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Research Article

A new generalization of Young type inequality and applications

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Abstract

In this paper, we give refinements of the classical Young inequality for scalars and we use these refinements to establish refined some Young type inequalities for the trace, determinants, and norms of positive definite matrices.

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1. Introduction

Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices and let $|||.|||$ denote any unitarily invariant (or symmetric) norm on $M_n(\mathbb{C})$ and the Hilbert–Schmidt norm $\|A\|_2 = \left(\sum_{j=1}^{n} s_j^2\right)^{\frac{1}{2}}$, where $s_1(A) \geq \cdots \geq s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$.

Through out the paper of A is an $n \times n$ matrix, we write trA to denote the trace of A and $detA$ for the determinate of A.

The classical Young inequality for two scalars is the α-weighted arithmetic–geometric mean inequality, which is a fundamental relation between two nonnegative real numbers. This inequality says that if $a, b \geq 0$ *and* $0 \leq \alpha \leq 1$, then

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)\, b \tag{1.1}$$

with equality if and only if $a = b$. If $\alpha = \frac{1}{2}$, we obtain the arithmetic-geometric mean inequality.

$$\sqrt{ab} \leq \frac{a+b}{2} \tag{1.2}$$

The first refinements of Young inequality is the squared version, proved in [4] as follow:

$$\left(a^{\alpha}b^{1-\alpha}\right)^2 + min\{\alpha, 1-\alpha\}^2(a-b)^2 \leq (\alpha a + (1-\alpha)\, b)^2. \tag{1.3}$$

Later, Kittaneh and Manasrah in [2] , obtained another interesting refinement of Young inequality as follow:

$$a^{\alpha}b^{1-\alpha} + \min\{\alpha, 1-\alpha\}\left(\sqrt{a} - \sqrt{b}\right)^2 \leq \alpha a + (1-\alpha)\, b \tag{1.4}$$

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Recently, Al-Manasrah and Kittaneh in [3] gave a generalization of two refined Young inequalities (1.3) and (1.4) as follow: For $m = 1, 2, 3, \cdots$

$$\left(a^{\alpha}b^{1-\alpha}\right)^{m} + {r_0}^{m}\left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^{2} \leq \left(\alpha a^{r} + (1-\alpha)\, b^{r}\right)^{\frac{m}{r}}, \quad r \geq 1 \tag{1.5}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

These inequalities are generalized to the space of operators on finite dimensional Hilbert spaces. In the setting of matrices, the above refinement is read as follow [3]

$$|||\ A^{\alpha}XB^{1-\alpha}\ |||^{m} + r_0^{m}\left(|||\ AX\ |||^{\ \frac{m}{2}} - |||\ BX\ |||^{\ \frac{m}{2}}\right)^{2} \leq (\alpha|||\ AX\ ||| + (1-\alpha)|||\ XB\ |||\ )^{m} \tag{1.6}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Very recently, Ighachane and Akkouchi in [1] gave a new generalization of the refined Young inequalities (1.3) and (1.4) as follow: For $m = 1, 2, 3, \ldots,$

$$\left(a^{\alpha}b^{1-\alpha}\right)^{m} + {r_0}^{m}\left(\frac{b^{m+1} - a^{m+1}}{b-a} - (m+1)\,(ab)^{\frac{m}{2}}\right) \leq (\alpha a + (1-\alpha)\, b)^{m} \tag{1.7}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

The authors in [1] gave the reverse of scalar Young type inequality (1.6) as follows

$${r_0}^{m}\left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^{2} \leq {r_0}^{m}\left(\frac{b^{m+1} - a^{m+1}}{b-a} - (m+1)\,(ab)^{\frac{m}{2}}\right) \leq (\alpha a + (1-\alpha)\, b)^{m} - \left(a^{\alpha}b^{1-\alpha}\right)^{m} \tag{1.8}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Motivated by the refinements of the well-known Young inequality, in this paper, we apply this inequality to obtain considerable generalizations and refinements of the Young inequality for positive scalars. Applications to unitarily invariant norm inequalities involving positive semidefinite matrices are also given.

2. Refinements of the scalar Young's inequality

We begin this section with the following refinements of the scalar Young's inequality (1.3).

