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A NOVEL ANALYSIS OF INTEGRAL INEQUALITIES IN THE FRAME OF FRACTIONAL CALCULUS

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Abstract. In this paper, we define and explore the new family of exponentially convex functions which are called exponentially s–convex functions. We attain the amazing examples and algebraic properties of this newly introduced function. In addition, we find a novel version of Hermite-Hadamard type inequality in the support of this newly introduced concept via the frame of classical and fractional calculus (non-conformable and Riemann-Liouville integrals operator). Furthermore, we investigate refinement of Hermite-Hadamard type inequality by using exponentially s–convex functions via fractional integral operator. Finally, we elaborate some Ostrowski type inequalities in the frame of fractional calculus. These new results yield us some generalizations of the prior results.

1. Introduction

Convex functions are significant in the hypothesis of numerical inequalities, some notable outcomes are immediate ramifications of these functions. The ideas of different sorts of new convex functions are developed from the basic definition of a convex function. The generalizations, extensions and refinements of these functions are proved to be very beneficial in mathematical analysis, financial mathematics, mathematical statistics, optimization theory, etc. For the attention of readers, see the reference [1–4].

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In literature, the celebrated Hermite-Hadamard double Inequality [5] for convex function on an interval of the real line, discovered by C. Hermite and J. Hadamard individually, has been the hot topic for extensive research, which is stated as,

Let a function $\varphi : \mathbb{A} \longrightarrow \mathbb{R}$ is a convex function on $\mathbb{A}$ in $\mathbb{R}$ and $\delta_1, \delta_2 \in \mathbb{A}$ with $\delta_1 < \delta_2$, then

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\varphi(x)dx \leq \frac{\varphi(\delta_1)+\varphi(\delta_2)}{2}. \quad (1)$$

Let φ is a concave function, then this inequality 1 is reversed. The inequality 1 provides upper and lower bounds for the integral means of the convex function φ. The inequality 1 has different kinds of forms with correspondence to different kinds of convexities such as s–convex function, h–convex function, p–convex function, log–convex function, exponential convex function, exponential type convex function, MT–convex function, tgs–convex function, n–polynomial convex function, preinvex functions etc.

Recently, few researchers have been studying on the properties and applications of exponential type convexity, for more information, we refer interested readers to go through [6–9].

In the literature of inequalities the Hadamard inequality and Ostrowski type inequality appear in different forms for various convex functions. In [10] and [11] for the first time, Hermite-Hadamard inequality and Ostrowski inequality was studied for Riemann-Liouville fractional integrals respectively and after it, researchers started to get many versions of these for different kinds of fractional integral operators and functions.

In the literature, Ostroswki Inequality [12] is defined as follows:

Let $\varphi : I \subset [0,\infty) \longrightarrow R$ be a differentiable mapping on I°, the interior of the interval I, such that $\varphi' \in \mathcal{L}[\delta_1, \delta_2]$, where $\delta_1, \delta_2 \in I$ with $\delta_1 < \delta_2$. If $|\varphi'(z)| \leq \mathbb{M}$, for all $x \in [\delta_1, \delta_2]$, then the following inequality holds:

$$\left|\varphi(z) - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\varphi(x)\,dx\right| \leq \frac{\mathbb{M}}{(\delta_2-\delta_1)}\left[\frac{(z-\delta_1)^2+(\delta_2-z)^2}{2}\right], \quad (2)$$

which gives an upper bound for approximations of the integral average by $\varphi(z)$ at point $z \in [\delta_1, \delta_2]$

For some recent generalizations about this inequality please see [13], [14] and the references therein. In [15], the authors have established some Ostrowski type inequalities for s-convex function in the second sense. In the recent past many generalizations for Ostrowski type inequality have been performed via different directions like on coordinates, on quantum calculus, on different fractional integral operators like Riemann -Liouville, Katugampola, Caputo, Caputo Fabrizio, ψ- generalized fractional operator, etc.

2. Preliminaries

In this section, we recall some known concepts.

Definition 1. *[16] Let $\varphi : \mathbb{A} \to \mathbb{R}$ be a real valued function. A function φ is said to be convex, if*

$$\varphi\left(\lambda\delta_1 + (1-\lambda)\,\delta_2\right) \leq \lambda\varphi\left(\delta_1\right) + (1-\lambda)\,\varphi\left(\delta_2\right), \tag{3}$$

holds for all $\delta_1, \delta_2 \in \mathbb{A}$ and $\lambda \in [0,1]$.

Definition 2. *[17] Let $s \in (0,1]$, then $\varphi : [0,+\infty) \to \mathbb{R}$ is known as s–convex in the 2^{nd} sense, if*

$$\varphi\left(\lambda\delta_1 + (1-\lambda)\,\delta_2\right) \leq \lambda^s\varphi\left(\delta_1\right) + (1-\lambda)^s\,\varphi\left(\delta_2\right), \tag{4}$$

holds $\forall\ \delta_1, \delta_2 \in [0,+\infty)$ and $\lambda \in [0,1]$.

