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Some Results on $\mathscr{D}$-Homothetic Deformation On Almost Paracontact Metric Manifolds

Mehmet Solgun

Department of Mathematics, Faculty of Science and Arts, Bilecik Şeyh Edebali University, Bilecik, Turkey

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Abstract

In this paper, we investigate the effect of $\mathscr{D}$-homothetic deformation on almost para-contact metric manifolds. The main results of the paper are about some classes of almost paracontact metric manifolds in which the characteristic vector field is parallel. It is shown that certain classes are invariant under the $\mathscr{D}$-homothetic deformation.

1. Introduction

Almost paracontact structures were first studied by [1] (Kaneyuki, 1985) and after the work of Zamkovoy in [2] (Zamkovoy, 2009), many authors have made contributions to the subject. In the literature, there are many studies on almost paracontact manifolds from different perspectives in various dimensions. For recent studies, see [3]-[8]. In [9], Zamkovoy and Nakova classified almost paracontact metric structures into the 2^{12} classes by considering the covariant derivative of the fundamental 2-form Φ of the structure with respect to the Levi-Civita connection. The main goal of this work is to study $\mathscr{D}$-homothetic deformations on these structures. We examine the almost paracontact metric structure after the deformation and investigate some certain classes after the deformation. Mostly, we focused on the classes having parallel characteristic vector fields.

2. Almost paracontact metric structures

Definition 2.1. *A differentiable manifold M of dimension $(2n+1)$ is said to be have an almost paracontact structure (ϕ,ξ,η), if it has an endomorphism ϕ, a 1-form η and a vector field ξ such that*

$$\phi^2 = I-\eta\otimes\xi, \quad \eta(\xi)=1, \quad \phi(\xi)=0, \quad \eta\circ\phi=0, \tag{2.1}$$

and there exists a distribution $\mathbb{D}: p\in M \longrightarrow \mathbb{D}_p = Ker\eta$ such that an almost paracomplex structure is induced by the tensor field ϕ. The vector field ξ is said to be the Reeb (or characteristic) vector field of (ϕ,ξ,η).

For each $p\in M$, the tangent space T_pM can be stated as the direct sum

$$T_pM = \mathbb{D}_p \oplus Span_{\mathbb{R}}\{\xi(p)\}$$

and a vector $U\in T_pM$ can be uniquely decomposed as

$$u = hU + vU,$$

Email address and ORCID number: **mehmet.solgun@bilecik.edu.tr, 0000-0002-2275-7763**

where $hU = \phi^2 U \in \mathbb{D}_p$ and $vU = \eta(U)\xi(p) \in Span_{\mathbb{R}}\{\xi(p)\}$ [9]. Let g be a semi-Riemannian metric of signature $(n, n+1)$ on an almost paracontact manifold M with

$$g(\phi U, \phi V) = -g(U,V) + \eta(U)\eta(V). \tag{2.2}$$

Then the metric g is said to be a compatible metric and the quadruple (ϕ, ξ, η, g) is called an almost paracontact metric structure on M. The 2-form Φ given with

$$\Phi(U,V) := g(\phi U, V)$$

is called the fundamental 2-form of the structure.
The basis (namely, ϕ-basis) $\{e_1, \phi e_1, \cdots, e_n, \phi e_n, \xi\}$ with

$$g(e_i, e_j) = -g(\phi e_i, \phi e_j) = \delta_{ij}, \quad g(e_i, \phi e_j) = 0, \quad i, j = 1, \cdots, n,$$

is an orthonormal basis on (ϕ, ξ, η, g) see [2]. For the almost contact case, see [10]. It can be seen that the $(0,3)$- tensor F (the fundamental tensor) given with

$$F(U,V,W) = (\nabla_U \Phi)(V,W) = g((\nabla_U \phi)V, W),$$

satisfies the followings

$$\begin{aligned} F(U,V,W) &= -F(U,W,V), \\ F(U,\phi V,\phi W) &= F(U,V,W) + \eta(V)F(U,W,\xi) - \eta(W)F(U,V,\xi), \end{aligned} \tag{2.3}$$

for any $U,V,W \in TM$. In [9], Zamkovoy and Nakova classified almost paracontact metric manifolds by considering the space $\mathscr{F}$ of tensors F which satisfy (2.3). Initially, they decomposed this space into four subspaces $\mathscr{W}_i$ $(i = 1,2,3,4)$, i.e.

