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Research Article

Approximately Cohen-Macaulay modules

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Abstract

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. There is a variety of nice results about approximately Cohen-Macaulay rings. These results were done by Goto. In this paper we prove some these results for modules and generalize the concept of approximately Cohen-Macaulay rings to approximately Cohen-Macaulay modules. It is seen that when M is an approximately Cohen-Macaulay module, for any proper ideal I we have $\operatorname{grade}(I, M) \geq \dim_R M - \dim_R M/IM - 1$. Specially when M is R itself, we obtain an interval for $\operatorname{grade}(I, R)$. We also give a definition for these modules in case that R is not necessarily local and show that approximately Cohen-Macaulay modules are in close relationship with perfect modules. Finally we consider the behaviour of these modules under faithful flat extensions.

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1. Introduction

Let R denote a commutative Noetherian ring (with identity) and I be an ideal of R. The local cohomology modules $\mathrm{H}^i_I(M)$, $i = 0, 1, 2, \cdots$, of an R-module M with respect to I were introduced by Grothendieck, [6]. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I, i.e., $\Gamma_I(M) = \bigcup_{n=1}^{\infty}(0 :_M I^n)$. We refer the reader to [6] or [2], for more details about local cohomology.

For a finitely generated R-module M over a commutative Noetherian local ring $(R, \mathfrak{m})$, let δ be the largest submodule of M with $\dim_R \delta < \dim_R M$. Because M is a Noetherian R-module, δ is well-defined. Suppose that

$$\operatorname{Assh}_R M = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \dim R/\mathfrak{p} = \dim_R M\}$$

and put

$$U_M(0) = \bigcap_{\mathfrak{p} \in \operatorname{Assh}_R M} Q(\mathfrak{p}),$$

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where $0 = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M} Q(\mathfrak{p})$ denotes a minimal primary decomposition of 0 in M. It is seen that $U_M(0) = \delta$, see 2.2. We denote the common length of maximal regular M-sequence in ideal I, by $\mathrm{grade}(I, M)$ and if $(R, \mathfrak{m})$ is local we denote the $\mathrm{grade}(\mathfrak{m}, M)$ by $\mathrm{depth}_R M$. We also denote the height of I by $\mathrm{ht}(I)$.

The concept of approximately Cohen-Macaulay rings was introduced first by Goto in [5]. The local ring $(R, \mathfrak{m})$ of dimension d is called an approximately Cohen-Macaulay ring if either $d = 0$ or there exists an element a of $\mathfrak{m}$ such that R/a^nR is a Cohen-Macaulay ring of dimension $d-1$ for every integer $n > 0$. Schenzel in [11, Definition 4.4], inspired by Goto's idea employed [5, Theorem 1.1], (without proof) to introduce approximately Cohen-Macaulay modules. Then he considered these modules as a subset of Cohen-Macaulay filtered modules that in their dimension filteration appear only two modules, M_{d-1} and M_d. As a first part of our investigations we prepare a proof, in modules mode, for Goto's Theorem which guarantees the Schenzel's definition and generalizes the concept of approximately Cohen-Macaulay, see 2.1 to 2.7. In addition, in case that $(R, \mathfrak{m})$ is the homomorphic image of a local Gorenstein ring $(R', \mathfrak{n})$, we describe a relation between approximately Cohen-Macaulay R-modules and their canonical and deficiency modules, see 2.9.

It turns out, see 3.1 and 3.2, that approximately Cohen-Macaulay property is stable under finite direct sum and specialization.

Pournaki, Tousi and Yassemi in [9], investigated the behaviour of approximately Cohen-Macaulay rings and algebras under tensor product operations. They showed if R is an approximately Cohen-Macaulay ring, then so is the ring $R_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$. As an analogue for modules, we present Theorem 3.4 and show if M is an approximately Cohen-Macaulay module, then so is $M_\mathfrak{p}$ for any prime ideal $\mathfrak{p} \in \mathrm{Supp}_R M$. Therefore, approximately Cohen-Macaulay property can be extended from modules over local rings, to modules over not necessarily local rings.

For every ideal I we study the relation between $\mathrm{grade}(I, M)$ and $\dim_R M$, whenever M is an approximately Cohen-Macaulay module. It is seen that if R is an approximately Cohen-Macaulay ring, $\mathrm{grade}(I, R)$ can take only two values $\mathrm{ht}(I)$ or $\mathrm{ht}(I) - 1$. In addition if R is local, $\mathrm{ht}(I) + \dim R/I$ can take only two values $\dim R$ or $\dim R - 1$, see Lemmas 3.3 and 3.7.

A finitely generated R-module M is said to be perfect, if it's projective dimension is equal to $\mathrm{grade}(\mathrm{Ann}_R M, R)$. There is an interesting relation between perfect modules and Cohen-Macaulay modules over Cohen-Macaulay rings presented in [3, Theorem 2.1.5]. Lemmas 3.3 and 3.7, help us to probe this relation between perfect modules and approximately Cohen-Macaulay modules over approximately Cohen-Macaulay rings, see 3.8.

It is shown in [5, Example 3.5], a local ring R is approximately Cohen-Macaulay if and only if so is the formal power series ring $R[[x]]$. This raises the following two natural questions for approximately Cohen-Macaulay modules over (non)local rings:

1. If R is not necessarily local, is it true that M is an approximately Cohen-Macaulay R-module if and only if so is $M[[x]]$ as $R[[x]]$-module?

2. What can we say about $M[x]$? If R is not necessarily local, is it true that M is an approximately Cohen-Macaulay R-module if and only if so is $M[x]$ as $R[x]$-module?

In order to give answers for the above questions we first need to study the behaviour of approximately Cohen-Macaulay modules, under faithful flat extensions in 4.1. Fortunately this helps us in 4.3, to find the positive answers for both questions.

2. Approximately Cohen-Macaulay modules

Throughout this section, M is a finitely generated module over a commutative Noetherian lacal ring $(R, \mathfrak{m})$.

Lemma 2.1. *Let $M \neq 0$ be an R-module of dimension d. Then the set*

$$\Sigma := \{N | N \text{ is a submodule of } M \text{ and } \dim_R N < d\}$$

has a unique largest element with respect to inclusion, δ say. Set $G := M/\delta$. Then

(i) $\dim_R G = d$;
(ii) *G has no non-zero submodule of dimension less than d;*
(iii) $\operatorname{Ass}_R G = \{\mathfrak{p} \in \operatorname{Ass}_R M | \dim R/\mathfrak{p} = d\}$;
(iv) $\mathrm{H}^d_{\mathfrak{m}}(M) \cong \mathrm{H}^d_{\mathfrak{m}}(G)$.

Proof. See [2, Lemma 7.3.1]. □

In the following for an R-module M, we characterize the submodule δ in term of the minimal primary decomposition of 0 in M. To this end, let

$$\mathfrak{g} := \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M \smallsetminus \operatorname{Assh}_R M} \mathfrak{p}$$

and in the case that $\operatorname{Ass}_R M = \operatorname{Assh}_R M$ assume $\mathfrak{g} = R$. As mentioned already in section 1, we set $U_M(0) := \bigcap_{\mathfrak{p} \in \operatorname{Assh}_R M} Q(\mathfrak{p})$, where $0 = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M} Q(\mathfrak{p})$ denotes a minimal primary decomposition of 0 in M. It is easy to see that $\operatorname{Ass}_R M = \operatorname{Assh}_R M$ if and only if $U_M(0) = 0$.