Dragomir et al. investigated and explored a novel version of Hadamard's inequality in the mode of s–convex functions in the 2^{nd} sense in the published article [18]. In the last few decades or so, fractional calculus can be seen gaining a lot of attention as the most researched subject of mathematics. Its importance is prominent from the fact that many real-life problems are well interpreted and modeled using the theory of fractional calculus. It is also seen that various branches of engineering and applied science have been using the tools and techniques of fractional calculus. It is mainly due to the two mathematicians, L'Hospital and Leibnitz that fractional calculus is so popular nowadays. After this many mathematicians developed different new types of fractional operators and worked upon them to generalize inequalities like Hermite-Hadamard, Ostrowski, Opial, Jensen, Hermite-Hadamard-mercer, Oslen type, etc. The authors examined and celebrated conformable and non-conformable derivative in the published articles [19] and [20]. Both fractional integral operators have a lot of meaningful and useful applications, see the references [21–29].

Definition 3. *Let $\varphi : \mathbb{A} \subseteq [0,\infty) \to \mathbb{R}$ be a real valued function, then the non–conformable derivative of φ is defined by*

$${}_{N_3^{\alpha}}\varphi(x) = \lim_{\epsilon\to 0} \frac{\varphi(\lambda + \epsilon\lambda^{\alpha}) - \varphi(\lambda)}{\epsilon},$$

where $\alpha \in (0,1)$ and $\lambda \in \mathbb{A}$.

If $\exists\ {}_{N_3^{\alpha}}\varphi(\lambda)$ and is finite, then φ is a α–differentiable at λ.
If φ at λ is a differentiable, then

$${}_{N_3^{\alpha}}\varphi(\lambda) = \lambda^{\alpha}\varphi^{'}(\lambda).$$

Definition 4. *[30] For each $\varphi \in L[\delta_1, \delta_2]$ and $0 < \delta_1 < \delta_2$, we define*

$${}_{N_3}J_u^{\alpha}\varphi(x) = \int_u^x \lambda^{-\alpha}\varphi(\lambda)d\lambda,$$

for every $x, u \in [\delta_1, \delta_2]$ and $\alpha \in \mathbb{R}$.

Definition 5. *[30] For each function $\varphi \in L[\delta_1, \delta_2]$ and $\delta_1 < \delta_2$, we define the fractional integrals*

$$ {}_{N_3}J^{\alpha}_{\delta_1^+}\varphi(x) = \int_{\delta_1}^{x} (x-\lambda)^{-\alpha} \varphi(\lambda) d\lambda, $$

$$ {}_{N_3}J^{\alpha}_{\gamma_2^-}\varphi(x) = \int_{x}^{\delta_2} (\lambda - x)^{-\alpha} \varphi(\lambda) d\lambda, $$

for every $x \in [\gamma_1, \gamma_2]$ and $\alpha \in \mathbb{R}$.

Remark 1. *In the above definitions, if we put $\alpha = 0$ then we get the classical integrals which is represented by ${}_{N_3}J^{\alpha}_{\delta_1^+}\varphi(x) = {}_{N_3}J^{\alpha}_{\delta_2^-}\varphi(x) = \int_{\delta_1}^{\delta_2} \varphi(\lambda) d\lambda$.*

It is remarkable that M.Z. Sarikaya et al. (see in [10]) proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *[10] Suppose $\varphi : \mathbb{A} = [\delta_1, \delta_2] \to \mathbb{R}$ is a positive mapping with $\delta_2 > \delta_1$ and $\varphi \in L[\delta_1, \delta_2]$. If φ is a convex function on $[\delta_1, \delta_2]$, then the following inequalities for fractional integrals holds:*

$$ \varphi\left(\frac{\delta_1 + \delta_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\delta_2 - \delta_1)^{\alpha}} \left\{ J^{\alpha}_{\delta_1^+} \varphi(\delta_2) + J^{\alpha}_{\delta_2^-} \varphi(\delta_1) \right\} \leq \frac{\varphi(\delta_1) + \varphi(\delta_2)}{2}, $$

with $\alpha > 0$.

Definition 6. *[10] Let $\varphi \in L[\delta_1, \delta_2]$. Then Riemann–Liouville fractional integrals of order $\alpha > 0$ with $\delta_1 \geq 0$ are defined as follows:*

$$ J^{\alpha}_{\delta_1^+}\varphi(z) = \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{z} (z-\lambda)^{\alpha-1} \varphi(\lambda) \, d\lambda, \quad z > \delta_1 $$

and

$$ J^{\alpha}_{\delta_2^-}\varphi(z) = \frac{1}{\Gamma(\alpha)} \int_{z}^{\delta_2} (\lambda - z)^{\alpha-1} \varphi(\lambda) \, d\lambda, \quad z < \delta_2. \tag{5} $$

where $\Gamma(r)$ is the Gamma function defined by

$$ \Gamma(r) = \int_0^{\infty} e^{-y} y^{r-1} dy. $$

Since $a(a > 0)$ will stand for the parameter of the incomplete gamma function (see [31]:8.2.1)

$$ \gamma(a, r) = \int_0^{r} e^{-y} y^{a-1} dy. $$

For further details one may, see [32, 33, 35].

We compose the paper in the following manner, In section 3, we will give the idea of exponentially s–convex functions, examples, and its properties. In section 4, we will give the generalizations of (H-H)–type inequality in the support of the newly introduced idea. In section 5, we will investigate the new version of Hermite–Hadamard type inequality and its refinements for exponentially s–convex function via a fractional integral operator. In section 6, we will also obtain some Ostrowski type inequalities for the exponentially s–convex function φ for fractional integral inequalities. In section 7, a brief conclusion will be given as well.

3. Exponentially s–Convex Function and Its Properties

The main aim of this section is to define the new family of convex functions, which are called exponentially s–convex functions. In the manner of this newly introduced concept, we obtain some examples and algebraic properties.