$$\mathscr{F} = \mathscr{W}_1 \oplus \mathscr{W}_2 \oplus \mathscr{W}_3 \oplus \mathscr{W}_4,$$

where $\mathscr{W}_i$'s are defined by

$$\mathscr{W}_1 = \{F \in \mathscr{F} | F(U,V,W) = F(hU, hV, hW)\},$$

$$\mathscr{W}_2 = \{F \in \mathscr{F} | F(U,V,W) = -\eta(V)F(hU, hW, \xi) + \eta(W)F(hU, hV, \xi)\},$$

$$\mathscr{W}_3 = \mathscr{G}_{11} = \{F \in \mathscr{F} | F(U,V,W) = \eta(U)F(\xi, hV, hW)\},$$

$$\mathscr{W}_4 = \mathscr{G}_{12} = \{F \in \mathscr{F} | F(U,V,W) = \eta(U)[(\eta(V)F(\xi,\xi,hW) - \eta(W)F(\xi,\xi,hV)]\}.$$

Then $\mathscr{W}_1$ and $\mathscr{W}_2$ are written as sums of $U(n) \times 1$ irreducible components $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \mathscr{G}_4$ and $\mathscr{G}_5, \cdots, \mathscr{G}_{10}$ respectively, where $U(n)$ is the paraunitary group, with the following relations [9]:

$\mathscr{G}_1$	$F(U,V,W) = \frac{1}{2n-1}[g(U,\phi V)\theta_F(\phi W) - g(U,\phi W)\theta_F(\phi V) - g(\phi U,\phi V)\theta_F(hW) + g(\phi U,\phi W)\theta_F(hU)]$
$\mathscr{G}_2$	$F(\phi U,\phi V,W) = -F(U,V,W), \quad \theta_F = 0$
$\mathscr{G}_3$	$F(\xi,V,W) = F(U,\xi,W) = 0, \quad F(U,V,W) = -F(V,U,W)$
$\mathscr{G}_4$	$F(\xi,V,W) = F(U,\xi,W) = 0, \quad \mathfrak{S}_{(U,V,W)}F(U,V,W) = 0$
$\mathscr{G}_5$	$F(U,V,W) = \frac{\theta_F(\xi)}{2n}[\eta(V)g(\phi U,\phi W) - \eta(W)g(\phi U,\phi V)]$
$\mathscr{G}_6$	$F(U,V,W) = -\frac{\theta_F^*(\xi)}{2n}[\eta(V)g(U,\phi W) - \eta(W)g(U,\phi V)]$
$\mathscr{G}_7$	$F(U,V,W) = -\eta(V)F(U,W,\xi) + \eta(W)F(U,V,\xi),; F(U,V,\xi) = -F(V,U,\xi) = -F(\phi U,\phi V,\xi), \quad \theta_F^*(\xi) = 0$
$\mathscr{G}_8$	$F(U,V,W) = -\eta(V)F(U,W,\xi) + \eta(W)F(U,V,\xi),; F(U,V,\xi) = F(V,U,\xi) = -F(\phi U,\phi V,\xi), \quad \theta_F(\xi) = 0$
$\mathscr{G}_9$	$F(U,V,W) = -\eta(V)F(U,W,\xi) + \eta(W)F(U,V,\xi),; F(U,V,\xi) = -F(V,U,\xi) = F(\phi U,\phi V,\xi)$
$\mathscr{G}_{10}$	$F(U,V,W) = -\eta(V)F(U,W,\xi) + \eta(W)F(U,V,\xi); F(U,V,\xi) = F(V,U,\xi) = F(\phi U,\phi V,\xi)$
$\mathscr{G}_{11}$	$F(U,V,W) = \eta(U)F(\xi,\phi V,\phi W)$
$\mathscr{G}_{12}$	$F(U,V,W) = \eta(U)[\eta(V)F(\xi,\xi,W) - \eta(W)F(\xi,\xi,V)]$

where $\theta_F(U) = g^{ij}F(e_i, e_j, U)$, $\theta_F^*(U) = g^{ij}F(e_i, \phi e_j, U)$, (called Lee forms of the structure).