Theorem 2.1. *If $a, b \geq 0,\ 0 \leq \alpha \leq 1$ and $r > 0$, then*

$$a^{2r\alpha}b^{2r(1-\alpha)} + r_0(a^{r} - b^{r})^{2} \leq \alpha a^{2r} + (1-\alpha)b^{2r} \tag{2.1}$$

where $r_0 = min\,\{\alpha, 1-\alpha\}$.

Proof. If $\alpha = \frac{1}{2}$, the inequality (2.1) becomes an equality.

Assume that $\alpha < \frac{1}{2}$. Then, by the inequality (1.1) we have

$$\begin{aligned}
&\alpha a^{2r} + (1-\alpha)\, b^{2r} - \alpha\,(a^{r} - b^{r})^{2} \\
&= 2\alpha a^{r}b^{r} + (1-2\alpha)\, b^{2r} \\
&\geq (ab)^{2r\alpha}b^{2r(1-2\alpha)} \\
&= a^{2r\alpha}b^{2r(1-\alpha)}
\end{aligned}$$

and so

$$\alpha a^{2r} + (1-\alpha)\, b^{2r} \geq \alpha\,(a^{r} - b^{r})^{2} + a^{2r\alpha}b^{2r(1-\alpha)}$$

If $\alpha > \frac{1}{2}$, then

$$\begin{aligned}&\alpha a^{2r} + (1-\alpha)\, b^{2r} - (1-\alpha)\,(a^r - b^r)^2\\&= (2\alpha - 1)\, a^{2r} + 2\,(1-\alpha)\, a^r b^r\\&\geq a^{2r(2\alpha-1)}(ab)^{2r(1-\alpha)}\\&= a^{2r\alpha} b^{2r(1-\alpha)}\end{aligned}$$

and so,

$$\alpha a^{2r} + (1-\alpha)\, b^{2r} \geq (1-\alpha)\,(a^r - b^r)^2 + a^{2r\alpha} b^{2r(1-\alpha)}$$

Hence,

$$a^{2r\alpha} b^{2r(1-\alpha)} + r_0 (a^r - b^r)^2 \leq \ \alpha a^{2r} + (1-\alpha)\, b^{2r}$$

This completes the proof. □

Theorem 2.2. *If $a, b \geq 0$ and $-1 \leq \alpha \leq 1$ then,*

$$a^{\frac{1-\alpha}{2}} b^{\frac{1+\alpha}{2}} + r_0 (\sqrt{a} - \sqrt{b})^2 \leq \frac{1-\alpha}{2} a + \frac{1+\alpha}{2} \mathrm{b} \tag{2.2}$$

where $r_0 = min\left\{\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right\}$.

Proof. If $a = b$, then the inequality (2.2) becomes an equality.
Assume that $\alpha < 0$. Then, by the inequality (1.1) we have

$$\begin{aligned}\frac{1-\alpha}{2} a + \frac{1+\alpha}{2} \mathrm{b} - \frac{1+\alpha}{2} \left(\sqrt{\mathrm{a}} - \sqrt{b}\right)^2 &= -\alpha a + (1+\alpha)\, \sqrt{ab}\\&\geq a^{-\alpha} (ab)^{\frac{1+\alpha}{2}} = a^{\frac{1-\alpha}{2}} b^{\frac{1+\alpha}{2}}\end{aligned}$$

and so,

$$\frac{1-\alpha}{2} a + \frac{1+\alpha}{2} \mathrm{b} \geq \frac{1+\alpha}{2} \left(\sqrt{\mathrm{a}} - \sqrt{b}\right)^2 + a^{\frac{1-\alpha}{2}} b^{\frac{1+\alpha}{2}}$$

If $\alpha > 0$ then,

$$\begin{aligned}&\frac{1-\alpha}{2} a + \frac{1+\alpha}{2} \mathrm{b} - \frac{1-\alpha}{2} \left(\sqrt{\mathrm{a}} - \sqrt{b}\right)^2\\&= \alpha b + (1-\alpha)\, \sqrt{ab}\\&\geq b^{\alpha} (ab)^{\frac{1-\alpha}{2}}\\&= a^{\frac{1-\alpha}{2}} b^{\frac{1+\alpha}{2}}\end{aligned}$$

and so,

$$\frac{1-\alpha}{2} a + \frac{1+\alpha}{2} \mathrm{b} \geq \frac{1-\alpha}{2} \left(\sqrt{\mathrm{a}} - \sqrt{b}\right)^2 + a^{\frac{1-\alpha}{2}} b^{\frac{1+\alpha}{2}}$$