Definition 7. *Let $s \in [\ln 2.4, 1]$. Then $\varphi : \mathbb{A} \subset \mathbb{R} \to \mathbb{R}$ is known to be exponentially s–convex function, if*

$$\varphi\left(\lambda\delta_1 + (1-\lambda)\delta_2\right) \leq \left(e^{s\lambda} - 1\right)\varphi\left(\delta_1\right) + \left(e^{s(1-\lambda)} - 1\right)\varphi\left(\delta_2\right), \tag{6}$$

holds $\forall \delta_1, \delta_2 \in \mathbb{R}$ and $\lambda \in [0,1]$.

Remark 2. *In above Definition 7, if $s = 1$, then we get exponential type convexity given by İşcan in [6].*

Remark 3. *The range of the exponentially s–convex functions for some fixed $s \in [\ln 2.4, 1]$ is $[0, +\infty)$.*

Lemma 1. *The following inequalities $(e^{s\lambda} - 1) \geq \lambda^s$ and $(e^{s(1-\lambda)} - 1) \geq (1-\lambda)^s$ are holds, if for all $\lambda \in [0,1]$ and for some fixed $s \in [\ln 2.4, 1]$*

Proof. The proof is evident. □

Proposition 1. *Every nonnegative s–convex function is exponentially s–convex function for $s \in [\ln 2.4, 1]$.*

Proof. By using Lemma 1, for $s \in [\ln 2.4, 1]$, we have

$$\begin{aligned}\varphi\left(\lambda\delta_1 + (1-\lambda)\delta_2\right) &\leq \lambda^s\varphi\left(\delta_1\right) + (1-\lambda)^s\varphi\left(\delta_2\right) \\ &\leq \left(e^{s\lambda} - 1\right)\varphi\left(\delta_1\right) + \left(e^{(1-\lambda)s} - 1\right)\varphi\left(\delta_2\right).\end{aligned}$$

□

Proposition 2. *Every exponentially s–convex function for $s \in [\ln 2.4, 1]$ is an h–convex function with $h(\lambda) = \left(e^{\lambda} - 1\right)$*

Proof.

$$\begin{aligned}\varphi\left(\lambda\delta_1 + (1-\lambda)\delta_2\right) &\leq \left(e^{\lambda} - 1\right)\varphi\left(\delta_1\right) + \left(e^{1-\lambda} - 1\right)\varphi\left(\delta_2\right) \\ &\leq h(\lambda)\varphi(\delta_1) + h(1-\lambda)\varphi(\wp_2).\end{aligned}$$

□

Example 1. *Dragomir have investigated that in the published article [18], the non-negative function* $\varphi(x) = x^{ls}$, $x > 0$ *is* s*–convex function for the all mention conditions* $s \in (0,1)$, *where* $1 \leq l \leq \frac{1}{s}$, . *Then according to Proposition 2, it is also exponentially* s*–convex function for some fixed* $s \in [\ln 2.4, 1)$.

Example 2. $\varphi(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$ *for* $m > 1$, $q \geq 1$ *is a non-negative* s*-convex function. Then according to Proposition 2, it is also exponentially* s*–convex function for some fixed* $s \in [\ln 2.4, 1)$.

Theorem 2. *Let* $\varphi : [0,\delta] \to \mathbb{A}$ *be* s*–convex function for* $s \in [\ln 2.4, 1]$ *and* $\phi : \mathbb{A} \to \mathbb{R}$ *is non-decreasing and exponentially convex function. Then the function* $\phi \circ \varphi : [0,\delta] \to \mathbb{R}$ *is exponentially* s*–convex function.*

Proof. For all $\delta_1, \delta_2 \in [0,\delta]$ and $\lambda \in [0,1]$, and for $s \in [\ln 2.4, 1]$, we have

$$\begin{aligned}(\phi \circ \varphi)(\lambda\delta_1 + (1-\lambda)\delta_2) &= \phi(\varphi(\lambda\delta_1 + (1-\lambda)\delta_2)) \leq \phi(\lambda^s \varphi(\delta_1) + (1-\lambda)^s \varphi(\delta_2)) \\ &\leq \left(e^{s\lambda} - 1\right)(\phi \circ \varphi)(\delta_1) + \left(e^{(1-\lambda)s} - 1\right)(\phi \circ \varphi)(\delta_2).\end{aligned}$$

□

Remark 4. *If we choose* $s = 1$ *in above Theorem (2), then we get Theorem (2.2) in [6].*

4. New Generalizations of (H–H) Type Inequality Using Exponentially s–Convex Functions

The aim of this section is to find the new generalization of Hermite–Hadamard type inequality for the exponentially s–convex function for φ in the frame of simple calculus and also we attain the novel version of Hermite–Hadamard type inequality in the manner of newly introduced idea in the frame of fractional calculus by the non-conformable integral operator.

