walrave and S. Yaprak have considered minimal translation surfaces in n-dimensional Euclidian space.

The concept of graph translation surfaces in $\mathbb{E}^3$ has been generlized in the three dimensional Lie group, in particular, homogenous manifolds. In [9], J. Inoguchi, R. Lopez and M.I. Munteanu, clssified minimal translation surfaces in the three dimensional Heisenberg group Nil_3. In [10], R, Lopez and M.I. Munteanu studied minimal translation surfaces in Sol_3 space. In [11], Dj. Bensikaddour, L. Belarbi studied minimal translation surfaces in Lorentz-Heisenberg 3-space $\mathbb{H}_3$.

In [12], the second author, M. Bekkar and C. Baba Hamed have observed that in the 3-dimensional Lorentz-Minkowski space, translation surfaces are eigenfunction, component functions of their Laplace operator. In [13], Yoon who considered, within the 3-dimensional Minkowski space, the Gauss map G that comply with the condition $\Delta G = AG$, $A \in Mat(3,\mathbb{R})$, where $Delta$ represent Laplacien of the surfaces with regard to the induced Metric $Mat(3,\mathbb{R})$ the set of 3×3 real matrix. In [14], M. I. Munteanu and A. I. Nistor have studied the second fundamental form of translation surface in the Euclidean space $\mathbb{E}^3$. They have introduced a non-existence polynominal translation surfaces in $\mathbb{E}^3$ results, with fading second Gauss curvature K_{II}. They have ranked those translation surfaces for which K_{II} and H are proportional.

Most recently, in [15] the second author, A. Azzi and M. Bekkar classified surfaces graph of function in $SL(2,\mathbb{R})$, which has finite type immersion.

On the other hand, in [16] and [17] the authors showed that modulo an automorphism of the Lie algebra, the three dimensional Heisenberg group $\mathbb{H}_3$ has the following classes of left-invariant Lorentz metrics:

$$\begin{aligned} g_1 &= -dx^2 + dy^2 + (xdy + dz)^2, \\ g_2 &= dx^2 + dy^2 - (xdy + dz)^2, \\ g_3 &= dx^2 + (xdy + dz)^2 - [(1-x)\,dy - dz]^2. \end{aligned}$$

They proved that the metrics g_1, g_2, g_3 are non-isometrics and g_3 is flat.

In the present study, we are mainly interested in the graph translation surfaces in Lorentz-Heisenberg 3-space $\mathbb{H}_3$ endowed with the left invariant flat metric g_3. We describe such surfaces in $\mathbb{H}_3$ with H and K being constants.

2. Preliminaries

The Heisenberg group $\mathbb{H}_3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x,y,z) * (\bar{x},\bar{y},\bar{z}) = (x+\bar{x}, y+\bar{y}, z+\bar{z} - x\bar{y}).$$

The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-xy-z)$. The left invariant Lorentz metric on $\mathbb{H}_3$ is

$$g_3 = dx^2 + (xdy + dz)^2 - [(1-x)\,dy - dz]^2.$$

The following set of left-invariant vector fields forms an pseudo-orthonormal for corresponding Lie-algebra

$$B = \left\{ e_1 = \frac{\partial}{\partial x},\quad e_2 = \frac{\partial}{\partial y} + (1-x)\frac{\partial}{\partial z},\quad e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \right\}.$$

The characterizing properties of this algebra are the following commutation relations :

$$[e_2, e_3] = 0,\ [e_3, e_1] = e_2 - e_3,\ [e_2, e_1] = e_2 - e_3,$$

with

$$g_3(e_1,e_1) = 1,\ g_3(e_2,e_2) = 1,\ g_3(e_3,e_3) = -1.$$

If ∇ is the Levi-Civita connection and R is the curvature tensor of ∇, we have

$$\begin{aligned}\nabla_{e_1}e_1 &= \nabla_{e_1}e_2=\nabla_{e_1}e_3=0,\\ \nabla_{e_2}e_1 &= \nabla_{e_3}e_1=e_2-e_3,\\ \nabla_{e_2}e_2 &= \nabla_{e_2}e_3=\nabla_{e_3}e_2=\nabla_{e_3}e_3=-e_1.\end{aligned}$$