Hence,

$$a^{\frac{1-\alpha}{2}} b^{\frac{1+\alpha}{2}} + r_0 \left(\sqrt{\mathrm{a}} - \sqrt{b}\right)^2 \leq \ \frac{1-\alpha}{2} a + \frac{1+\alpha}{2} \mathrm{b}$$

This completes the proof. □

3. Improving Young inequality for matrices

In this section we give some refined Young type inequalities for traces, determinants, and norms of positive definite matrices based on the refined Young Inequalities (2.1) and (2.2). Now we give a refinement of both the trace and the determinant versions of Young's Inequality. To do this, we need the following Lemma ([2]).

Lemma 3.1. *Let $A, B \in M_n(\mathbb{C})$. Then*

$$\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A)s_j(B)$$

Theorem 3.2. *Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite, if $0 \leq \alpha \leq 1$ and $r > 0$, then*

$$tr\left|A^{2r\alpha}B^{2r(1-\alpha)}\right| + r_0(trA^r - trB^r)^2 \leq tr(\alpha A^{2r} + (1-\alpha)B^{2r}) \tag{3.1}$$

where $r_0 = min\{\alpha, 1-\alpha\}$

Proof. By the inequality (2.1) we have

$$\alpha {s_j}^{2r}(A) + (1-\alpha){s_j}^{2r}(B) \geq {s_j}^{2r\alpha}(A){s_j}^{2r(1-\alpha)}(B) + r_0({s_j}^r(A) - {s_j}^r(B))^2$$

for $j = 1, \ldots\ldots n$

Thus, by Lemma 3.1 and the Cauchy-Schwarz inequality, we have

$$tr\left(\alpha A^{2r} + (1-\alpha)B^{2r}\right) = \alpha tr\left(A^{2r}\right) + (1-\alpha)tr\left(B^{2r}\right) = \sum_{j=1}^{n}\left(\alpha {s_j}^{2r}(A) + (1-\alpha){s_j}^{2r}(B)\right)$$

$$\geq \sum_{j=1}^{n} s_j\left(A^{2r\alpha}\right)s_j\left(B^{2r(1-\alpha)}\right) + r_0\left(\sum_{j=1}^{n}{s_j}^{2r}(A) + \sum_{j=1}^{n}{s_j}^{2r}(B) - 2\sum_{j=1}^{n}{s_j}^r(A){s_j}^r(B)\right)$$

$$\geq \sum_{j=1}^{n} s_j\left(A^{2r\alpha}B^{2r(1-\alpha)}\right) + r_0\left(tr\left(A^{2r}\right) + tr\left(B^{2r}\right) - 2\left(\sum_{j=1}^{n}s_j\left(A^{2r}\right)\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}s_j\left(B^{2r}\right)\right)^{\frac{1}{2}}\right)$$

$$= \sum_{j=1}^{n} s_j\left(A^{2r\alpha}B^{2r(1-\alpha)}\right) + r_0\left(trA^{2r} + trB^{2r} - 2\left(trA^{2r}\right)^{\frac{1}{2}}\left(trB^{2r}\right)^{\frac{1}{2}}\right)$$

$$= tr\left|A^{2r\alpha}B^{2r(1-\alpha)}\right| + r_0(trA^r - trB^r)^2$$

This completes the proof of the trace inequality. □

Theorem 3.3. *Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite and if $-1 \leq \alpha \leq 1$, then*

$$tr\left|A^{\frac{1-\alpha}{2}}B^{\frac{1+\alpha}{2}}\right| + r_0(\sqrt{tr\mathrm{A}} - \sqrt{tr\mathrm{B}})^2 \leq tr(\frac{1-\alpha}{2}\mathrm{A} + \frac{1+\alpha}{2}B) \tag{3.2}$$

where $r_0 = min\left\{\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right\}$.

Proof. By the inequality (2.2) we have

$$\frac{1-\alpha}{2}s_j(A) + \frac{1+\alpha}{2}s_j(B) \geq s_j\left(A^{\frac{1-\alpha}{2}}\right)\ s_j\ \left(B^{\frac{1+\alpha}{2}}\right) + r_0({s_j}^{\frac{1}{2}}(A) - {s_j}^{\frac{1}{2}}(B))^2$$

for $j = 1, \ldots\ldots, n$.