Theorem 3. *Suppose* $s \in [\ln 2.4, 1]$, $\alpha \in (0,1]$, $\delta_2 > \delta_1$ *and* $\varphi : \mathbb{A} = [\delta_1, \delta_2] \to \mathbb{R}$ *is exponentially* s*–convex function such that* $\varphi \in L[\delta_1, \delta_2]$. *Then one has*

$$\begin{aligned}&\frac{1}{2\left(e^{\frac{s}{2}} - 1\right)} \varphi\left(\frac{\delta_1 + \delta_2}{2}\right) \leq \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \varphi(u)\; du \\ &\leq \left(\frac{e^s - s - 1}{s}\right) \Big[\varphi(\delta_1) + \varphi(\delta_2)\Big]\;.\end{aligned} \tag{7}$$

Proof. Let $z_1, z_2 \in \mathbb{A}$. Then it follows from the exponentially s–convex function for φ on $\mathbb{A}$ that

$$\varphi\left(\frac{z_1 + z_2}{2}\right) \leq \left(e^{\frac{s}{2}} - 1\right)\left[\varphi(z_1) + \varphi(z_2)\right] \tag{8}$$

Suppose

$$z_1 = \lambda\delta_2 + (1-\lambda)\delta_1 \quad \text{and} \quad z_2 = \lambda\delta_1 + (1-\lambda)\delta_2.$$

Then (8) leads to

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \left(e^{\frac{s}{2}}-1\right)\left[\varphi\left(\lambda\delta_2+(1-\lambda)\,\delta_1\right)+\varphi\left(\lambda\delta_1+(1-\lambda)\,\delta_2\right)\right]. \tag{9}$$

Now integrating on both sides in the last inequality with respect to λ from 0 to 1, we obtain

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \left(e^{\frac{s}{2}}-1\right)\left[\int_0^1 \varphi\left(\lambda\delta_2+(1-\lambda)\,\delta_1\right)d\lambda+\int_0^1 \varphi\left(\lambda\delta_1+(1-\lambda)\,\delta_2\right)\,d\lambda\right].$$

$$\frac{1}{2\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\varphi(u)\,\,du,$$

which gives the proof of first part of inequality of (7). Next, we show the second part of inequality of (7). Let $\lambda\in[0,1]$. Then from the fact that φ is exponentially s–convex function, we obtain

$$\varphi\left(\lambda\delta_2+(1-\lambda)\,\delta_1\right) \leq \left(e^{s\lambda}-1\right)\varphi(\delta_2)+\left(e^{s(1-\lambda)}-1\right)\varphi(\delta_1) \tag{10}$$

and

$$\varphi\left(\lambda\delta_1+(1-\lambda)\,\delta_2\right) \leq \left(e^{s\lambda}-1\right)\varphi(\delta_1)+\left(e^{s(1-\lambda)}-1\right)\varphi(\delta_2). \tag{11}$$

By adding the above inequalities, we obtain

$$\begin{aligned} &\varphi\left(\lambda\delta_2+(1-\lambda)\,\delta_1\right)+\varphi\left(\lambda\delta_1+(1-\lambda)\,\delta_2\right) \\ &\leq\ [\varphi(\delta_1)+\varphi(\delta_2)]\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}. \end{aligned} \tag{12}$$

Now integrating on both sides by above equation with respect to λ from 0 to 1, then making the change of variable, we obtain

$$\begin{aligned} &2\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\varphi(u)\,\,du. \\ &\leq\ [\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda, \end{aligned}$$

which leads to the conclusion that

$$\leq 2\left(\frac{e^s-s-1}{s}\right)[\varphi(\delta_1)+\varphi(\delta_2)].$$

The proof is completed. □

Remark 5. *If we choose $s=1$, then Theorem 3 becomes to* $\big[$*Theorem 3.1, [6]*$\big]$.

Theorem 4. *Let $\varphi:\mathbb{A}=[\delta_1,\delta_2]\to\mathbb{R}$ be a positive function with $0\leq\delta_1\leq\delta_2$ and φ be a integrable function on closed interval set δ_1 and δ_2. If φ is exponentially*

s–convex function, then the following inequalities for fractional integral namely non-conformable integral operator with $\alpha < 0$ and $s \in [\ln 2.4, 1]$ holds:

$$\frac{1}{\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{1-\alpha}{(\delta_2-\delta_1)^{1-\alpha}}\left[\,_{N_3}J^{\alpha}_{a^+}\varphi(x) +_{N_3} J^{\alpha}_{b^-}\varphi(x)\right] \tag{13}$$
$$\leq (1-\alpha)[\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1 \lambda^{-\alpha}\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda.$$

Proof. Let $\sigma_1, \sigma_2 \in \mathbb{A}$. Then it follows from the exponentially s–convex function for φ on $\mathbb{A}$ that

$$\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right) \leq \left(e^{\frac{s}{2}}-1\right)[\varphi(\sigma_1)+\varphi(\sigma_2)]. \tag{14}$$

Suppose

$$\sigma_1 = \lambda\delta_2 + (1-\lambda)\delta_1 \quad \text{and} \quad \sigma_2 = \lambda\delta_1 + (1-\lambda)\delta_2.$$

Then (14) leads to

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \left(e^{\frac{s}{2}}-1\right)[\varphi(\lambda\delta_2+(1-\lambda)\delta_1) + \varphi(\lambda\delta_1+(1-\lambda)\delta_2)]. \tag{15}$$