So we obtain that

$$R(e_1,e_3)=R(e_1,e_2)=R(e_2,e_3)=0.$$

Now, let $p=(x,y,z)$ be a point in $\mathbb{H}_3$, and $T=t_1\partial x+t_2\partial y+t_3\partial z$ be a tangent vector at p. Then, T can be written, with respect to the pseudo-orthonormal basis $B=\{e_1,e_2,e_3\}$ as follows :

$$T=t_1e_1+(xt_2+t_3)e_2+((1-x)t_2-t_3)e_3.$$

Graph Surface in $\mathbb{H}_3$

Let M^2 be a surface in the Lorentz-Heisenberg 3-space $\mathbb{H}_3$ which represents the graph of the function $z=h(x,y)$, parametrized by

$$\begin{array}{cccc} r:U\subset\mathbb{R}^2 & \longrightarrow & \mathbb{H}_3 \\ (x,y) & \longmapsto & (x,y,h(x,y)), \end{array} \tag{2}$$

where $r(x,y)=(x,y,h(x,y))$ is the position vector. Hence,

$$\begin{aligned} r_x &= (1,0,h_x)=\partial_x+h_x\partial_z,\\ r_y &= (0,1,h_y)=\partial_y+h_y\partial_z.\end{aligned}$$

Therefore,

$$\begin{aligned} &r_x=e_1+h_xe_2-h_xe_3,\\ &r_y=(x+h_y)e_2+(1-x-h_y)e_3.\end{aligned} \tag{3}$$

The formes fundamentals I and II of the surface M^2 are given respectively by

$$\begin{aligned} I &= Edx^2+2Fdxdy+Gdy^2,\\ II &= Ldx^2+2Mdxdy+Ndy^2,\end{aligned}$$

with

$$E=\ g_3(r_x,r_x)=1,\quad F=\ g_3(r_x,r_y)=h_x,\quad G=\ g_3(r_y,r_y)=(2h_y+2x-1),$$

and

$$L=g_3(\nabla_{r_x}r_x,\aleph),\quad M=g_3(\nabla_{r_x}r_y,\aleph),\quad N=g_3\left(\nabla_{r_y}r_y,\aleph\right),$$

where $\aleph$ is a unit vector field normal on M^2, which satisfies the following system

$$\begin{cases} g_3(r_x,\aleph)=0,\\ g_3(r_y,\aleph)=0,\\ g_3(\aleph,\aleph)=-1.\end{cases}$$

Hence

$$\begin{aligned} &\nabla_{r_x}r_x=h_{xx}e_2-h_{xx}e_3,\\ &\nabla_{r_x}r_y=(h_{xy}+1)e_2-(h_{xy}+1)e_3,\\ &\nabla_{r_y}r_y=-e_1+h_{yy}e_2-h_{yy}e_3.\end{aligned} \tag{4}$$

The normal vector is then given by

$$\begin{aligned}\aleph &= \frac{(-h_x,(1-x-h_y),(x+h_y))}{W}\\ &= \frac{-h_x}{W}e_1+\frac{(1-x-h_y)}{W}e_2+\frac{(x+h_y)}{W}e_3,\end{aligned}$$

with

$$W=\sqrt{EG-F^2}=\sqrt{2\left(h_y+x\right)-1-h_x^2}>0.$$

Therefore

$$L=\tfrac{1}{W}h_{xx}\ ,\quad M=\tfrac{1}{W}\left(1+h_{xy}\right)\ ,\quad N=\tfrac{1}{W}\left(h_x+h_{yy}\right)\ . \tag{5}$$