3. The projection maps of the structure tensor F

In this section, we recall the projection maps of the tensor F. The vector space $\mathscr{F}$ is decomposed as the direct sums of the subspaces $\mathscr{W}_i$ $(i=1,2,3,4)$ and $\mathscr{G}_j$ $(j=1,...,12)$ mean that any $F\in\mathscr{F}$ can be uniquely represented in the form

$$F(U,V,W)=\sum_{i=1}^{4}F^{\mathscr{W}_i}(U,V,W),$$

and

$$F(U,V,W)=\sum_{j=1}^{12}F^{i}(U,V,W)$$

respectively, where $F^{\mathscr{W}_i}\in\mathscr{W}_i$ and $F^j\in\mathscr{G}_j$. Thereby, $M\in\mathscr{G}_i\oplus\mathscr{G}_j\oplus...$ if and only if the structure tensor F of M satisfies $F=F^i+F^j+....$ The projections $(F^i(i=1,...,12))$ are defined as follows [9]

$$\begin{aligned}
F^1(U,V,W) &= \frac{1}{2n-1}[g(U,\phi V)\theta_{F^1}(\phi W)-g(U,\phi W)\theta_{F^1}(\phi V)\\
&\quad -g(\phi U,\phi V)\theta_{F^1}(\phi^2 W)+g(\phi U,\phi W)\theta_F(\phi^2 V)],
\end{aligned}$$

$$\begin{aligned}
F^2(U,V,W) &= \frac{1}{2}[F(\phi^2U,\phi^2V,\phi^2W)-F(\phi U,\phi^2V,\phi W)]\\
&\quad -\frac{1}{2n-1}[g(U,\phi V)\theta_{F^1}(\phi W)-g(U,\phi W)\theta_{F^1}(\phi V)\\
&\quad -g(\phi U,\phi V)\theta_{F^1}(\phi^2 W)+g(\phi U,\phi W)\theta_F(\phi^2 V)],
\end{aligned}$$

$$\begin{aligned}
F^3(U,V,W) &= \frac{1}{6}[F(\phi^2U,\phi^2V,\phi^2W)+F(\phi U,\phi^2V,\phi W)\\
&\quad +F(\phi^2V,\phi^2W,\phi^2U)+F(\phi V,\phi^2W,\phi U)\\
&\quad F(\phi^2W,\phi^2U,\phi^2V)+F(\phi W,\phi^2U,\phi V)],
\end{aligned}$$

$$\begin{aligned}
F^4(U,V,W) &= \frac{1}{2}[F(\phi^2U,\phi^2V,\phi^2W)+F(\phi U,\phi^2V,\phi W)]\\
&\quad -\frac{1}{6}[F(\phi^2U,\phi^2V,\phi^2W)+F(\phi U,\phi^2V,\phi W)\\
&\quad +F(\phi^2V,\phi^2W,\phi^2U)+F(\phi V,\phi^2W,\phi U)\\
&\quad F(\phi^2W,\phi^2U,\phi^2V)+F(\phi W,\phi^2U,\phi V)],
\end{aligned}$$

$$F^5(U,V,W) = \frac{\theta_{F^5}(\xi)}{2n}[\eta(V)g(\phi U,\phi W)-\eta(W)g(\phi U,\phi V)],$$

$$F^6(U,V,W) = -\frac{\theta^*_{F^6}(\xi)}{2n}[\eta(V)g(U,\phi W)-\eta(W)g(U,\phi V)],$$