Proposition 2.2. *Let $M \neq 0$ be an R-module of dimension d. Then*

(i) $U_M(0) = \Gamma_{\mathfrak{g}}(M)$;
(ii) *$U_M(0)$ is the largest element of Σ, introduced in Lemma 2.1.*

Proof. (i) The proof is clear in case that $\operatorname{Ass}_R M = \operatorname{Assh}_R M$ because $\mathfrak{g} = R$ and $U_M(0) = 0$. So let $\operatorname{Assh}_R M \subsetneq \operatorname{Ass}_R M$ and put $I := \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M \smallsetminus \operatorname{Assh}_R M} \operatorname{Ann}_R(M/Q(\mathfrak{p}))$ and $K := \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M \smallsetminus \operatorname{Assh}_R M} Q(\mathfrak{p})$, where the submodules $Q(\mathfrak{p})$ are primary components of 0 in M. Assume that $x \in U_M(0)$. Then $Ix \subseteq U_M(0) \cap K = 0$. This leads to $x \in \Gamma_{\mathfrak{g}}(M)$ because $\sqrt{I} = \mathfrak{g}$. Conversely if $y \in \Gamma_{\mathfrak{g}}(M)$, there exists $t \in \mathbb{N}$, such that $\mathfrak{g}^t y \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Assh}_R M} Q(\mathfrak{p})$. Therefore $\mathfrak{g}^t y \subseteq Q(\mathfrak{p})$ for every primary component $Q(\mathfrak{p})$ which $\mathfrak{p} \in \operatorname{Assh}_R M$. Moreover $\mathfrak{g}^t \nsubseteq \sqrt{\operatorname{Ann}_R(M/Q(\mathfrak{p}))}$ for all such components. This guarantees that $y \in U_M(0)$.

(ii) First note that $\Sigma = \{N \leqslant M | N_{\mathfrak{p}} = 0 \quad \forall \mathfrak{p} \in \operatorname{Supp}_R M \text{ with } \dim R/\mathfrak{p} = d\}$. Let $\Gamma_{\mathfrak{g}}(M) \notin \Sigma$. Then there exist $\mathfrak{p}^* \in \operatorname{Supp}_R M$ with $\dim R/\mathfrak{p}^* = d$ such that $(\Gamma_{\mathfrak{g}}(M))_{\mathfrak{p}^*} \neq 0$. By Flat Base Change theorem, see [2, Corollary 4.3.2], we can pass this statement to the $\Gamma_{\mathfrak{g}R_{\mathfrak{p}^*}}(M_{\mathfrak{p}^*}) \neq 0$ and get $\mathfrak{g} \subseteq \mathfrak{p}^*$, while it is a contradiction. Therefore by view of part (i), $U_M(0) \in \Sigma$.

Now suppose that δ is the largest element of Σ with respect to inclusion and that $x \in \delta$ is arbitrary. Since for all components $Q(\mathfrak{p})$ in the primary decomposition of 0 with $\dim_R M/Q(\mathfrak{p}) = d$ we have $(\operatorname{Ann}_R x).x \subseteq Q(\mathfrak{p})$ and $\operatorname{Ann}_R x \nsubseteq \mathfrak{p}$, we must have $x \in Q(\mathfrak{p})$. Hence it follows that $\delta \subseteq U_M(0)$ and the proof is complete. □

From both the previous lemma and proposition, we immediately get the following corollary.

Corollary 2.3. *Let $M \neq 0$ be an R-module of dimension d. Then $U_M(0)$ is the largest submodule of M contained in Σ, introduced in Lemma 2.1. Moreover*

(i) $\dim_R M/U_M(0) = d$;
(ii) *$M/U_M(0)$ has no non-zero submodule of dimension less than d;*
(iii) $\operatorname{Ass}_R M/U_M(0) = \{\mathfrak{p} \in \operatorname{Ass}_R M | \dim R/\mathfrak{p} = d\}$;
(iv) $\mathrm{H}^d_{\mathfrak{m}}(M) \cong \mathrm{H}^d_{\mathfrak{m}}(M/U_M(0))$.

The following lemma which is quite useful in the proof of the main result of this section, deals with a special element $a \in \mathfrak{m}$ with the property $(0 :_M a) = (0 :_M a^2)$. It should be mentioned, this property is equivalent to a being a d-sequence of length 1 on the module M, as defined by Huneke in [7, Definition 1.1 and Remark 4].

Lemma 2.4. *Let $M \neq 0$ be an R-module of dimension d. Let $a \neq 0$ be an element of $\mathfrak{m}$ and put $N := (0 :_M a)$. Assume that $(0 :_M a) = (0 :_M a^2) \neq 0$ and that $\operatorname{depth}_R M/a^2M \geq d-1$. Then*

(i) *M/N is a Cohen-Macaulay R-module of dimension d;*
(ii) $\operatorname{depth}_R M/aM \geq d-1$*;*
(iii) $\operatorname{depth}_R N \geq d-1$*;*
(iv) $\operatorname{depth}_R M \geq d-1$.

Proof. (i) We know that $\operatorname{depth}_R M/N \leq \dim_R M/N \leq d$. So it is enough to show that $\operatorname{depth}_R M/N > d-1$. If $\operatorname{depth}_R M/N := t \leq d-1$, then we get $\operatorname{depth}_R M/aM+N = \operatorname{depth}_R M/a^2M+N = t-1$ since a is regular on M/N. Consequently by considering the exact sequences

$$(a) \qquad 0 \longrightarrow N \longrightarrow M/a^2M \longrightarrow M/a^2M+N \longrightarrow 0,$$

$$(b) \qquad 0 \longrightarrow N \longrightarrow M/aM \longrightarrow M/aM+N \longrightarrow 0,$$

which exist since $aM \cap N = 0$, we conclude that $\operatorname{depth}_R N \geq t$ and $\operatorname{depth}_R M/aM \geq t-1$. On the other hand, the natural surjective homomorphism $f : M \to aM$ yields $M/N \cong aM$ and so $aM/a^2M \cong M/aM+N$. These lead to the following exact sequence

$$(c) \quad 0 \longrightarrow M/aM+N \longrightarrow M/a^2M \longrightarrow M/aM \longrightarrow 0,$$

and therefore $\operatorname{depth}_R M/aM+N \geq t$. A contradiction, since $\operatorname{depth}_R M/aM+N = t-1$.

(ii) We employ the exact sequences of part (i) and by using a same argument as above prove the statement. To this end assume that $\operatorname{depth}_R M/aM := t < d-1$. Therefore $\operatorname{depth}_R M/a^2M \geq t+1$ and it follows that $\operatorname{depth}_R M/N \geq t+2$ because by the exact sequence (c) we get $\operatorname{depth}_R M/aM+N \geq t+1$.