Thus, by Lemma 3.1 and the Cauchy-Schwarz inequality, we have

$$tr\left(\frac{1-\alpha}{2}\mathrm{A} + \frac{1+\alpha}{2}\mathrm{B}\right) = \frac{1-\alpha}{2}tr(\mathrm{A}) + \frac{1+\alpha}{2}tr(\mathrm{B}) = \sum_{j=1}^{n}\left(\frac{1-\alpha}{2}s_j(A) + \frac{1+\alpha}{2}s_j(B)\right)$$

$$\geq \sum_{j=1}^{n} s_j\left(A^{\frac{1-\alpha}{2}}\right)\ s_j\ \left(B^{\frac{1+\alpha}{2}}\right) + r_0\left(\sum_{j=1}^{n}s_j(A) + \sum_{j=1}^{n}s_j(B) - 2\sum_{j=1}^{n}{s_j}^{\frac{1}{2}}(A){s_j}^{\frac{1}{2}}(B)\right)$$

$$\geq \sum_{j=1}^{n} s_j\left(A^{\frac{1-\alpha}{2}}B^{\frac{1+\alpha}{2}}\right) + r_0\left(tr(\mathrm{A}) + tr(\mathrm{B}) - 2\left(\sum_{j=1}^{n}s_j(A)\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}s_j(B)\right)^{\frac{1}{2}}\right)$$

$$= \sum_{j=1}^{n} s_j\left(A^{\frac{1-\alpha}{2}}B^{\frac{1+\alpha}{2}}\right) + r_0\left(tr\mathrm{A} + tr\mathrm{B} - 2(tr\mathrm{A})^{\frac{1}{2}}(tr\mathrm{B})^{\frac{1}{2}}\right)$$

$$= tr\left|A^{\frac{1-\alpha}{2}}B^{\frac{1+\alpha}{2}}\right| + r_0\left(\sqrt{trA}-\sqrt{tr\mathrm{B}}\right)^2$$

This completes the proof of the trace inequality. □

Theorem 3.4. *Let $A, B \in M_n(\mathbb{C})$ be positive definite, if $0 \le \alpha \le 1$ and $r > 0$, then*

$$\begin{aligned}&\det\left(A^{2r\alpha}B^{2r(1-\alpha)}\right) + {r_0}^n \det\left(A^{2r}+B^{2r}-2\left(B^r\left(B^{-r}A^{2r}B^{-r}\right)^{1/2}B^r\right)\right)\\&\le \det\left(\alpha A^{2r}+(1-\alpha)B^{2r}\right)\end{aligned} \tag{3.3}$$

where $r_0 = min\{\alpha, 1-\alpha\}$

Proof. By the inequality (2.1) we have

$$\alpha s_j\left(B^{-r}A^{2r}B^{-r}\right)+(1-\alpha) \ge {s_j}^{\alpha}\left(B^{-r}A^{2r}B^{-r}\right)+r_0\left({s_j}^{1/2}\left(B^{-r}A^{2r}B^{-r}\right)-1\right)^2$$

for $j = 1, \ldots\ldots, n$. Thus

$$\det\left(\alpha B^{-r}A^{2r}B^{-r}+(1-\alpha)I\right) = \prod_{j=1}^{n}\left(\alpha s_j\left(B^{-r}A^{2r}B^{-r}\right)+1-\alpha\right)$$

$$\ge \prod_{j=1}^{n}\left[{s_j}^{\alpha}\left(B^{-r}A^{2r}B^{-r}\right)+r_0\left({s_j}^{1/2}\left(B^{-r}A^{2r}B^{-r}\right)-1\right)^2\right]$$

$$\ge \prod_{j=1}^{n}{s_j}^{\alpha}\left(B^{-r}A^{2r}B^{-r}\right)+{r_0}^n\prod_{j=1}^{n}\left({s_j}^{1/2}\left(B^{-r}A^{2r}B^{-r}\right)-1\right)^2$$

$$= \det\left(B^{-r}A^{2r}B^{-r}\right)^{\alpha}+{r_0}^n\det\left(\left(B^{-r}A^{2r}B^{-r}\right)^{1/2}-I\right)^2$$