$$\begin{aligned}
F^7(U,V,W) &= -\frac{1}{4}\eta(Y)[F(\phi^2U,\phi^2W,\xi)-F(\phi U,\phi W,\xi)\\
&\quad -F(\phi^2W,\phi^2U,\xi)+F(\phi W,\phi U,\xi)]+\frac{1}{4}\eta(W)[F(\phi^2U,\phi^2V,\xi)\\
&\quad -F(\phi U,\phi V,\xi)-F(\phi^2V,\phi^2U,\xi)+F(\phi V,\phi U,\xi)]\\
&\quad +\frac{\theta^*_{F^6}(\xi)}{2n}[\eta(V)g(U,\phi W)-\eta(W)g(U,\phi V)],
\end{aligned}$$

$$\begin{aligned}
F^8(U,V,W) &= -\frac{1}{4}\eta(V)[F(\phi^2U,\phi^2W,\xi)-F(\phi U,\phi W,\xi)\\
&\quad +F(\phi^2W,\phi^2U,\xi)-F(\phi W,\phi U,\xi)]+\frac{1}{4}\eta(W)[F(\phi^2U,\phi^2V,\xi)\\
&\quad -F(\phi U,\phi V,\xi)+F(\phi^2V,\phi^2U,\xi)-F(\phi V,\phi U,\xi)]\\
&\quad -\frac{\theta_{F^5}(\xi)}{2n}[\eta(V)g(\phi U,\phi W)-\eta(W)g(\phi U,\phi V)],
\end{aligned}$$

$$\begin{aligned} F^9(U,V,W) &= -\frac{1}{4}\eta(V)[F(\phi^2U,\phi^2W,\xi)+F(\phi U,\phi W,\xi) \\ &\quad -F(\phi^2W,\phi^2U,\xi)-F(\phi W,\phi U,\xi)]+\frac{1}{4}\eta(W)[F(\phi^2U,\phi^2V,\xi) \\ &\quad +F(\phi U,\phi V,\xi)-F(\phi^2V,\phi^2U,\xi)-F(\phi V,\phi U,\xi)], \end{aligned}$$

$$\begin{aligned} F^{10}(U,V,W) &= -\frac{1}{4}\eta(V)[F(\phi^2U,\phi^2W,\xi)+F(\phi U,\phi W,\xi) \\ &\quad +F(\phi^2W,\phi^2U,\xi)+F(\phi W,\phi U,\xi)]+\frac{1}{4}\eta(W)[F(\phi^2U,\phi^2V,\xi) \\ &\quad +F(\phi U,\phi V,\xi)+F(\phi^2V,\phi^2U,\xi)+F(\phi V,\phi U,\xi)], \end{aligned}$$

$$F^{11}(U,V,W) = \eta(U)F(\xi,\phi^2V,\phi^2W),$$

$$F^{12}(U,V,W) = \eta(U)[\eta(V)F(\xi,\xi,\phi^2W)-\eta(W)F(\xi,\xi,\phi^2V)].$$

4. Almost paracontact metric structures with parallel Reeb vector field

This section is dedicated to investigating the almost paracontact metric structures equipped with parallel Reeb vector field ξ. In [9], it is stated that the vector field ξ is Killing only in the classes $\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3,\mathscr{G}_4,\mathscr{G}_5,\mathscr{G}_8,\mathscr{G}_9,\mathscr{G}_{11}$ and in their direct sums. As it is known, the vector field ξ is said to be parallel if $\nabla_U\xi=0$, and Killing if $g(\nabla_U\xi,V)+g(\nabla_V\xi,U)=0$, for any vector field U,V. So, as a natural result of these definitions, we can say that if a vector field is not Killing, then it is not parallel. Thus, the characteristic vector field ξ of the classes $\mathscr{G}_6,\mathscr{G}_7,\mathscr{G}_{10},\mathscr{G}_{12}$ and of their direct sums can not be parallel. So, let us consider the remaining classes.