Thus regularity of a on M/N implies that $\operatorname{depth}_R M/a^2M+N \geq t+1$. Now by using the exact sequence (a) we see that $\operatorname{depth}_R N \geq t+1$ and hence by (b) we must have $\operatorname{depth}_R M/aM \geq t+1$. This is a contradiction.

(iii) We have by (i) that $\operatorname{depth}_R M/a^2M+N = d-1$. Thus the assertion follows from the exact sequence (a).

(iv) This immediately follows from the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0.$$

□

Lemma 2.5. *Let N be a submodule of d-dimensional R-module M. Assume that N is Cohen-Macaulay of dimension $d-1$ and that M/N is Cohen-Macaulay of dimension d. Let a be an element of $\mathfrak{m}$ such that $\dim_R M/aM = d-1$. Then M/aM is a Cohen-Macaulay R-module of dimension $d-1$, provided $aN = 0$.*

Proof. Since $\operatorname{Ass}_R M/N \subseteq \operatorname{Assh}_R M$ we find that a is M/N-regular. Therefore $aM \cap N = 0$ provides the following exact sequence

$$0 \longrightarrow N \longrightarrow M/aM \longrightarrow M/aM+N \longrightarrow 0,$$

which implies $\operatorname{depth}_R M/aM \geq d-1$. Thus M/aM is a Cohen-Macaulay R-module of dimension $d-1$. □

In the following we define the notion of approximately Cohen-Macaulay modules inspired by definition of approximately Cohen-Macaulay rings. It should be mentioned that, this is a generalization of the definition provided with N. T. Cuong and D. T. Cuong in [4, Definition 4.4].

Definition 2.6. A finitely generated module M over a Noetherian local ring $(R, \mathfrak{m})$ is called an approximately Cohen-Macaulay module if either $\dim_R M = 0$ or there exists an element a of $\mathfrak{m}$ such that M/a^nM is Cohen-Macaulay of dimension $d-1$ for every integer $n > 0$.

Note that every Cohen-Macaulay module is approximately Cohen-Macaulay module. So we may consider the zero module to be approximately Cohen-Macaulay.

We are now in the position to present the main result of this section. The following theorem gives us some equivalent conditions for the approximately Cohen-Macaulay concept. In case that M is a non Cohen-Macaulay R-module, the equivalence of (i) and (ii) is shown in [4, Proposition 4.5]. Moreover, in [11, Definition 4.4], Schenzel considered the equivalence condition (iv), as the definition of approximately Cohen-Macaulay modules. In addition to prove the equivalence of these conditions in a more general case, we also mention another equivalent condition for approximately Cohen-Macaulay modules.

Theorem 2.7. *Let M be an R-module of dimension $d > 0$. Then the following are equivalent:*

(i) *M is an approximately Cohen-Macaulay module;*
(ii) *There is an element $a \in \mathfrak{m}$ such that $(0 :_M a) = (0 :_M a^2)$ and M/a^2M is a Cohen-Macaulay module of dimension $d-1$;*
(iii) *M contains a submodule N such that M/N is Cohen-Macaulay of dimension d and N is either zero or Cohen-Macaulay of dimension $d-1$;*
(iv) *$M/U_M(0)$ is Cohen-Macaulay of dimension d and $\operatorname{depth}_R M \geqslant d-1$.*

Proof. (i)$\Rightarrow$(ii): Since M is Noetherian, there exists an integer $n > 0$ such that $(0 :_M a^n) = (0 :_M a^{2n})$. It is enough to replace a with a^n. Then the assertion (ii) follows immediately.

(ii)$\Rightarrow$(iii): We put $N = (0 :_M a)$. In case that M is not Cohen-Macaulay, $N \neq 0$. Thus by Lemma 2.4, we have that M/N is a Cohen-Macaulay R-module of dimension d and that $\operatorname{depth}_R N \geq d-1$. Moreover it follows that $\dim_R N \neq d$ because $aN = 0$ and $\dim_R M/aM = d-1$. So we get that N is a Cohen-Macaulay R-module of dimension $d-1$, as required. In the case in which M is Cohen-Macaulay, $N = 0$ and the assertion follows immediately.

(iii)$\Rightarrow$(iv): If $N = 0$, M is a Cohen-Macaulay module and therefore $U_M(0) = 0$ as $\operatorname{Ass}_R M = \operatorname{Assh}_R M$. Hence there is nothing to prove in this case. When $N \neq 0$, by using the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0;$$

we get $\operatorname{depth}_R M \geq d-1$. So it remains to prove that $U_M(0) = N$.

First note that because of $\dim_R N < d$, from the Corollary 2.3, we find N as a submodule of $U_M(0)$. Also from the above exact sequence we have $\operatorname{Assh}_R M \subseteq \operatorname{Ass}_R N \cup \operatorname{Ass}_R M/N$. This implies that $\operatorname{Assh}_R M \subseteq \operatorname{Ass}_R M/N$ since $\operatorname{Assh}_R M \cap \operatorname{Ass}_R N = \Phi$. On the other hand because M/N is a Cohen-Macaulay R-module of dimension d, it is well known that $\operatorname{Ass}_R M/N = \operatorname{Assh}_R M/N \subseteq \operatorname{Assh}_R M$. Hence we get that $\operatorname{Ass}_R M/N = \operatorname{Assh}_R M$. So by view of definition of $U_M(0)$, this yields that $U_M(0)/N = 0$.

(iv)$\Rightarrow$(i): If M is a Cohen-Macaulay module, $U_M(0) = 0$. Therefore we immediately have the assertion (i) by [3, Theorem 2.1.2 (c)]. In case that M is not Cohen-Macaulay module, let us apply Lemma 2.5, to the situation $N = U_M(0)$.

From which the fact that $U_M(0) \neq 0$, we can consider the exact sequence

$$0 \longrightarrow U_M(0) \longrightarrow M \longrightarrow M/U_M(0) \longrightarrow 0\ ,$$

which implies $\operatorname{depth}_R U_M(0) \geq d-1$. Hence by Corollary 2.3, we find $U_M(0)$ as a Cohen-Macaulay R-module of dimension $d-1$. Let $0 = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M} Q(\mathfrak{p})$ denotes a minimal primary decomposition of 0 in M. Then with the notations K and $\mathfrak{g}$ were introduced in Proposition 2.2, we have $K \neq 0$ since $\operatorname{Ass}_R M \neq \operatorname{Assh}_R M$. Moreover it follows that $\mathfrak{g} \neq 0$ immediately. So we can take an element $b \neq 0$ of $\mathfrak{g}$ not contained in $\bigcup_{\mathfrak{p} \in \operatorname{Assh}_R M} \mathfrak{p}$. It is straightforward to see that there exists an integer $t > 0$ such that $b^t M \subseteq K$. Now we can put $a = b^t$ and get $aU_M(0) = 0$, because $K \cap U_M(0) = 0$. Finally by Lemma 2.5, the proof is completed. $\square$

As we saw above, the submodule N in assertion (iii) is uniquely determined and is exactly equal to $U_M(0)$.