Consequently,

$$\begin{aligned}&\det\left(A^{2r\alpha}B^{2r(1-\alpha)}\right)+{r_0}^n\det\left(A^{2r}+B^{2r}-2\left(B^r\left(B^{-r}A^{2r}B^{-r}\right)^{1/2}\right)B^r\right)\\&\le\det\left(\alpha A^{2r}+(1-\alpha)B^{2r}\right)\end{aligned}$$

This complete the proof of the determinant inequality. □

Theorem 3.5. *Let $A, B \in M_n(\mathbb{C})$ be positive definite and if $-1 \le \alpha \le 1$, then*

$$\begin{aligned}&\det\left(A^{\frac{1-\alpha}{2}}B^{\frac{1+\alpha}{2}}\right)+{r_0}^n\det\left(\mathrm{A}+\mathrm{B}-2\left(B^{\frac{1}{2}}\left(B^{\frac{-1}{2}}\mathrm{A}B^{\frac{-1}{2}}\right)^{\frac{1}{2}}B^{\frac{1}{2}}\right)\right)\\&\le\det\left(\frac{1-\alpha}{2}\mathrm{A}+\frac{1+\alpha}{2}\mathrm{B}\right)\end{aligned} \tag{3.4}$$

where $r_0 = min\left\{\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right\}$

Proof. By the inequality (2.2) we have

$$\frac{1-\alpha}{2}s_j\left(B^{\frac{-1}{2}}\mathrm{A}B^{\frac{-1}{2}}\right)+\frac{1+\alpha}{2} \ge {s_j}^{\frac{1-\alpha}{2}}\left(B^{\frac{-1}{2}}\mathrm{A}B^{\frac{-1}{2}}\right)+r_0\left({s_j}^{\frac{1}{2}}\left(B^{\frac{-1}{2}}\mathrm{A}B^{\frac{-1}{2}}\right)-1\right)^2$$

for $j = 1, \ldots\ldots, n$. Thus

$$\det\left(\frac{1-\alpha}{2}B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + \frac{1+\alpha}{2}I\right) = \prod_{j=1}^{n}\left(\frac{1-\alpha}{2}s_j\left(B^{\frac{-1}{2}}AB^{\frac{-1}{2}}\right) + \frac{1+\alpha}{2}\right)$$

$$\geq \prod_{j=1}^{n}\left[s_j^{\frac{1-\alpha}{2}}\left(B^{\frac{-1}{2}}AB^{\frac{-1}{2}}\right) + r_0\left(s_j^{\frac{1}{2}}\left(B^{\frac{-1}{2}}AB^{\frac{-1}{2}}\right) - 1\right)^2\right]$$

$$\geq \prod_{j=1}^{n} s_j^{\frac{1-\alpha}{2}}\left(B^{\frac{-1}{2}}AB^{\frac{-1}{2}}\right) + r_0{}^n\prod_{j=1}^{n}\left(s_j^{\frac{1}{2}}\left(B^{\frac{-1}{2}}AB^{\frac{-1}{2}}\right) - 1\right)^2$$

$$= \det\left(B^{\frac{-1}{2}}AB^{\frac{-1}{2}}\right)^{\frac{1-\alpha}{2}} + r_0{}^n\det\left(\left(B^{\frac{-1}{2}}AB^{\frac{-1}{2}}\right)^{\frac{1}{2}} - I\right)^2$$

Consequently,

$$\det\left(A^{\frac{1-\alpha}{2}}B^{\frac{1+\alpha}{2}}\right) + r_0{}^n\det\left(A + B - 2\left(B^{\frac{1}{2}}\left(B^{\frac{-1}{2}}AB^{\frac{-1}{2}}\right)^{\frac{1}{2}}B^{\frac{1}{2}}\right)\right) \leq \det\left(\frac{1-\alpha}{2}A + \frac{1+\alpha}{2}B\right)$$

This completes the proof of the determinant inequality. □

In the next result, we give an improved arithmetic-geometric mean inequality for the Hilbert-Schmidt norm whose proof based on the spectral theorem.