For the classes $\mathscr{G}_i$ $(i=1,2,3,4,11)$, set $V=\xi$ and substitute W with ϕW. Then, we get

$$F^i(U,\xi,\phi W)=g((\nabla_U\phi)(\xi),\phi W)=0.$$

Since $\eta(\nabla_U\xi)=0$ for any U, and from the equation (2.2), we get $g(\nabla_U\xi,W)=0$, which means $\nabla\xi=0$, since g is non-degenerate.
For the class $\mathscr{G}_5$, set $V=\xi$ in the defining relation of $\mathscr{G}_5$. Then we get

$$g(\phi(\nabla_U\xi),W)=\frac{\theta_F(\xi)}{2n}g(\phi^2U,W).$$

From the equation (2.1), we get

$$\nabla_U\xi=\frac{\theta_F(\xi)}{2n}\phi U,$$

which is non-zero since the class $\mathscr{G}_5$ is non-trivial. Thus, the vector field ξ is not parallel in $\mathscr{G}_5$.
For the classes $\mathscr{G}_i$, $(i=8,9)$, assume that the vector field ξ is parallel. Under this assumption, one can easily see that $F^i=0$. However, since these classes are non-trivial, we come up with the result that ξ is not parallel in these classes.
In addition, it is known from [9] that, if an almost paracontact metric structure is of the classes $\mathscr{G}_i\oplus\mathscr{G}_j\oplus ...$, then the structure tensor F is of the form $F=F^i+F^j+....$ So, it is clear that a class, which is a direct sum of some classes having a parallel characteristic vector field, is also equipped with a parallel characteristic vector field.
After all, we can give the following theorem:

Theorem 4.1. *The characteristic vector field ξ is parallel only in the classes $\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3,\mathscr{G}_4,\mathscr{G}_{11}$ and in their direct sums.*

5. $\mathscr{D}$-homothetic deformation on an almost paracontact metric structure

The idea of a $\mathscr{D}$-homothetic deformation on a contact metric manifold (especially on Sasakian and K-contact structures) was introduced by Tanno ([11], [12]).
Let (ϕ,ξ,η,g) be an almost paracontact metric structure on a $(2n+1)-$dimensional manifold M and $\lambda\neq 0$ be a positive constant. Set,

$$\bar\phi=\phi,\quad \bar\xi=\frac{1}{\lambda}\xi,\quad \bar\eta=\lambda\eta,\quad \bar g=-\lambda g+\lambda(\lambda+1)\eta\otimes\eta.$$

Then, it can be seen that

$$Ker\bar\eta=Ker\eta,\quad \bar\phi^2=I-\bar\eta\otimes\bar\xi,\quad \bar\eta(\bar\xi)=1$$

and for any $U,V \in \mathfrak{X}(M)$,

$$\bar{g}(\bar{\phi}U,\bar{\phi}V) = -\bar{g}(U,V) + \bar{\eta}(U)\bar{\eta}(V).$$

Hence, $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also an almost paracontact metric structure on M [2].
This is called a $\mathscr{D}-$homothetic deformation of (ϕ,ξ,η,g). In this paper, we consider this deformation. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections of the metrics g and $\bar{g}$, respectively. Then by the Koszul formula and the definition of $\bar{g}$,

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_U V,W) &= -2\lambda g(\bar{\nabla}_U V,W) + 2\lambda(\lambda+1)\eta(\bar{\nabla}_U V)\eta(W) \\ &= -2\lambda g(\nabla_U V,W) + \lambda(\lambda+1)[2g(\nabla_U V,\xi)\eta(W) \\ &\quad +\eta(U)(g(\nabla_V\xi,W) - g(\nabla_W\xi,V)) \\ &\quad +\eta(V)(g(\nabla_U\xi,W) - g(\nabla_W\xi,U)) \\ &\quad +\eta(W)(g(\nabla_U\xi,V) + g(\nabla_V\xi,U))]. \end{aligned} \tag{5.1}$$