Definition 2.8. [10, Section 1.2] Suppose that the local ring $(R, \mathfrak{m})$ is the homomorphic image of a local Gorenstein ring $(R', \mathfrak{n})$. Let M be a finitely generated R-module of dimension d. For an integer $i \in \mathbb{Z}$, define

$$K^i(M) := \operatorname{Ext}_{R'}^{n'-i}(M, R'),$$

where $n' = \dim R'$. Then the module $K(M) := K^d(M)$ is called the canonical module of M and for $i \neq d$ the modules $K^i(M)$ are called the modules of deficiency of M.

Theorem 2.9. *Let $(R, \mathfrak{m})$ denote a complete local ring and suppose that M is a d-dimensional R-module which is not Cohen-Macaulay. Then the following are equivalent:*

(i) *M is approximately Cohen-Macaulay;*
(ii) *$K^d(M), K^{d-1}(M)$ are Cohen-Macaulay R-modules of dimension $d, d-1$ respectively and $K^i(M) = 0$ for all $i \neq d, d-1$.*

Proof. (i)⇒(ii): Applying the fact that $M/U_M(0)$ is a Cohen-Macaulay R-module of dimension d together with

$$\operatorname{depth}_R M = \operatorname{depth}_R U_M(0) = \dim_R U_M(0) = d-1,$$

to the following induced exact sequence

$$\cdots \longrightarrow \mathrm{H}_{\mathfrak{m}}^{d-2}(M/U_M(0)) \longrightarrow \mathrm{H}_{\mathfrak{m}}^{d-1}(U_M(0)) \longrightarrow \mathrm{H}_{\mathfrak{m}}^{d-1}(M)) \longrightarrow \mathrm{H}_{\mathfrak{m}}^{d-1}(M/U_M(0)) \longrightarrow$$

$$\mathrm{H}_{\mathfrak{m}}^{d}(U_M(0)) \longrightarrow \mathrm{H}_{\mathfrak{m}}^{d}(M) \longrightarrow \mathrm{H}_{\mathfrak{m}}^{d}(M/U_M(0)) \longrightarrow \mathrm{H}_{\mathfrak{m}}^{d+1}(U_M(0)) \longrightarrow \cdots ;$$

leads us to obtain $\mathrm{H}_{\mathfrak{m}}^{i}(M) = 0$ for all $i \neq d, d-1$ and

$$\mathrm{H}_{\mathfrak{m}}^{d-1}(M) \cong \mathrm{H}_{\mathfrak{m}}^{d-1}(U_M(0)), \qquad \mathrm{H}_{\mathfrak{m}}^{d}(M) \cong \mathrm{H}_{\mathfrak{m}}^{d}(M/U_M(0)).$$

Note that we may express R as a homomorphic image of a local Gorenstein ring R' with $\dim R' = n'$, see [8, Theorem 29.4]. Hence by view of Matlis Duality theorem [3, Theorem 3.2.13], and Local Duality theorem [2, Theorem 11.2.6], we have the following isomorphisms

$$K^i(M) = \operatorname{Ext}_{R'}^{n'-i}(M, R') \cong \operatorname{Hom}_R\Big(\operatorname{Hom}_R(\operatorname{Ext}_{R'}^{n'-i}(M, R'), \mathrm{E}(R/\mathfrak{m})), \mathrm{E}(R/\mathfrak{m})\Big)$$

$$\cong \operatorname{Hom}_R(\mathrm{H}_{\mathfrak{m}}^{i}(M), \mathrm{E}(R/\mathfrak{m})),$$

which imply $K^i(M) = 0$ for all $i \neq d, d-1$. (Here $\mathrm{E}(R/\mathfrak{m})$ denotes the injective hull of $R/\mathfrak{m}$). However, by putting $i = d$ in above, we find that

$$
\begin{aligned}
K^d(M) &\cong \mathrm{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(M), \mathrm{E}(R/\mathfrak{m})) \cong \mathrm{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(M/U_M(0)), \mathrm{E}(R/\mathfrak{m}))\\
&\cong \mathrm{Hom}_R\Big(\mathrm{Hom}_R(\mathrm{Ext}^{n'-d}_{R'}(M/U_M(0), R'), \mathrm{E}(R/\mathfrak{m})), \mathrm{E}(R/\mathfrak{m})\Big)\\
&\cong \mathrm{Ext}^{n'-d}_{R'}(M/U_M(0), R') = K^d(M/U_M(0)).
\end{aligned}
$$

Similarly in case $i = d-1$ it is straightforward to obtain $K^{d-1}(M) \cong K^{d-1}(U_M(0))$. So by [11, Proposition 3.2], the proof is complete.

(ii)⇒(i): Since for all $0 \leq i \leq d$ the R-modules $K^i(M)$ are either zero or i-dimensional Cohen-Macaulay modules, we find by [11, Theorem 5.5], that M is a Cohen-Macaulay filtered module (in the sense of [11, Definition 4.1]). We claim that $\mathrm{depth}_R M = d-1$. To this end first note, it is well known that for every R-module M, $\mathrm{Hom}_R(M, \mathrm{E}(R/\mathfrak{m})) \neq 0$ if and only if $M \neq 0$. So it follows from Local Duality theorem [2, Theorem 11.2.6], and Matlis Duality theorem [3, Theorem 3.2.13], that $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i \neq d, d-1$. Moreover $\mathrm{H}^{d-1}_{\mathfrak{m}}(M) \neq 0$ because $K^{d-1}(M) \neq 0$. This guarantees $\mathrm{depth}_R M = d-1$.

Now it follows immediately from [11, Proposition 4.5], that M is an approximately Cohen-Macaulay module. □

Let $\widehat{R}$ and $\widehat{M}$ denote the $\mathfrak{m}$-adic completions of R and M respectively. At the end of this section we collect some preliminary properties of approximately Cohen-Macaulay modules.

Corollary 2.10. *Suppose that M is an approximately Cohen-Macaulay R-module of dimension d. Then*

(i) $\dim R/\mathfrak{p} \geq \dim_R M - 1$ *for all* $\mathfrak{p} \in \mathrm{Ass}_R M$;
Moreover if M is not Cohen-Macaulay

(ii) $\mathrm{H}^i_{\mathfrak{m}}(M) \neq 0$ *for* $i = d, d-1$ *and it is zero for all* $0 \leq i < d-1$;

(iii) $K^d(\widehat{M}), K^{d-1}(\widehat{M})$ *are Cohen-Macaulay $\widehat{R}$-modules of dimension $d, d-1$ respectively and* $K^i(\widehat{M}) = 0$ *for all* $i \neq d, d-1$.

Proof. (i) This is an immediately consequence of [3, Proposition 1.2.13], because of $\mathrm{depth}_R M \geq d-1$.

(ii) This is trivial by view of [2, Corollary 6.2.8]

(iii) Note that $\widehat{R}$ is a complete ring, $H^{d-1}_{\mathfrak{m}\widehat{R}}(\widehat{M}) \cong H^{d-1}_{\mathfrak{m}\widehat{R}}(\widehat{U_M(0)})$ and $H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}/\widehat{U_M(0)})$. Thus with a similar argument presented for Theorem 2.9, we find $K^{d-1}(\widehat{M}) \cong K^{d-1}(\widehat{U_M(0)})$ and $K^d(\widehat{M}) \cong K^d(\widehat{M}/\widehat{U_M(0)})$. Moreover it follows that $K^i(\widehat{M}) = 0$ for all $i \neq d, d-1$ because $H^i_{\mathfrak{m}\widehat{R}}(\widehat{M}) = 0$ for all $i \neq d, d-1$. Now we invoke [11, Proposition 3.2] and complete the proof.