Now integrating on both sides in the last inequality with respect to λ from 0 to 1 and multiply both sides by $\lambda^{-\alpha}$, we obtain

$$\begin{aligned}\frac{1}{1-\alpha}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \left(e^{\frac{s}{2}}-1\right)\left[\int_0^1 \lambda^{-\alpha}\varphi(\lambda\delta_2+(1-\lambda)\delta_1)\,d\lambda\right.\\ &\qquad \left.+\int_0^1 \lambda^{-\alpha}\varphi(\lambda\delta_1+(1-\lambda)\delta_2)\,d\lambda\right],\\ \frac{1}{\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \frac{1-\alpha}{(\delta_2-\delta_1)^{1-\alpha}}\left[N_3J^{\alpha}_{a^+}\varphi(x) + N_3J^{\alpha}_{b^-}\varphi(x)\right],\end{aligned}$$

which gives the proof of first part of inequality of (13). Next, we show the second part of inequality of (13). Let $\lambda \in [0,1]$. Then from the fact that φ is exponentially s–convex function, we obtain

$$\varphi(\lambda\delta_2+(1-\lambda)\delta_1) \leq \left(e^{s\lambda}-1\right)\varphi(\delta_2) + \left(e^{s(1-\lambda)}-1\right)\varphi(\delta_1) \tag{16}$$

and

$$\varphi(\lambda\delta_1+(1-\lambda)\delta_2) \leq \left(e^{s\lambda}-1\right)\varphi(\delta_1) + \left(e^{s(1-\lambda)}-1\right)\varphi(\delta_2). \tag{17}$$

By adding the above inequalities, we obtain

$$\varphi(\lambda\delta_2+(1-\lambda)\delta_1) + \varphi(\lambda\delta_1+(1-\lambda)\delta_2) \tag{18}$$
$$\leq [\varphi(\delta_1)+\varphi(\delta_2)]\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}.$$

Now integrating on both sides by above equation with respect to λ from 0 to 1 and multiply $\lambda^{-\alpha}$ both sides, then making the change of variable, we obtain

$$\frac{1}{(\delta_2-\delta_1)^{1-\alpha}}\left[\,{}_{N_3}J^{\alpha}_{a^+}\varphi(x)+{}_{N_3}\,J^{\alpha}_{b^-}\varphi(x)\right]$$
$$\leq\ [\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1\lambda^{-\alpha}\Big\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\Big\}d\lambda,$$

which leads to the conclusion that

$$\frac{1-\alpha}{(\delta_2-\delta_1)^{1-\alpha}}\left[\,{}_{N_3}J^{\alpha}_{a^+}\varphi(x)+{}_{N_3}\,J^{\alpha}_{b^-}\varphi(x)\right]$$
$$\leq(1-\alpha)[\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1\lambda^{-\alpha}\Big\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\Big\}d\lambda.$$

The proof is completed. □

Remark 6. (i) *If we choose $s=1$ and $\alpha=0$ then Theorem 3 becomes to $\big[$Theorem 3.1, [6]$\big]$.*
(ii) *If we choose $\alpha=0$ then Theorem 4, then we attain the Theorem 3.*

5. Hermite–Hadamard Type Inequality and Its Refinements For Exponentially s–Convex Function Via Fractional Integral Operator

The main key of this section is to obtain the new sort of Hermite–Hadamard inequality in the manner of new introduced concept in the frame of fractional calculus namely Riemann-Liouville integral operator. Also we attain the refinement of this inequality.

Theorem 5. *Let $\varphi:\mathbb{A}=[\delta_1,\delta_2]\to\mathbb{R}$ be a positive function with $0\leq\delta_1\leq\delta_2$ and φ be a integrable function on closed interval set δ_1 and δ_2. If φ is exponentially s–convex function, then the following inequalities for fractional integral namely Riemann-Liouville with $\alpha>0$ and $s\in[\ln 2.4,1]$ holds:*

$$\frac{1}{\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right)\leq\frac{\Gamma(\alpha+1)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right]\tag{19}$$
$$\leq\alpha[\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1\lambda^{\alpha-1}\Big\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\Big\}d\lambda.$$

Proof. Let $\sigma_1,\sigma_2\in\mathbb{A}$. Then it follows from the exponentially s–convex function for φ on $\mathbb{A}$ that

$$\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right)\leq\ \left(e^{\frac{s}{2}}-1\right)[\varphi(\sigma_1)+\varphi(\sigma_2)]\tag{20}$$

Suppose $\sigma_1=\lambda\delta_2+(1-\lambda)\,\delta_1$ *and* $\sigma_2=\lambda\delta_1+(1-\lambda)\,\delta_2$.

Then (20) leads to

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \left(e^{\frac{s}{2}}-1\right)\left[\varphi\left(\lambda\delta_2+(1-\lambda)\delta_1\right)+\varphi\left(\lambda\delta_1+(1-\lambda)\delta_2\right)\right]. \tag{21}$$

Now integrating on both sides in the last inequality with respect to λ from 0 to 1 and multiply both sides by $\lambda^{\alpha-1}$, we obtain

$$\begin{aligned}\frac{1}{\alpha}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \left(e^{\frac{s}{2}}-1\right)\left[\int_0^1 \lambda^{\alpha-1}\varphi\left(\lambda\delta_2+(1-\lambda)\delta_1\right)d\lambda\right.\\ &\quad\left.+\int_0^1 \lambda^{\alpha-1}\varphi\left(\lambda\delta_1+(1-\lambda)\delta_2\right)\, d\lambda\right],\\ \frac{1}{\alpha\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \frac{\Gamma(\alpha)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right],\\ \frac{1}{\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right],\end{aligned}$$

which gives the proof of first part of inequality of (19).
Next, we show the second part of inequality of (19). Let $\lambda\in[0,1]$. Then from the fact that φ is exponentially s–convex function, we obtain

$$\varphi\left(\lambda\delta_2+(1-\lambda)\delta_1\right) \leq \left(e^{s\lambda}-1\right)\varphi(\delta_2)+\left(e^{s(1-\lambda)}-1\right)\varphi(\delta_1) \tag{22}$$

and

$$\varphi\left(\lambda\delta_1+(1-\lambda)\delta_2\right) \leq \left(e^{s\lambda}-1\right)\varphi(\delta_1)+\left(e^{s(1-\lambda)}-1\right)\varphi(\delta_2). \tag{23}$$