To obtain the relation between $\eta(\bar{\nabla}_U V)$ and $\eta(\nabla_U V)$, take $W=\xi$ in the equation (5.1). So we get,

$$\begin{aligned} \eta(\bar{\nabla}_U V) &= \eta(\nabla_U V) + \frac{\lambda+1}{2\lambda}[-\eta(U)g(\nabla_\xi\xi,V) \\ &\quad -\eta(V)g(\nabla_\xi\xi,U) + g(\nabla_U\xi,V) + g(\nabla_V\xi,U)]. \end{aligned} \tag{5.2}$$

If we apply the equation (5.2) into the equation (5.1), we get the following

$$\begin{aligned} g(\bar{\nabla}_U V,W) &= g(\nabla_U V,W) + \frac{(\lambda+1)^2}{2\lambda}\eta(W)[-\eta(U)g(\nabla_\xi\xi,V) \\ &\quad -\eta(V)g(\nabla_\xi\xi,U) + g(\nabla_U\xi,V) + g(\nabla_V\xi,U)] \\ &\quad -\frac{\lambda+1}{2}[\eta(U)(g(\nabla_V\xi,W) - g(\nabla_W\xi,V)) \\ &\quad +\eta(V)(g(\nabla_U\xi,W) - g(\nabla_W\xi,U))) \\ &\quad +\eta(W)(g(\nabla_U\xi,V) + g(\nabla_V\xi,U))]. \end{aligned} \tag{5.3}$$

By means of the equation (5.3), we may obtain the relations between $\bar{\nabla}$ and ∇ under some certain assumptions. The next section is devoted to studying the $\mathscr{D}-$ homothetic deformations of the structure with a parallel characteristic vector field.

6. $\mathscr{D}$-homothetic deformations of the structures with parallel Reeb vector field

In this section, we examine the $\mathscr{D}-$homothetic deformations of the almost paracontact structures with a parallel characteristic vector field.

Let (M,ϕ,ξ,η,g) be an almost paracontact metric manifold with parallel characteristic vector field ξ (i.e. $\nabla_U\xi = 0$, for any $U \in \mathfrak{X}(M)$ and $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ be the $\mathscr{D}$-homothetic deformed structure as defined above. Then we state the followings:

Proposition 6.1. *Let (M,ϕ,ξ,η,g) be an almost paracontact metric manifold with parallel characteristic vector field ξ. Then the followings hold*

i $\bar{\nabla}_U V = \nabla_U V$,
ii $\bar{F}(U,V,W) = -\lambda F(U,V,W)$,
iii $\bar{\theta}_{\bar{F}}(U) = \theta_F(U)$,

for any $U,V,W \in \mathfrak{X}(M)$, where $\bar{F}$ and $\bar{\theta}$ are the fundamental tensor and the Lee form of the deformed structure, respectively.

Proof. By assuming $\nabla\xi = 0$ in the equation (5.3), we directly get the equation (i).
For the equation (ii), we have the following

$$\begin{aligned} \bar{F}(U,V,W) &= \bar{g}((\bar{\nabla}_U\bar{\phi})(V),W) = -\lambda g((\nabla_U\phi)(V),W) + \lambda(\lambda+1)\eta((\nabla_U\phi)(V))\eta(W) \\ &= -\lambda F(U,V,W) + \lambda(\lambda+1)\eta((\nabla_U\phi)(V))\eta(W). \end{aligned}$$

On the other hand, since

$$0 = U[g(\phi V,\xi)] = g(\nabla_U\phi V,\xi) + g(\phi V,\nabla_U\xi) \Rightarrow g(\nabla_U\phi V,\xi) = 0,$$

we have

$$\eta((\nabla_U\phi)(V)) = g((\nabla_U\phi)(V),\xi) = g(\nabla_U\phi V,\xi) - g(\phi(\nabla_U V),\xi) = 0.$$

Thus, the equation (ii) is proved.