□

3. Some results

In this section we shall investigate some properties of approximately Cohen-Macaulay modules. Throughout this section unless we say otherwise, the Noetherian ring R is local with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module.

Proposition 3.1. *A direct sum of finitely many approximately Cohen-Macaulay R-modules with equal dimension d is approximately Cohen-Macaulay.*

Proof. By induction, it is enough to prove for a direct sum of two approximately Cohen-Macaulay R-modules. Let $M = M_1 \oplus M_2$, where M_1 and M_2 are approximately Cohen-Macaulay modules of dimension d. Then by Theorem 2.7, there exist Cohen-Macaulay submodules $N_1 \leq M_1$ and $N_2 \leq M_2$ such that M_1/N_1 and M_2/N_2 are Cohen-Macualay of dimension d. We may assume that N_1 and N_2 are not zero. Thus, it follows easily from the exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_1 \oplus N_2 \longrightarrow N_2 \longrightarrow 0,$$

that $N_1 \oplus N_2$ is Cohen-Macaulay of dimension $d-1$. Moreover $(M_1 \oplus M_2)/(N_1 \oplus N_2)$ is Cohen-Macaulay of dimension d because it is isomorphic to $(M_1/N_1) \oplus (M_2/N_2)$. Hence M is approximately Cohen-Macaulay. □

Lemma 3.2. *Let M be an approximately Cohen-Macaulay R-module. Suppose that $\boldsymbol{x} = x_1, x_2, \cdots, x_n$ is an M-sequence in $\mathfrak{m}$. Then $M/\boldsymbol{x}M$ is also approximately Cohen-Macaulay (over both R and $R/(\boldsymbol{x})$).*

Proof. We may assume that M is not Cohen-Macaulay and $\dim_R M = d$. By the hypothesis, there exists a submodule N of M such that N is a Cohen-Macaulay R-module of dimension $d-1$ and M/N is a Cohen-Macaulay R-module of dimension d. Let $n = 1$ be considered, that is $\mathbf{x} = x_1$ is an M-sequence of length one. Therefore N/x_1N is a $(d-2)$-dimensional Cohen-Macaulay submodule of M/x_1M (over both R and $R/(x_1)$). So by view of Theorem 2.7, it is enough to show that $\dfrac{M/x_1M}{N/x_1N}$ is a Cohen-Macaulay module of dimension $d-1$ (over both R and $R/(x_1)$).

Obviously x_1 is regular over M/N because $\operatorname{Ass}_R M/N = \operatorname{Assh}_R M$. Thus $M/x_1M + N$ is a Cohen-Macaulay module of dimension $d-1$. On the other hand, from which the fact that the submodule N is exactly $U_M(0)$ itself, we get $N \cap x_1M = x_1N$. This implies the isomorphism $\dfrac{M/x_1M}{N/x_1N} \cong M/x_1M + N$ and completes the proof in case $n = 1$. Now we can get the sentence by induction on n. □

In the following lemma for an ideal I in R, we prepare a relation between $\operatorname{grade}(I, M)$ and $\dim_R M$.

Lemma 3.3. *Suppose that M is an approximately Cohen-Macaulay R-module and that $I \subseteq \mathfrak{m}$ is an ideal of R. Then*

$$\operatorname{grade}(I, M) \geq \dim_R M - \dim_R M/IM - 1.$$

Proof. If $\dim_R M \leq 0$, there is nothing to prove. So we put $\dim_R M > 0$ and prove the assertion by induction on $\operatorname{grade}(I, M)$. In the first step suppose that $\operatorname{grade}(I, M) = 0$. Then there exists a prime $\mathfrak{p} \in \operatorname{Ass}_R M$ with $I + \operatorname{Ann}_R M \subseteq \mathfrak{p}$. Therefore it follows from Corollary 2.10 part (i)

$$\dim_R M - 1 \leq \dim R/\mathfrak{p} \leq \dim R/(I + \operatorname{Ann}_R M) = \dim_R M/IM,$$

which proves the first step.

Now let $\operatorname{grade}(I, M) > 0$. Then we can choose an M-regular element $x \in I$. It should be pointed $\operatorname{grade}(I, M/xM) = \operatorname{grade}(I, M) - 1$ and $\dim_R M/xM = \dim_R M - 1$. Moreover M/xM is an approximately Cohen-Macaulay R-module by Lemma 3.2. Hence in view of the inductive hypothesis

$$\operatorname{grade}(I, M/xM) \geq \dim_R M/xM - \dim_R M/IM - 1.$$

This completes the proof. □

Theorem 3.4. *Let M be an approximately Cohen-Macaulay R-module. Then*

(i) $\dim_{R_\mathfrak{p}} M_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \leq 1$ *for any* $\mathfrak{p} \in \operatorname{Spec} R$.

(ii) *Suppose that M is not Cohen-Macaulay. Then the following hold for any $\mathfrak{p} \in \operatorname{Supp}_R M$ such that $M_\mathfrak{p}$ is not Cohen-Macaulay:*

(a) $\dim_{R_\mathfrak{p}} M_\mathfrak{p} + \dim_R M/\mathfrak{p}M = \dim_R M$;

(b) $\operatorname{grade}(\mathfrak{p}, M) = \dim_R M - \dim_R M/\mathfrak{p}M - 1 = \operatorname{depth}_R M - \dim_R M/\mathfrak{p}M$;

(c) $\operatorname{grade}(\mathfrak{p}, M) = \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$.

(iii) *Suppose that M is not Cohen-Macaulay. Then $(U_M(0))_{\mathfrak{p}} = U_{M_{\mathfrak{p}}}(0)$, for any $\mathfrak{p} \in \operatorname{Supp}_R M$ such that $M_{\mathfrak{p}}$ is not Cohen-Macaulay.*

(iv) *$M_{\mathfrak{p}}$ is an approximately Cohen-Macaulay $R_{\mathfrak{p}}$-module, for any $\mathfrak{p} \in \operatorname{Supp}_R M$.*

Proof. (i) It is straightforward to see that $\operatorname{depth}_R M \leq \operatorname{grade}(\mathfrak{p}, M) + \dim_R M/\mathfrak{p}M$, for every $\mathfrak{p} \in \operatorname{Spec} R$. Applying this together with the fact that $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim_R M/\mathfrak{p}M \leq \dim_R M$ for every $\mathfrak{p} \in \operatorname{Spec} R$, we can write

$$\begin{aligned} 1 \geq \dim_R M - \operatorname{depth}_R M &\geq \dim_R M - \operatorname{grade}(\mathfrak{p}, M) - \dim_R M/\mathfrak{p}M \\ &\geq \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \operatorname{grade}(\mathfrak{p}, M) \\ &\geq \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}. \end{aligned}$$

(ii) Let $\mathfrak{p}$ be a prime in $\operatorname{Supp}_R(M)$ such that $M_{\mathfrak{p}}$ is not Cohen-Macaulay. Then by (i), $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - 1$. Now in view of Lemma 3.3, we have

$$\begin{aligned} \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim_R M/\mathfrak{p}M &= \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - 1 + \dim_R M/\mathfrak{p}M \\ &= \operatorname{ht}(\mathfrak{p}/\operatorname{Ann}_R M) - 1 + \dim R/\mathfrak{p} \\ &\leq \dim R/\operatorname{Ann}_R M - 1 \\ &= \dim_R M - 1 \\ &\leq \operatorname{grade}(\mathfrak{p}, M) + \dim_R M/\mathfrak{p}M \\ &\leq \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim_R M/\mathfrak{p}M. \end{aligned}$$

This implies all equations (a), (b) and (c) immediately.