Theorem 3.6. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, if $0 \leq \alpha \leq 1$ and $r > 0$, then*

$$\begin{aligned}&\left\|A^{r\alpha}XB^{r(1-\alpha)}\right\|_2^2 + r_0\left\|A^rX - XB^r\right\|_2^2 + 2\sqrt{\alpha}\sqrt{1-\alpha}\left\|\sqrt{A^r}X\sqrt{B^r}\right\|_2^2\\ &\leq \left\|\sqrt{\alpha}A^rX + \sqrt{1-\alpha}XB^r\right\|_2^2\end{aligned} \tag{3.5}$$

where $r_0 = min\{\alpha, 1-\alpha\}$

Proof. Since A *and* B are positive semidefinite, it is followed by spectral theorem that there exist unitary matrices $U, V \in M_n(\mathbb{C})$, such that $A = U\Lambda_1U^*$, $B = V\Lambda_2V^*$, where $\Lambda_1 = diag(\lambda_1, \lambda_2, \ldots\ldots, \lambda_n)$, $\Lambda_2 = diag(\mu_1, \mu_2, \ldots\ldots\ldots, \mu_n)$, $\lambda_i, \mu_j \geq 0$, $i, j = 1, 2, \ldots\ldots, n$. For our computation, let $Y = U^*XV = [y_{ij}]$. Then we have

$$\sqrt{\alpha}A^rX + \sqrt{1-\alpha}XB^r = U\left[(\sqrt{\alpha}\lambda_i{}^r + \sqrt{1-\alpha}\mu_j{}^r)y_{ij}\right]V^*,$$

$$A^{r\alpha}XB^{r(1-\alpha)} = U\left[(\lambda_i{}^{r\alpha}\mu_j{}^{r(1-\alpha)})y_{ij}\right]V^*,$$

$$A^rX - XB^r = U\left[(\lambda_i{}^r - \mu_j{}^r)y_{ij}\right]V^*,$$

$$A^{\frac{r}{2}}XB^{\frac{r}{2}} = U\left[(\lambda_i{}^{\frac{r}{2}}\mu_j{}^{\frac{r}{2}})y_{ij}\right]V^*$$

Now by (2.1), we have

$$\left\|\sqrt{\alpha}A^rX + \sqrt{1-\alpha}XB^r\right\|_2^2 = \sum_{i,j=1}^{n}\left(\sqrt{\alpha}\lambda_i{}^r + \sqrt{1-\alpha}\mu_j{}^r\right)^2|y_{ij}|^2$$

$$= \sum_{i,j=1}^{n}\left(\alpha\lambda_i{}^{2r} + (1-\alpha)\mu_j{}^{2r} + 2\sqrt{\alpha}\sqrt{1-\alpha}\lambda_i{}^r\mu_j{}^r\right)|y_{ij}|^2$$

$$\geq \sum_{i,j=1}^{n}\left[\left(\lambda_i{}^{2r\alpha}\right).\left(\mu_j{}^{2r(1-\alpha)}\right) + r_0(\lambda_i{}^r - \mu_j{}^r)^2 + 2\sqrt{\alpha}\sqrt{1-\alpha}\lambda_i{}^r\mu_j{}^r\right]|y_{ij}|^2$$

$$= \sum_{i,j=1}^{n}\left(\lambda_i{}^{r\alpha}\mu_j{}^{r(1-\alpha)}\right)^2|y_{ij}|^2 + r_0\sum_{i,j=1}^{n}(\lambda_i{}^r - \mu_j{}^r)^2|y_{ij}|^2 + 2\sqrt{\alpha}\sqrt{1-\alpha}\sum_{i,j=1}^{n}\left(\lambda_i{}^{\frac{r}{2}}\mu_j{}^{\frac{r}{2}}\right)^2|y_{ij}|^2$$

$$= \left\|A^{r\alpha}XB^{r(1-\alpha)}\right\|_2^2 + r_0\|A^rX - XB^r\|_2^2 + 2\sqrt{\alpha}\sqrt{1-\alpha}\left\|\sqrt{A^r}X\sqrt{B^r}\right\|_2^2$$