By adding the above inequalities, we obtain

$$\begin{aligned}&\varphi\left(\lambda\delta_2+(1-\lambda)\delta_1\right)+\varphi\left(\lambda\delta_1+(1-\lambda)\delta_2\right)\\ &\leq [\varphi(\delta_1)+\varphi(\delta_2)]\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}.\end{aligned} \tag{24}$$

Now integrating on both sides by above equation with respect to λ from 0 to 1 and multiply $\lambda^{\alpha-1}$ both sides, then making the change of variable, we obtain

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right]\\ &\leq [\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1 \lambda^{\alpha-1}\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda,\end{aligned}$$

which leads to the conclusion that

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right]\\ &\leq \alpha[\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1 \lambda^{\alpha-1}\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda.\end{aligned}$$

The proof is completed. □

Remark 7. (i) *If we choose* $s = 1$ *and* $\alpha = 1$ *then Theorem 3 becomes to* $\Big[$*Theorem 3.1, [6]* $\Big]$.
(ii) *If we choose* $\alpha = 1$ *then Theorem 5, then we attain the Theorem 3.*

Next we find the refinement of Hermite–Hadamard type inequality using exponentially s–convex function via fractional integral operator. In order to obtain the following result, we need the following lemma.

Lemma 2. *[34] Let* $\varphi : \mathbb{A} \longrightarrow \mathbb{R}$ *be a differentiable mapping on* $\mathbb{A}^\circ$, *where* $\delta_1, \delta_2 \in \mathbb{A}^\circ$ *with* $0 \leq \delta_1 \leq \delta_2$. *If* $\varphi' \in \mathcal{L}[\delta_1, \delta_2]$, *then the following equality for fractional integral holds*

$$\frac{\varphi(\delta_1) + \varphi(\delta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta_2 - \delta_1)^\alpha}\left[I^\alpha_{\delta_1^+}\varphi(\delta_2) + I^\alpha_{\delta_2^-}\varphi(\delta_1)\right]$$
$$= \frac{(\delta_2 - \delta_1)}{2}\left\{\int_0^1 [(1-\lambda)^\alpha - \lambda^\alpha]\,\varphi'(\lambda\delta_1 + (1-\lambda)\delta_2) d\lambda\right\}.$$

Theorem 6. *Let* $\varphi : \mathbb{A} = [\delta_1, \delta_2] \subset [0, \infty) \longrightarrow \mathbb{R}$ *be a differentiable mapping on* (δ_1, δ_2) *with* $\delta_1 < \delta_2$ *such that* $\varphi' \in L[\delta_1, \delta_2]$. *If* $|\varphi'|^q$ *is an exponentially s-convex function on* $[\delta_1, \delta_2]$*for some fixed* $s \in [ln2.4, 1]$ *and* $q \geq 1$. *Then the following fractional inequality holds true.*

$$\left|\tfrac{\varphi(\delta_1)+\varphi(\delta_2)}{2} - \tfrac{\Gamma(\alpha+1)}{2(\delta_2-\delta1)^\alpha}\left[I^\alpha_{\delta_1^+}\varphi(\delta_2) + I^\alpha_{\delta_2^-}\varphi(\delta1)\right]\right|$$
$$\leq \tfrac{(\delta_2-\delta_1)}{2}\left[\tfrac{2}{\alpha+1}\left(1 - \tfrac{1}{2^\alpha}\right)\right]^{\frac{q-1}{q}}$$
$$\times\Bigg\{|\varphi'(\delta_1)|^q \tfrac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)} - |\varphi'(\delta_1)|^q \tfrac{2^{(-2-\alpha)}[-2(2+\alpha)+(1+\alpha)e^s]}{(1+\alpha)(2+\alpha)}$$
$$+|\varphi'(\delta_2)|^q \tfrac{2^{(-2-\alpha)}\left[4+2\alpha-e^s-\alpha e^s+2^{(2+\alpha)}(-2-\alpha+(1+\alpha)e^s)\right]}{(1+\alpha)(2+\alpha)} - |\varphi'(\delta_2)|^q \tfrac{2^{(-2-\alpha)}[-2(2+\alpha)+(3+\alpha)e^s]}{(1+\alpha)(2+\alpha)}$$
$$+|\varphi'(\delta_1)|^q \tfrac{1}{4}\left(\tfrac{2(-2+2^{-\alpha})}{1+\alpha} + \tfrac{(4-2^{-\alpha})e^\alpha}{2+\alpha}\right) - |\varphi'(\delta_1)|^q \tfrac{2^{(-2-\alpha)}[-2(2+\alpha)+(3+\alpha)e^s]}{(1+\alpha)(2+\alpha)}$$
$$+|\varphi'(\delta_2)|^q \tfrac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)} - |\varphi'(\delta_2)|^q \tfrac{2^{(-2-\alpha)}[-2(2+\alpha)+(1+\alpha)e^s]}{(1+\alpha)(2+\alpha)}\Bigg\}^{\frac{1}{q}}.$$