(iii) Let $\mathfrak{p} \in \operatorname{Supp}_R M$ such that $M_{\mathfrak{p}}$ is not Cohen-Macaulay $R_{\mathfrak{p}}$-module. Suppose that $0 = \bigcap_{\mathfrak{q} \in \operatorname{Ass}_R M} Q(\mathfrak{q})$ denotes a minimal primary decomposition of 0 in M. Then $(Q(\mathfrak{q}))_{\mathfrak{p}}$ is a $\mathfrak{q}R_{\mathfrak{p}}$-primary submodule of $M_{\mathfrak{p}}$ for any $\mathfrak{q} \in \operatorname{Ass}_R M$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Moreover it is obvious that $(Q(\mathfrak{q}))_{\mathfrak{p}} = M_{\mathfrak{p}}$ for all $\mathfrak{q} \in \operatorname{Ass}_R M$ such that $\mathfrak{q} \not\subseteq \mathfrak{p}$. Therefore

$$0 = \bigcap_{\substack{\mathfrak{q} \in \operatorname{Ass}_R M \\ \mathfrak{q} \subseteq \mathfrak{p}}} (Q(\mathfrak{q}))_{\mathfrak{p}}$$

is a minimal primary decomposition for the zero submodule of $M_{\mathfrak{p}}$. Thus by definition of $U_M(0)$, it is enough to show that

$$\operatorname{Assh}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \{\mathfrak{q}R_{\mathfrak{p}} \mid \mathfrak{q} \in \operatorname{Assh}_R M, \quad \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Let $\mathfrak{q} \in \operatorname{Supp}_R M$. Then by view of (ii), we have

$$\begin{aligned} \mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Assh}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} &\Longleftrightarrow \mathfrak{q} \subseteq \mathfrak{p} \quad \text{and} \quad \dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}} = \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\Longleftrightarrow \mathfrak{q} \subseteq \mathfrak{p} \quad \text{and} \quad \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) = \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\Longleftrightarrow \mathfrak{q} \subseteq \mathfrak{p} \quad \text{and} \quad \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim R/\mathfrak{p} = \dim_R M. \end{aligned}$$

Let $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Assh}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. Since

$$\dim_R M = \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim R/\mathfrak{p} \leq \dim R/\mathfrak{q} \leq \dim_R M,$$

$\dim R/\mathfrak{q} = \dim_R M$ and hence $\mathfrak{q} \in \operatorname{Assh}_R M$.

Conversely, let $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \operatorname{Assh}_R M$. Since M is approximately Cohen-Macaulay, by view of [11, Propositions 4.5 and 4.6], $\operatorname{Supp}_R M$ is a catenary subset of $\operatorname{Spec} R$. Consequently $R/\mathfrak{q}$ is a catenary integral domain, because $\operatorname{Supp}_R R/\mathfrak{q} \subseteq \operatorname{Supp}_R M$. Now by [8, Theorem 31.4], we have $\operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim R/\mathfrak{p} = \dim R/\mathfrak{q}$. Thus $\operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim R/\mathfrak{p} = \dim_R M$ and we conclude that $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Assh}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$.

(iv) Let $\mathfrak{p} \in \operatorname{Supp}_R M$. If $M_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module, then it is an approximately Cohen-Macaulay module. If $M_{\mathfrak{p}}$ is not a Cohen-Macaulay module, then M is not Cohen-Macaulay. On the other hand, $M/U_M(0)$ is a Cohen-Macaulay

R-module by Theorem 2.7. Hence by (iii), $M_{\mathfrak{p}}/U_{M_{\mathfrak{p}}}(0)$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module. Now the assertion follows from Theorem 2.7, because $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - 1$, by (i).

□

In the following by using Theorem 3.4, we can extend the concept of approximately Cohen-Macaulay modules over a local ring R, to those finitely generated R-modules that R is not necessarily local.

Definition 3.5. Let R be a ring which is not necessarily local. A finitely generated module M over R is said to be an approximately Cohen-Macaulay R-module if for every prime ideal $\mathfrak{p} \in \operatorname{Supp}_R M$, $M_{\mathfrak{p}}$ is an approximately Cohen-Macaulay $R_{\mathfrak{p}}$-module. In the same way, if R itself is an approximately Cohen-Macaulay module, then it is called an approximately Cohen-Macaulay ring.

Remark 3.6. Let $\operatorname{Max} R$ denotes the set of all maximal ideals in R. Since for every $\mathfrak{p} \in \operatorname{Supp}_R M$ there exists $\mathfrak{m} \in \operatorname{Max} R$ with $\mathfrak{p} \subseteq \mathfrak{m}$ and hence $M_{\mathfrak{p}} \cong (M_{\mathfrak{m}})_{\mathfrak{p}R_{\mathfrak{m}}}$, therefore in case that R is not necessarily local, can be asserted M is approximately Cohen-Macaulay if and only if so is $M_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Max} R$.

Now assume that M is an approximately Cohen-Macaulay R-module and that $\mathbf{x} = x_1, x_2, \cdots, x_n$ is an M-sequence in R. Then $M/\mathbf{x}M$ is an approximately Cohen-Macaulay module (over both R and $R/(\mathbf{x})$). In fact, it is well known that $\mathbf{x}R_{\mathfrak{m}}$ is an $M_{\mathfrak{m}}$-sequence for all $\mathfrak{m} \in \operatorname{Max} R$ with $\mathbf{x}R \subseteq \mathfrak{m}$, see [3, Corollary 1.1.3]. So by Lemma 3.2, $M_{\mathfrak{m}}/\mathbf{x}R_{\mathfrak{m}}M_{\mathfrak{m}}$ is an approximately Cohen-Macaulay module for every $\mathfrak{m} \in \operatorname{Max} R$.

Lemma 3.7. *Let R be an approximately Cohen-Macaulay ring which is not necessarily local and $I \neq R$ an ideal. Then*

$$\operatorname{ht}(I) - 1 \leq \operatorname{grade}(I, R) \leq \operatorname{ht}(I)$$

and if R is local, then

$$\dim R - \dim R/I - 1 \leq \operatorname{grade}(I, R) \leq \dim R - \dim R/I \ ;$$

$$\dim R - 1 \leq \operatorname{ht}(I) + \dim R/I \leq \dim R \ .$$

Proof. For an ideal $I \neq R$ one has $\operatorname{grade}(I, R) = min\{\operatorname{depth} R_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec} R\,, I \subseteq \mathfrak{p}\}$ and furthermore $\operatorname{ht}(I) = min\{\dim R_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec} R\,, I \subseteq \mathfrak{p}\}$. On the other hand, Theorem 2.7 yields $\operatorname{depth} R_{\mathfrak{p}} \geq \dim R_{\mathfrak{p}} - 1$ because $R_{\mathfrak{p}}$ is an approximately Cohen-Macaulay local ring for any $\mathfrak{p} \in \operatorname{Spec} R$. This proves the first two inequalities.