The curvatures H and K are respectively defined by

$$H=\frac{EN-2FM+GL}{2\left(EG-F^2\right)}=\frac{1}{2W^3}\left[h_{yy}+\left(2\left(h_y+x\right)-1\right)h_{xx}-2h_xh_{xy}-h_x\right], \tag{6}$$

and

$$K=\frac{LN-M^2}{EG-F^2}=\frac{h_{xx}\left(h_x+h_{yy}\right)-\left(1+h_{xy}\right)^2}{W^4}. \tag{7}$$

3. Graph translation surfaces with constant mean curvature

In what follows, we consider the graph translation surface in H_3 parameterized by

$$r(x,y)=\left(0,y,g\left(y\right)\right)*\left(x,0,f\left(x\right)\right)=\left(x,y,f\left(x\right)+g\left(y\right)\right). \tag{8}$$

Hence, we get from (6) that

$$H=\frac{\left[g''+\left(2\left(g'+x\right)-1\right)f''-f'\right]}{2\left(1+f'^2-2\left(g'+x\right)\right)^{\frac{3}{2}}}. \tag{9}$$

Theorem 1 *A graph translation surface in Lorentz-Heisenberg 3-space has constant mean curvature H_0 if and only if one of the following statements hold true:*

1. If $H_0=0$, then

a. $z(x,y)=c_1x+\frac{c_1}{2}y^2+c_2y+c_3$,

b. $z(x,y)=\frac{c_1}{3}\left(2x-1+2c_2\right)^{\frac{3}{2}}+c_2y+c_3$.

2. Otherwise, i.e. $H_0\neq 0$,

$z(x,y)=\int\frac{(-4H_0x+d_3)\sqrt{2x-1+2d_2}}{\sqrt{(-4H_0x+d_3)^2-4}}dx+c_2y+c_3$,

where c_1,c_2,c_3,d_1,d_2,d_3 are constants.

Proof 1

First let us separate the cases.

Case A: Let $H_0=0$. Then (9), reduces to

$$g''+\left(2\left(g'+x\right)-1\right)f''-f'=0. \tag{10}$$

Case A.1. Let $f(u)=c_1x+c_2$, $c_1,c_2\in\mathbb{R}$. Then by (10) we get

$$g(y)=\frac{c_1}{2}y^2+c_3y+c_4\ ,\ c_3,c_4\in\mathbb{R}.$$

Case A.2. Let $g(y)=c_5y+c_6$, $c_5,c_6\in\mathbb{R}$. Then (10) becomes

$$\left(2\left(c_5+x\right)-1\right)f''-f'=0. \tag{11}$$

Solving it gives

$$f(x)=\frac{c_7}{3}\left(2x+2c_5-1\right)^{\frac{3}{2}}+c_8,\ c_7,c_8\in\mathbb{R},c_7\neq 0.$$

Case A.3. Let $f''g''\neq 0$. Taking partial derivative in (10) with respect to y, we find

$$g'''+2f''g''=0. \tag{12}$$

Then (12) can be rewritten as

$$\frac{g'''}{2g''}=-f''. \tag{13}$$

The left hand side of (13) is a function of y, and the right hand side is a function of x. Then both sides have to be equal a nonzero constant, i.e.

$$\frac{g'''}{2g''}=\lambda_1=-f'',$$

which gives that $f(x)=-\frac{\lambda_1}{2}x^2+\lambda_2 x+\lambda_3$ where $\lambda_2,\lambda_3\in\mathbb{R}$.

By substituting this in (10) we get

$$g''-\left(2g'-1\right)\lambda_1=\lambda_1 x+\lambda_2. \tag{14}$$

The right hand side in (14) is a function of x while the other side is either a constant or function of y. This is not possible.

Case B: $H=H_0\neq 0$. Then (9), can be rewritten as

$$2H_0\left(1+f'^2-2\left(g'+x\right)\right)^{\frac{3}{2}}=\left[g''+\left(2\left(g'+x\right)-1\right)f''-f'\right]. \tag{15}$$

We have three cases to solve (15).