Proof. Suppose that $q = 1$. From lemma (2) and using properties of modulus, we have

$$\left|\frac{\varphi(\delta_1) + \varphi(\delta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta_2 - \delta_1)^\alpha}\left[I^\alpha_{\delta_1^+}\varphi(\delta_2) + I^\alpha_{\delta_2^-}\varphi(\delta_1)\right]\right|$$
$$= \frac{(\delta_2 - \delta_1)}{2}\left\{\int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha|\,|\varphi'(\lambda\delta_1 + (1-\lambda)\delta_2)| d\lambda\right\}$$
$$\leq \frac{(\delta_2 - \delta_1)}{2}\left\{\int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha|\left[(e^{s\lambda} - 1)|\varphi'(\delta_1)| + (e^{s(1-\lambda)} - 1)|\varphi'(\delta_2)|\right] d\lambda\right\}$$
$$\leq \frac{(\delta_2 - \delta_1)}{2}\left\{\int_0^{1/2} [(1-\lambda)^\alpha - \lambda^\alpha]\left[(e^{s\lambda} - 1)|\varphi'(\delta_1)| + (e^{s(1-\lambda)} - 1)|\varphi'(\delta_2)|\right] d\lambda\right.$$

$$+\int_{1/2}^{1}\left[\lambda^{\alpha}-(1-\lambda)^{\alpha}\right]\left[(e^{s\lambda}-1)|\varphi'(\delta_1)|+(e^{s(1-\lambda)}-1)|\varphi'(\delta_2)|\right]d\lambda\Bigg\}$$

$$=\frac{(\delta_2-\delta_1)}{2}\Bigg\{|\varphi'(\delta_1)|\int_0^{1/2}(1-\lambda)^{\alpha}(e^{s\lambda}-1)d\lambda-|\varphi'(\delta_1)|\int_0^{1/2}\lambda^{\alpha}(e^{s\lambda}-1)d\lambda$$

$$+|\varphi'(\delta_2)|\int_0^{1/2}(1-\lambda)^{\alpha}(e^{s(1-\lambda)}-1)d\lambda-|\varphi'(\delta_2)|\int_0^{1/2}\lambda^{\alpha}(e^{s(1-\lambda)}-1)d\lambda$$

$$+|\varphi'(\delta_1)|\int_{1/2}^{1}\lambda^{\alpha}(e^{s\lambda}-1)d\lambda-|\varphi'(\delta_1)|\int_{1/2}^{1}(1-\lambda)^{\alpha}(e^{s\lambda}-1)d\lambda$$

$$+|\varphi'(\delta_2)|\int_{1/2}^{1}\lambda^{\alpha}(e^{s(1-\lambda)}-1)d\lambda-|\varphi'(\delta_2)|\int_{1/2}^{1}(1-\lambda)^{\alpha}(e^{s(1-\lambda)}-1)d\lambda\Bigg\}$$

$$=\frac{(\delta_2-\delta_1)}{2}\Bigg\{|\varphi'(\delta_1)|\frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$-|\varphi'(\delta_1)|\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$

$$+|\varphi'(\delta_2)|\frac{2^{(-2-\alpha)}\left[4+2\alpha-e^s-\alpha e^s+2^{(2+\alpha)}(-2-\alpha+(1+\alpha)e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$-|\varphi'(\delta_2)|\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$

$$+|\varphi'(\delta_1)|\frac{1}{4}\left(\frac{2(-2+2^{-\alpha})}{1+\alpha}+\frac{(4-2^{-\alpha})e^{\alpha}}{2+\alpha}\right)$$

$$-|\varphi'(\delta_1)|\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$

$$+|\varphi'(\delta_2)|\frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$-|\varphi'(\delta_2)|\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)},$$

where,

$$\int_0^{1/2}(1-\lambda)^{\alpha}(e^{s(1-\lambda)}-1)\,d\lambda=\frac{2^{(-2-\alpha)}\left[4+2\alpha-e^s-\alpha e^s+2^{(2+\alpha)}(-2-\alpha+(1+\alpha)e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$\int_0^{1/2}\lambda^{\alpha}(e^{s(1-\lambda)}-1)\,d\lambda=\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$

$$\int_0^{1/2}(1-\lambda)^{\alpha}(e^{s\lambda}-1)\,d\lambda=\frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$\int_0^{1/2} \lambda^\alpha (e^{s\lambda}-1)\,d\lambda = \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$
$$\int_{1/2}^{1} (1-\lambda)^\alpha (e^{s\lambda}-1)\,d\lambda = \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$
$$\int_{1/2}^{1} \lambda^\alpha (e^{s\lambda}-1)\,d\lambda = \frac{1}{4}\left(\frac{2(-2+2^{-\alpha})}{1+\alpha}+\frac{(4-2^{-\alpha})e^\alpha}{2+\alpha}\right)$$
$$\int_{1/2}^{1} (1-\lambda)^\alpha (e^{s(1-\lambda)}-1)\,d\lambda = \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$
$$\int_{1/2}^{1} \lambda^\alpha (e^{s(1-\lambda)}-1)\,d\lambda = \frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}.$$

This completes the proof of this case. Suppose now that $q>1$, since $|\varphi|^q$ is an exponential s-convex function, we have

$$|\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)|^q \le (e^{s\lambda}-1)|\varphi(\delta_1)|^q + (e^{s(1-\lambda)}-1)|\varphi(\delta_2)|^q$$