For the proof of (iii), consider the $\phi-$ basis $\{e_i,\phi e_i,\xi\}(i=1,...,n)$ for the structure (ϕ,ξ,η,g). Then $\{\bar{e}_i,\bar{\phi e}_i,\bar{\xi}\}$ is the $\phi-$ basis for the structure $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$, where

$$\bar{e}_i=\frac{1}{\sqrt{\lambda}}e_i,\bar{\phi e}_i=\frac{1}{\sqrt{\lambda}}\phi e_i,\bar{\xi}=\frac{1}{\lambda}\xi \qquad (\lambda>0)$$

and

$$\bar{g}(\bar{e}_i,\bar{e}_i)=-\bar{g}(\bar{\phi e}_i,\bar{\phi e}_i)=-\bar{g}(\bar{\xi},\bar{\xi})=-1.$$

Since ξ is parallel, by (i) $\bar{\nabla}=\nabla$. So, by direct calculation we get

$$\bar{F}(\bar{e}_i,\bar{e}_i,U)=-F(e_i,e_i,U),$$

$$\bar{F}(\bar{\phi e}_i,\bar{\phi e}_i,U)=-F(\phi e_i,\phi e_i,U),$$

$$\bar{F}(\bar{\xi},\bar{\xi},U)=F(\xi,\xi,U)=0.$$

So, by the definition of the from θ, we have

$$\begin{aligned}\bar{\theta}_{\bar{F}}(U) &= -\sum_{i=1}^{n}\bar{F}(\bar{e}_i,\bar{e}_i,U)+\sum_{i=1}^{n}\bar{F}(\bar{\phi e}_i,\bar{\phi e}_i,U)\\ &= -\sum_{i=1}^{n}(-F(e_i,e_i,U))+\sum_{i=1}^{n}(-F(\phi e_i,\phi e_i,U))\\ &= \theta_F(U).\end{aligned}$$

□

Theorem 6.2. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_1$. Then $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_1$.*

Proof. Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_1$. Then the fundamental tensor F satisfied the defining relation of the class $\mathscr{G}_1$, that is

$$F(U,V,W)=\frac{1}{2n-1}[g(U,\phi V)\theta_F(\phi W)-g(U,\phi W)\theta_F(\phi V)-g(\phi U,\phi V)\theta_F(hW)+g(\phi U,\phi W)\theta_F(hV)]. \tag{6.1}$$

On the other hand, by the proposition (4.1), the vector field ξ is parallel and so, the equations in the proposition (6.1) hold. By routine calculation, it can be seen that $\bar{F}$ also satisfies the equation (6.1). Thus, $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_1$. □

Theorem 6.3. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_2$. Then $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_2$.*

Proof. Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_2$. Then the fundamental tensor F satisfies the defining relation of the class $\mathscr{G}_2$, that is

$$F(\phi U,\phi V,W)=-F(U,V,W), \quad \theta_F=0. \tag{6.2}$$

Since ξ is parallel, $\bar{F}(U,V,W)=-\lambda F(U,V,W)$ and $\bar{\theta}_{\bar{F}}(U)=\theta_F(U)$. Thus, $\bar{F}$ also satisfies the equation (6.2). □

Theorem 6.4. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_3$. Then $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_3$.*

Proof. Let F satisfies the defining relation of $\mathscr{G}_3$, that is,

$$F(\xi,V,W)=F(U,\xi,W)=0, \quad F(U,V,W)=-F(V,U,W). \tag{6.3}$$

Since ξ is parallel in the class $\mathscr{G}_3$, $\bar{F}(U,V,W)=-\lambda F(U,V,W)$ and so $\bar{F}$ also satisfies (6.3). □

Theorem 6.5. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_4$. Then $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_4$.*

Proof. It can be seen by direct calculation since ξ is parallel in $\mathscr{G}_4$. □

Theorem 6.6. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_{11}$. Then so is the structure $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$.*

Proof. It can see seen from the definition class and the proposition (6.1). □

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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