Now suppose that R is local. As was shown, $\operatorname{grade}(I, R)$ either equals to $\operatorname{ht}(I)$ or $\operatorname{ht}(I) - 1$. Moreover, it is well known that $\operatorname{ht}(I) + \dim R/I \leq \dim R$. Both these facts together with Lemma 3.3, imply that

$$\dim R - \dim R/I - 1 \leq \operatorname{grade}(I, R) \leq \dim R - \dim R/I.$$

Also by a similar argument and puting $\operatorname{ht}(I)$ or $\operatorname{ht}(I) - 1$ instead of $\operatorname{grade}(I, R)$ in Lemma 3.3, we can conclude immediately

$$\dim R - 1 \leq \operatorname{ht}(I) + \dim R/I \leq \dim R.$$

□

One says that a finitely generated R-module M is perfect if $\operatorname{pd}_R M = \operatorname{grade} M$. Here $\operatorname{pd}_R M$ denotes the projective dimension of M and $\operatorname{grade} M$ is $\operatorname{grade}(\operatorname{Ann}_R M, R)$, the length of all maximal R-sequences in $\operatorname{Ann}_R M$. For more details see [3, Definition 1.2.11]. In the following we compare the perfect modules with approximately Cohen-Macaulay modules.

Proposition 3.8. *Let R be an approximately Cohen-Macaulay ring which is not necessarily local and $M \neq 0$ a finitely generated R-module with $\mathrm{pd}_R M < \infty$.*

(i) *If M is perfect, $M_\mathfrak{p}$ is Cohen-Macaulay (so is approximately Cohen-Macaulay) or $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \dim_{R_\mathfrak{p}} M_\mathfrak{p} - 1$, for every $\mathfrak{p} \in \mathrm{Supp}_R M$.*

(ii) *If M is approximately Cohen-Macaulay, $M_\mathfrak{p}$ is perfect or $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = \mathrm{grade}\, M_\mathfrak{p} + 1$, for every $\mathfrak{p} \in \mathrm{Supp}_R M$. In particular, when M is an approximately Cohen-Macaulay module over a local ring R, then M is perfect or $\mathrm{pd}_R M = \mathrm{grade}\, M + 1$.*

Proof. (i) Let $\mathfrak{p}$ be a prime in $\mathrm{Supp}_R M$. Then $M_\mathfrak{p}$ is a module over the local ring $R_\mathfrak{p}$. For simplicity of writing, we would rather replace $R_\mathfrak{p}$ and $M_\mathfrak{p}$ with R and M. In this way we should prove M is a Cohen-Macaulay R-module or $\mathrm{depth}_R M = \dim_R M - 1$. Note that with these notations M is a finitely generated perfect module over the approximately Cohen-Macaulay local ring R and $\mathrm{pd}_R M < \infty$. Hence Auslander-Buchsbaum formula [3, Theoreme 1.3.3], gives $\mathrm{grade}\, M + \mathrm{depth}_R M = \mathrm{depth}\, R$ and follows frome Theorem 2.7, that $\mathrm{grade}\, M + \mathrm{depth}_R M \geq \dim R - 1$. Therefore

$$\mathrm{depth}_R M \geq \dim R - 1 - \mathrm{grade}(\mathrm{Ann}_R M, R).$$

According to Lemma 3.7, there are two possible values for $\mathrm{grade}(\mathrm{Ann}_R M, R)$. In case that $\mathrm{grade}(\mathrm{Ann}_R M, R) = \mathrm{ht}(\mathrm{Ann}_R M) - 1$ we have $\mathrm{depth}_R M \geq \dim R - \mathrm{ht}(\mathrm{Ann}_R M)$. Thus $\mathrm{depth}_R M \geq \dim R/\mathrm{Ann}_R M = \dim_R M$ due to $\mathrm{ht}(\mathrm{Ann}_R M) + \dim R/\mathrm{Ann}_R M \leq \dim R$. This yields that M is Cohen-Macaulay.

On the other hand in case that $\mathrm{grade}(\mathrm{Ann}_R M, R) = \mathrm{ht}(\mathrm{Ann}_R M)$ we obtain

$$\mathrm{depth}_R M \geq \dim R - 1 - \mathrm{ht}(\mathrm{Ann}_R M) \geq \dim R/\mathrm{Ann}_R M - 1 = \dim_R M - 1,$$

which leads $\mathrm{depth}_R M = \dim_R M - 1$ provided that M is not Cohen-Macaulay.

(ii) It is enough to show that $\mathrm{grade}\, M \leq \mathrm{pd}_R M \leq \mathrm{grade}\, M + 1$ whenever R is local. To this end we consider the following cases:

First, it follows from [3, Theorem 2.1.5], that $\mathrm{pd}_R M = \mathrm{grade}\, M$ provided that both of R and M are Cohen-Macaulay.

Second, in case that R is Cohen-Macaulay and M is not, Auslander-Buchsbaum formula [3, Theorem 1.3.3] and the fact that $\mathrm{depth}_R M = \dim_R M - 1$, give $\mathrm{pd}_R M = \dim R - \dim_R M + 1$. Therefore $\mathrm{pd}_R M = \dim R - \dim R/\mathrm{Ann}_R M + 1$ and by [3, Corollary 2.1.4], we have

$$\mathrm{pd}_R M = \mathrm{ht}(\mathrm{Ann}_R M) + 1 = \mathrm{grade}(\mathrm{Ann}_R M, R) + 1 = \mathrm{grade}\, M + 1.$$

Finally, suppose that R is not Cohen-Macaulay (and M is Cohen-Macaulay or not). Because $\mathrm{depth}\, R = \dim R - 1$ and $\mathrm{depth}_R M \geq \dim_R M - 1$, by a similar argument as above we find

$$(a) \qquad \mathrm{pd}_R M \leq \dim R - \dim R/\mathrm{Ann}_R M.$$

It is easy to see that we always have $\mathrm{grade}\, M \leq \mathrm{pd}_R M$. Moreover follows from Lemma 3.7, that $\dim R - \dim R/\mathrm{Ann}_R M$ is equal to $\mathrm{ht}(\mathrm{Ann}_R M)$ or $\mathrm{ht}(\mathrm{Ann}_R M) + 1$. Thus we have the following two possible inequalities:

$$(b) \qquad \mathrm{grade}(\mathrm{Ann}_R M, R) \leq \mathrm{pd}_R M \leq \mathrm{ht}(\mathrm{Ann}_R M);$$
$$(c) \qquad \mathrm{grade}(\mathrm{Ann}_R M, R) \leq \mathrm{pd}_R M \leq \mathrm{ht}(\mathrm{Ann}_R M) + 1.$$