Case B.1. Let $f'=d_1,d_1\in\mathbb{R},d_1\neq 0$. Then (15) reduces to

$$g''-d_1=2H_0\left(1+{d_1}^2-2\left(g'+x\right)\right)^{\frac{3}{2}}. \tag{16}$$

Taking the partial derivative in (16) with respect to x leads to

$$-6H_0\left(1+{d_1}^2-2\left(g'+x\right)\right)^{\frac{1}{2}}=0,$$

and this implies that $H_0=0$. This is a contradiction.

Case B.2. Let $g'=d_2,d_2\in\mathbb{R},d_2\neq 0$. By (15) we get

$$\frac{1-2\left(d_2+x\right)f''+f'}{\left(1+f'^2-2\left(d_2+x\right)\right)^{\frac{3}{2}}}=-2H_0. \tag{17}$$

Let us put $f'(x)=\varphi(x)$ in (17). Thus (17) can be rewritten as

$$\frac{1-2\left(d_2+x\right)\varphi'+\varphi}{\left(1+\varphi^2-2\left(d_2+x\right)\right)^{\frac{3}{2}}}=-2H_0. \tag{18}$$

After solving (18), we find

$$\varphi(x)=\frac{\left(-4H_0x+d_3\right)\sqrt{2x-1+2d_2}}{\sqrt{\left(-4H_0x+d_3\right)^2-4}},d_3\in\mathbb{R}. \tag{19}$$

Integrating (19) leads to

$$f(x)=\int\frac{\left(-4H_0x+d_3\right)\sqrt{2x-1+2d_2}}{\sqrt{\left(-4H_0x+d_3\right)^2-4}}dx. \tag{20}$$

Case B.3. Let $f''g'' \neq 0$. The partial derivatives of (15) with respect x and y, gives

$$g''\left[f'''(f'^2+1-2(g'+x))^{\frac{1}{2}}+3H_0(f'f''-1)\right]=0. \tag{21}$$

To solve (21), we distinguish two cases.

Case B.3.1. Let $f'''=0$ then $f'f''-1\neq 0$. By (21) we deduce $H_0=0$, which is not possible.

Case B.3.2. Let $f'''\neq 0$. By (21) we obtian

$$-2g'=\left(-3H_0\frac{f'f''-1}{f'''}\right)^2-f'^2-1+2x. \tag{22}$$

This implies that $g'=const$ so $g''=0$. It is a contradiction.

4. Graph translation surfaces with constant Gaussian curvature

Let us consider the graph translation surfaces given by (2) in $\mathbb{H}_3$ with constant Gaussian curvature K. Hence, we get from (7) that

$$K=\frac{[f''(f'+g'')-1]}{(1+f'^2-2(g'+x))^2}. \tag{23}$$

Theorem 2 *A graph translation surface in Lorentz-Heisenberg 3-space has constant Gaussian curvature K_0 if and only if one of the following statements hold true:*

1. If $K_0=0$, then

$$z(x,y)=\tfrac{c_1}{3}\left(\sqrt{2x+2c_2+c_3^2}-c_3\right)^2+\tfrac{1}{3}\left(\sqrt{2x+2c_2+c_3^2}-c_3\right)^3+\tfrac{c_3}{2}y^2+c_4y+c_5.$$

2. Otherwise, i.e. $K_0\neq 0$,

$$z(x,y)=\int\sqrt{2(c_1+x)-1-\tfrac{1}{2K_0x+c_2}}dx+c_1y+c_2,$$

where c_1,c_2,c_3,c_4 and c_5 are constants.

Proof 2

Let us assume that $K=K_0=const$. First we treat the case $K_0=0$.