□

Theorem 3.7. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite and if $-1 \le \alpha \le 1$, then*

$$\begin{aligned}&\left\|A^{\frac{1-\alpha}{2}}XB^{\frac{1+\alpha}{2}}\right\|_2^2+r_0\left\|\sqrt{\mathrm{A}}X-X\sqrt{\mathrm{B}}\right\|_2^2+2\sqrt{\frac{1-\alpha}{2}}\sqrt{\frac{1+\alpha}{2}}\left\|\sqrt{\mathrm{A}}X\sqrt{\mathrm{B}}\right\|_2^2\\&\le\left\|\sqrt{\frac{1-\alpha}{2}}\mathrm{A}X+\sqrt{\frac{1+\alpha}{2}}\mathrm{XB}\right\|_2^2\end{aligned}\tag{3.6}$$

where $r_0 = min\left\{\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right\}$

Proof. Since A *and* B are positive semidefinite, it follows by spectral theorem that there exist unitary matrices $U, V \in M_n(\mathbb{C})$, such that $A = U\Lambda_1U^*$, $B = V\Lambda_2V^*$, where $\Lambda_1 = diag(\lambda_1, \lambda_2, \ldots\ldots, \lambda_n)$, $\Lambda_2 = \mathrm{diag}(\mu_1, \mu_2, \ldots\ldots\ldots, \mu_n)$, $\lambda_i, \mu_j \ge 0$, $i, j = 1, 2, \ldots\ldots., n$. For our computation, let $Y = U^*XV = [y_{ij}]$. Then we have

$$\sqrt{\frac{1-\alpha}{2}}\mathrm{A}X+\sqrt{\frac{1+\alpha}{2}}\mathrm{XB}=U\left[(\sqrt{\frac{1-\alpha}{2}}\lambda_i+\sqrt{\frac{1+\alpha}{2}}\mu_j)y_{ij}\right]V^*,$$

$$A^{\frac{1-\alpha}{2}}XB^{\frac{1+\alpha}{2}}=U\left[(\lambda_i{}^{\frac{1-\alpha}{2}}\mu_j{}^{\frac{1+\alpha}{2}})y_{ij}\right]V^*,$$

$$\sqrt{\mathrm{A}}X-X\sqrt{\mathrm{B}}=U\left[(\sqrt{\lambda_i}-\sqrt{\mu_j})y_{ij}\right]V^*,$$

$$2\sqrt{\frac{1-\alpha}{2}}\sqrt{\frac{1+\alpha}{2}}(\sqrt{\mathrm{A}}\times\sqrt{\mathrm{B}})=U\left[(\sqrt{\frac{1-\alpha}{2}}\sqrt{\frac{1+\alpha}{2}}\ \sqrt{\lambda_i\mu_j})y_{ij}\right]V^*$$

Now by (2.2), we have

$$\begin{aligned}&\left\|\sqrt{\frac{1-\alpha}{2}}AX+\sqrt{\frac{1+\alpha}{2}}X\mathrm{B}\right\|_2^2=\sum_{i,j=1}^{n}\left(\sqrt{\frac{1-\alpha}{2}}\lambda_i+\sqrt{\frac{1+\alpha}{2}}\mu_j\right)^2|y_{ij}|^2\\&=\sum_{i,j=1}^{n}\left(\frac{1-\alpha}{2}\lambda_i{}^2+\frac{1+\alpha}{2}\mu_j{}^2+2\sqrt{\frac{1-\alpha}{2}}\sqrt{\frac{1+\alpha}{2}}\lambda_i\mu_j\right)|y_{ij}|^2\\&\ge\sum_{i,j=1}^{n}\left[\left(\lambda_i{}^{1-\alpha}\right).\left(\mu_j{}^{1+\alpha}\right)+r_0(\sqrt{\lambda_i}-\sqrt{\mu_j})^2+2\sqrt{\frac{1-\alpha}{2}}\sqrt{\frac{1+\alpha}{2}}\lambda_i\mu_j\right]|y_{ij}|^2\\&=\sum_{i,j=1}^{n}\left(\lambda_i{}^{\frac{1-\alpha}{2}}\mu_j{}^{\frac{1+\alpha}{2}}\right)^2|y_{ij}|^2+r_0\sum_{i,j=1}^{n}\left(\sqrt{\lambda_i}-\sqrt{\mu_j}\right)^2|y_{ij}|^2\\&+2\sqrt{\frac{1-\alpha}{2}}\sqrt{\frac{1+\alpha}{2}}\sum_{i,j=1}^{n}\left(\sqrt{\lambda_i}\sqrt{\mu_j}\right)^2|y_{ij}|^2\\&=\left\|A^{\frac{1-\alpha}{2}}XB^{\frac{1+\alpha}{2}}\right\|_2^2+r_0\left\|\sqrt{\mathrm{A}}X-X\sqrt{\mathrm{B}}\right\|_2^2+2\sqrt{\frac{1-\alpha}{2}}\sqrt{\frac{1+\alpha}{2}}\left\|\sqrt{\mathrm{A}}X\sqrt{\mathrm{B}}\right\|_2^2.\end{aligned}$$