On the other hand by Lemma 3.7 again, $\mathrm{ht}(\mathrm{Ann}_R M)$ can be equal to $\mathrm{grade}(\mathrm{Ann}_R M, R)$ or $\mathrm{grade}(\mathrm{Ann}_R M, R) + 1$. Hence we find by (b) and (c) in general that

$$\mathrm{grade}(\mathrm{Ann}_R M, R) \leq \mathrm{pd}_R M \leq \mathrm{grade}(\mathrm{Ann}_R M, R) + 2.$$

We claim that $\mathrm{pd}_R M \neq \mathrm{grade}(\mathrm{Ann}_R M, R) + 2$. Otherwise, by Lemma 3.3,

$$\mathrm{pd}_R M = \mathrm{grade}(\mathrm{Ann}_R M, R) + 2 \geq \dim R - \dim R/\mathrm{Ann}_R M + 1.$$

This means $\mathrm{pd}_R M > \dim R - \dim R/\mathrm{Ann}_R M$ which contradicts (a). Therefore in all cases we have $\mathrm{grade}(\mathrm{Ann}_R M, R) \leq \mathrm{pd}_R M \leq \mathrm{grade}(\mathrm{Ann}_R M, R) + 1$.

□

4. Faithful flat extensions

In the following we investigate how approximately Cohen-Macaulay modules behave under faithful flat local extensions. It is seen that they behave somehow similar to Cohen-Macaulay modules (see [3, Theorem 2.1.7]).

Theorem 4.1. *Let $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a homomorphism of Noetherian local rings. Suppose M is a finitely generated R-module and N is a finitely generated S-module which is faithfully flat over R. Then the following are equivalent:*

- (i) *M is an approximately Cohen-Macaulay R-module and $N/\mathfrak{m}N$ is a Cohen-Macaulay S-module;*
- (ii) *$M \otimes_R N$ is an approximately Cohen-Macaulay S-module and $U_M(0) \otimes_R N = U_{M\otimes_R N}(0)$.*

Proof. (i)⇒(ii): In case that M is a Cohen-Macaulay R-module, the assertion follows immediately from [3, Theorem 2.1.7]. Suppose that M is not Cohen-Macaulay. It follows again that $U_M(0) \otimes_R N$ is a Cohen-Macaulay S-module because $U_M(0)$ is a Cohen-Macaulay R-module of dimension $\dim_R M - 1$. Also we have

$$\begin{aligned}\dim_S(U_M(0) \otimes_R N) &= \dim_R(U_M(0)) + \dim_S N/\mathfrak{m}N \\ &= \dim_R M - 1 + \dim_S N/\mathfrak{m}N \\ &= \dim_S(M \otimes_R N) - 1.\end{aligned}$$

On the other hand, $M/U_M(0)$ is Cohen-Macaulay. Thus $(M \otimes_R N)/(U_M(0) \otimes_R N)$ is a Cohen-Macaulay S-module because

$$(M \otimes_R N)/(U_M(0) \otimes_R N) \cong (M/U_M(0)) \otimes_R N.$$

Moreover by view of Corollary 2.3,

$$\begin{aligned}\dim_S(M \otimes_R N) &\geq \dim_S(M \otimes_R N)/(U_M(0) \otimes_R N) \\ &\geq \dim_S\left((M \otimes_R N)/(U_{M\otimes_R N}(0))\right) \\ &= \dim_S(M \otimes_R N).\end{aligned}$$

Therefore by Theorem 2.7 part (iii), $M \otimes_R N$ is an approximately Cohen-Macaulay S-module. Hence the paragraph before Definition 2.8, implies that $U_M(0) \otimes_R N = U_{M\otimes_R N}(0)$.

(ii)⇒(i): We may assume that $\dim_S(M \otimes_R N) > 0$, because $M \otimes_R N$ is Cohen-Macaulay in case that $\dim_S(M \otimes_R N) = 0$.

Since $(M \otimes_R N)/(U_{M\otimes_R N}(0))$ is a Cohen-Macaulay S-module, therefore it is also $(M/U_M(0)) \otimes_R N$. This leads to $M/U_M(0)$ and $N/\mathfrak{m}N$ are Cohen-Macaulay modules over R and S respectively. Moreover we have

$$\begin{aligned}\dim_R M &= \dim_S(M \otimes_R N) - \dim_S N/\mathfrak{m}N \\ &\leq \operatorname{depth}_S(M \otimes_R N) + 1 - \operatorname{depth}_S N/\mathfrak{m}N \\ &= \operatorname{depth}_R M + 1.\end{aligned}$$

Hence by Theorem 2.7 part (iv), we find that M is an approximately Cohen-Macaulay module. □

Corollary 4.2. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$. Then M is approximately Cohen-Macaulay if and only if its $\mathfrak{m}$-adic completion $\widehat{M}$ is approximately Cohen-Macaulay and $U_{\widehat{M}}(0) = \widehat{U_M(0)}$.*

Proof. The extension $R \longrightarrow \widehat{R}$ is local and faithfully flat. So we can invoke Theorem 4.1 and conclude the proof. □

It should be mentioned that in general M is not approximately Cohen-Macaulay in case that $\widehat{M}$ is an approximately Cohen-Macaulay $\widehat{R}$-module. For this fact see [11, Example 6.1].

Theorem 4.3. *Let R be a ring which is not necessarily local, M a finitely generated R-module, and $S = R[X_1, \cdots, X_n]$ or $S = R[[X_1, \cdots, X_n]]$. Then $M \otimes_R S$ is an approximately Cohen-Macaulay S-module if and only if M is an approximately Cohen-Macaulay R-module*

Proof. We may assume $n = 1$, $X = X_1$ because the indeterminates can be adjoined successively. Suppose $M \otimes_R S$ is approximately Cohen-Macaulay. In both cases X is regular on $M \otimes_R S$, and $R \cong S/(X)$, $M \underset{R}{\cong} (M \otimes_R S)/X(M \otimes_R S)$. Therefore it follows from Remark 3.6, that M is an approximately Cohen-Macaulay module.

Conversely, let $\mathfrak{m}$ be a maximal ideal of S and set $\mathfrak{p} := \mathfrak{m} \cap R$. Then $S_\mathfrak{m}$ is an $R_\mathfrak{p}$-module by canonical homomorphism $\varphi : (R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p}) \longrightarrow (S_\mathfrak{m}, \mathfrak{m}S_\mathfrak{m})$. This leads to the following isomorphism

$$(M \otimes_R S)_\mathfrak{m} \underset{S_\mathfrak{m}}{\cong} M_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{m}.$$

So we may prove $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{m}$ is an approximately Cohen-Macaulay $S_\mathfrak{m}$-module. Since in both cases, S is a flat R-algebra, therefore $S_\mathfrak{m}$ is faithfully flat over $R_\mathfrak{p}$ by [1, Exercises 3.16 and 3.18]. Moreover the fiber $S_\mathfrak{m}/\mathfrak{p}S_\mathfrak{m}$ is a discrete valuation ring, and thus is Cohen-Macaulay (see outlined below [3, Theorem A.12]). Theorem 4.1, completes the proof. □

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