Case C: Let $K_0=0$. By (23), we get

$$f''\left(f'+g''\right)-1=0. \tag{24}$$

It should be noted that $f''\neq 0$. Then (24) can be rewritten as

$$g''=\frac{1}{f''}-f'. \tag{25}$$

Both sides of (25) are equal to some nonzero constant. More precisely

$$\frac{1}{f''}-f'=c \;\text{ and }\; g''=c, c\in\mathbb{R}. \tag{26}$$

From (26), we have

$$g(y)=\frac{c}{2}y^2+c_1y+c_2,\; c_1,c_2\in\mathbb{R}, \tag{27}$$

and
$$(f'+c)f''=1. \tag{28}$$
Let us put $f'(x)=h(x)$, in (28). Thus (28) can be rewritten in the form
$$h'(h+c)\frac{dh}{df}=1. \tag{29}$$
Solving the previous equation gives
$$c\frac{h^2}{2}+\frac{h^3}{3}=f+c_3, c_3\in\mathbb{R}, \tag{30}$$
which implies that
$$f(x)=-c_3+\frac{c}{2}f'^2(x)+\frac{1}{3}f'^3(x). \tag{31}$$
It is a Lagrange differential equation. Thus the solution given by
$$f(x)=-c_3+\frac{c}{3}\left(\sqrt{2x+2c_4+c^2}-c\right)^2+\frac{1}{3}\left(\sqrt{2x+2c_4+c^2}-c\right)^3,\ c_4\in\mathbb{R}. \tag{32}$$
Case D: Let $K=K_0\neq 0$. (23), can be rewritten as
$$f''\left(f'+g''\right)-1=K_0\left(1+f'^2-2\left(g'+x\right)\right)^2. \tag{33}$$

In order to solve (33), we have to consider three situations.

Case D.1. Let $f(x)=d_1x+d_2$, $d_1,d_2\in\mathbb{R}, d_1\neq 0$. It follows from (33) that
$$-\frac{1}{K_0}=\left(1+{d_1}^2-2\left(g'+x\right)\right)^2, \tag{34}$$

which implis that K_0 is negative and that
$$2x-1-{d_1}^2\sqrt{-\frac{1}{K_0}}=-2g'. \tag{35}$$
The left side in (35) is a function of x while the other side is either a constant or a function of y. Hence we have reached a contradiction.

Case D.2. Let $g(y)=d_3y+d_4$, $d_3,d_4\in\mathbb{R},\ d_3\neq 0$. Then (33) leads to
$$f''f'-1=K_0\left(1+f'^2-2\left(d_3+x\right)\right)^2. \tag{36}$$

We put $T(x)=1+f'^2-2\left(d_3+x\right)$. Then (36) can be rewritten in the form
$$\frac{T'}{2}=K_0T^2. \tag{37}$$
Then we obtain
$$T=\frac{-1}{2K_0x+d_5}, d_5\in\mathbb{R}. \tag{38}$$
Then we have
$$f(x)=\int\sqrt{2\left(d_3+x\right)-1-\frac{1}{2K_0x+d_5}}dx. \tag{39}$$
Case D.3. Let $f''g''\neq 0$. Taking partial derivative of (33) with respect to y leads to

$$f''g'''=-4K_0(f'^2+1-2(g'+x))g''. \tag{40}$$

Again, we have to discuss two cases.

Case D.3.1. Let $g''' = 0,\ g'' = d_6, d_6 \neq 0$. Hence from (40), we deduce

$$-4K_0 d_6(f'^2 + 1 - 2(g' + x)) = 0, \tag{41}$$

which gives rise to a similar type of contradiction as in Case D.1.

Case D.3.2. Let $g''' \neq 0$, Then taking partial derivative of (40) with respect to x gives

$$f''' g''' = -8K_0 g''(f' f'' - 1). \tag{42}$$

Thereby (42) can be arranged as

$$\frac{g'''}{g''} = -8K_0 \frac{f' f'' - 1}{f'''}. \tag{43}$$

Both sides of (43) are equal to some nonzero constant, namely

$$\frac{g'''}{g''} = d_7, \tag{44}$$

and

$$-8K_0 \frac{f' f'' - 1}{f'''} = d_7, d_7 \in \mathbb{R} - \{0\}. \tag{45}$$

Subtituting (44) in (40), we have

$$d_7 f'' = -4K_0(f'^2 + 1 - 2(g' + x)). \tag{46}$$

This equality is satisfied if g' is a constant so $g'' = 0$ which is contradiction.