□

The results of unitary norm are as follow:

Lemma 3.8. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, if $0 \le \alpha \le 1$ and $r > 0$, then*

$$\left|\left|\left|A^{2r\alpha}XB^{2r(1-\alpha)}\right|\right|\right|\le\left|\left|\left|A^{2r}X\right|\right|\right|^{\alpha}\left|\left|\left|XB^{2r}\right|\right|\right|^{(1-\alpha)}$$

Theorem 3.9. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, if $0 \leq \alpha \leq 1$ and $r > 0$, then*

$$\left|\left|\left|A^{2r\alpha} X B^{2r(1-\alpha)}\right|\right|\right| + r_0(|||A^r X||| - |||XB^r|||)^2 \leq \alpha \left|\left|\left|A^{2r} X\right|\right|\right| + (1-\alpha) \left|\left|\left|XB^{2r}\right|\right|\right| \quad (3.7)$$

where $r_0 = min\{\alpha, 1-\alpha\}$

Proof. By Lemma 3.8 and the inequality (2.1) we have

$$\begin{aligned}&\left|\left|\left|A^{2r\alpha} X B^{2r(1-\alpha)}\right|\right|\right| + r_0(|||A^r X||| - |||XB^r|||)^2 \\ &\leq \left|\left|\left|A^{2r} X\right|\right|\right|^{\alpha} \left|\left|\left|XB^{2r}\right|\right|\right|^{(1-\alpha)} + r_0(|||A^r X||| - |||XB^r|||)^2 \\ &\leq \alpha \left|\left|\left|A^{2r} X\right|\right|\right| + (1-\alpha) \left|\left|\left|XB^{2r}\right|\right|\right|\end{aligned}$$

□

Lemma 3.10. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite and if $-1 \leq \alpha \leq 1$, then*

$$\left|\left|\left|A^{\frac{1-\alpha}{2}} X B^{\frac{1+\alpha}{2}}\right|\right|\right| \leq |||\mathrm{A}X|||^{\frac{1-\alpha}{2}} |||\mathrm{XB}|||^{\frac{1+\alpha}{2}}.$$

Theorem 3.11. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite and if $-1 \leq \alpha \leq 1$, then*

$$\left|\left|\left|A^{\frac{1-\alpha}{2}} X B^{\frac{1+\alpha}{2}}\right|\right|\right| + r_0\left(\left|\left|\left|\sqrt{\mathrm{A}}X\right|\right|\right| - \left|\left|\left|\mathrm{X}\sqrt{\mathrm{B}}\right|\right|\right|\right)^2 \leq \frac{1-\alpha}{2} |||\mathrm{A}X||| + \frac{1+\alpha}{2} |||\mathrm{XB}||| \quad (3.8)$$

where $r_0 = min\left\{\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right\}$

Proof. By Lemma 3.10 and the inequality (2.2) we have

$$\begin{aligned}&\left|\left|\left|A^{\frac{1-\alpha}{2}} X B^{\frac{1+\alpha}{2}}\right|\right|\right| + r_0\left(\left|\left|\left|\sqrt{\mathrm{A}}X\right|\right|\right| - \left|\left|\left|\mathrm{X}\sqrt{\mathrm{B}}\right|\right|\right|\right)^2 \\ &\leq |||\mathrm{A}X|||^{\frac{1-\alpha}{2}} |||X\mathrm{B}|||^{\frac{1+\alpha}{2}} + r_0\left(\left|\left|\left|\sqrt{\mathrm{A}}X\right|\right|\right| - \left|\left|\left|X\sqrt{\mathrm{B}}\right|\right|\right|\right)^2 \\ &\leq \frac{1-\alpha}{2} |||AX||| + \frac{1+\alpha}{2} |||XB|||\end{aligned}$$

□

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