Conclusion

In this article, we outline the graph translation surfaces in the Lorentz-Heisenbeg space that have a contant mean and Gaussian curvatures. It appears that, as opposed to the Euclidean case there exist translation surfaces with constant Gaussian curvature K that are not cylindrical surfaces, and translation surfaces with constant mean curvature H wich are not settled.

Declaration of Competing Interest

The author(s), declares that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Brahim Medjahdi: Writing, Reviewing.
Hanifi Zoubir: Methodology, Supervision.

References

[1] A. D. Polyanin, W. E. Schiesser and A. I. Zhurov, "Partial differential equation," Scholarpedia, vol. 3, no. 10, 2008.

[2] V. Ushakov, "The explicit general solution of trivial Monge-Ampère equation," Comment. Math. Helv., vol. 75, no. 1, pp. 125-133, 2000.

[3] R. Lopez, "Separtion of variables in equations of mean-curvature type," Proc. Roy. Soc. Edinburgh Sect., vol. 146, no. 5, pp. 1017-1035, 2016.

[4] L. Simon, ”The minimal surface equation,” Geometry, Encyclopaedia Math. Sci. Springer Berlin, vol. 9, pp. 239-272, 1997.

[5] L. Simon, ”Equations of mean curvature type in 2 independent variables,” Pacific j. Math, vol.69, no. 1, pp. 245-268, 1977.

[6] H. F. Sherk, ”Bemerkungen uber die kleinste Flache innerhalb gegebener Grenzen,” J. Rein. Angew. Math., vol. 13, pp. 185-208, 1835.

[7] H. Liu, ”Translation surfaces with constant mean curvature in three dimensional spaces,” Journal Geom., vol. 64, pp. 141-149, 1999.

[8] L. Verstraelen, J.Walrave and S. Yaprak, ”The minimalt ranslation surfaces in Euclidian space,” Soochow J. Math., vol. 20, pp. 77-82, 1994.

[9] J. Inoguchi, R. Lopez and M. I. Munteanu, ”Minimal translation surfaces in the Heisenberg groub Nil3,” Geom. Dedicata., vol. 161, pp. 221-231, 2012.

[10] R. Lopez and M. I. Munteanu, ”Minimal translation surfaces in Sol_3,” J. Math. Soc. Japan., vol. 64, pp. 985-1003, 2012.

[11] Dj. Bensikaddour and L. Belarbi, ”Minimal translation surfaces in Lorentz Heisenberg 3-space,” Nolinear Studies, vol. 24, no. 4, pp. 859-867, 2017.

[12] C. Baba-Hamed, M. Bekkar and H. Zoubir, ”Translation surfaces of revolution in the 3-Dimensional Lorentz-Minkowski space satisfying $\Delta r_i = \boldsymbol{\lambda}_\mathbf{i} \mathbf{r}_i$,” Int. Journal of Math. Analysis, vol. 4, pp. 797-808, 2010.

[13] D. W. Yoon, ”On the Gauss map of translation surfaces in Minkowski 3-space,” Taiwanese Journal of Mathematics, vol. 6, no. 3, pp. 389-398, 2002.

[14] M. Munteanu, and A. I. Nistor, ”On the geometry of the second fundamental form of translation surfaces in E^3 ,” arXiv: 0812.3166 [math.DG]16 Dec. 2008.

[15] A. Azzi, H. Zoubir and M. Bekkar, ”Surfaces of finite type in $SL(2,\mathbb{R})$,” Nonlinear Studies, vol. 27, no. 3, pp. 783-794, 2020.

[16] N. Rahmani and S. Rahmani, ”Lorentzian geometry of the Heisenberg group,” Geom. Dedicata, vol. 118, pp. 133-140, 2006.

[17] N. Rahmani and S. Rahmani, ”Structures Homognes Lorentziennes sur le Groupe de Heisenberg,” J. Geom.Phys, vol. 13, pp. 254-258, 1994.