Now using Hölders Inequality for $\frac{1}{p}+\frac{1}{q}=1$

$$\left|\frac{\varphi(\delta_1)+\varphi(\delta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta_2-\delta_1)^\alpha}\left[I^\alpha_{\delta_1^+}\varphi(\delta_2)+I^\alpha_{\delta_2^-}\varphi(\delta_1)\right]\right|$$
$$\le \frac{(\delta_2-\delta_1)}{2}\left\{\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|\,|\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)|d\lambda\right\}$$
$$= \frac{(\delta_2-\delta_1)}{2}\left\{\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|^{1-1/q}\,|(1-\lambda)^\alpha-\lambda^\alpha|^{1/q}\,|\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)|d\lambda\right\}$$
$$\le \frac{(\delta_2-\delta_1)}{2}\left\{\left(\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|\,d\lambda\right)^{\frac{q-1}{q}}\left(\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|\,|\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)|^q d\lambda\right)^{\frac{1}{q}}\right\}$$
$$\le \frac{(\delta_2-\delta_1)}{2}\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\right]^{\frac{q-1}{q}}\left(\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|\left[(e^{s\lambda}-1)|\varphi(\delta_1)|^q\right.\right.$$
$$\left.\left.+(e^{s(1-\lambda)}-1)|\varphi(\delta_2)|^q\right]d\lambda\right)^{\frac{1}{q}} = \frac{(\delta_2-\delta_1)}{2}\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\right]^{\frac{q-1}{q}}$$
$$\times\left\{|\varphi'(\delta_1)|^q\frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}\right.$$
$$-|\varphi'(\delta_1)|^q\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$
$$+|\varphi'(\delta_2)|^q\frac{2^{(-2-\alpha)}\left[4+2\alpha-e^s-\alpha e^s+2^{(2+\alpha)}(-2-\alpha+(1+\alpha)e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$
\begin{aligned}
&-|\varphi'(\delta_2)|^q \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)} \\
&+|\varphi'(\delta_1)|^q \frac{1}{4}\left(\frac{2(-2+2^{-\alpha})}{1+\alpha}+\frac{(4-2^{-\alpha})e^{\alpha}}{2+\alpha}\right) \\
&-|\varphi'(\delta_1)|^q \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)} \\
&+|\varphi'(\delta_2)|^q \frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)} \\
&-|\varphi'(\delta_2)|^q \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}\Bigg\}^{\frac{1}{q}},
\end{aligned}
$$

which completes the proof of the Theorem. □

6. Ostrowski Type Inequalities for Exponentially s–Convexity via Fractional Integral

In this section, we established new Ostrowski type inequalities for exponentially s–convexity via Riemann-Liouville fractional integral. A useful and interesting feature of our results is that they provide new estimates on these types of inequalities for fractional integrals. In order to prove our results, we need the following identity.(see in [11, 35]).

Lemma 3. *Suppose a mapping $\varphi : \mathbb{A} \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{A}^o$, where $\delta_1, \delta_2 \in \mathbb{A}$ with $\delta_1 < \delta_2$. If $\varphi' \in L[\delta_1, \delta_2]$, for all $z \in [\delta_1, \delta_2]$ and $\alpha > 0$, then the following equality holds:*

$$
\begin{aligned}
&\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\left\{J_{z-}^{\alpha}\ \varphi(\delta_1)\ +\ J_{z+}^{\alpha}\ \varphi(\delta_2)\right\} \\
&= \frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1 \lambda^{\alpha}\ \varphi'(\lambda z+(1-\lambda)\delta_1)\ d\lambda \\
&- \frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1 \lambda^{\alpha}\ \varphi'(\lambda z+(1-\lambda)\delta_2)\ d\lambda, \qquad (25)
\end{aligned}
$$

where Γ is the Euler gamma function.

Theorem 7. *Suppose a mapping $\varphi : \mathbb{A} \subseteq R \to R$ is differentiable on $\mathbb{A}^o$, where $\delta_1, \delta_2 \in \mathbb{A}$ with $\delta_1 < \delta_2$. If $|\varphi'|$ is exponentially s–convex on $[\delta_1, \delta_2]$ for some $s \in [\ln 2.4, 1]$, $\varphi' \in L[\delta_1, \delta_2]$ and $|\varphi'(z)| \leq \mathbb{M}$, for all $z \in [\delta_1, \delta_2]$, $\alpha > 0$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\left\{J_{z-}^{\alpha}\ \varphi(\delta_1)\ +\ J_{z+}^{\alpha}\ \varphi(\delta_2)\right\}\right| \\
&\leq \frac{\mathbb{M}}{(\delta_2-\delta_1)}
\end{aligned}
$$

$$\times \Bigg[(z-\delta_1)^{\alpha+1} \Bigg\{ \left(\frac{\gamma(\alpha+1,-s) - \Gamma(\alpha+1)}{(-s)^\alpha s} - \frac{1}{\alpha+1} \right)$$
$$- \left(\frac{(\gamma(\alpha+1,s) - \Gamma(\alpha+1)) e^s}{s^{\alpha+1}} + \frac{1}{\alpha+1} \right) \Bigg\}$$
$$+ (\delta_2 - z)^{\alpha+1} \Bigg\{ \left(\frac{\gamma(\alpha+1,-s) - \Gamma(\alpha+1)}{(-s)^\alpha s} - \frac{1}{\alpha+1} \right)$$
$$- \left(\frac{(\gamma(\alpha+1,s) - \Gamma(\alpha+1)) e^s}{s^{\alpha+1}} + \frac{1}{\alpha+1} \right) \Bigg\} \Bigg]. \tag{26}$$

Proof. From Lemma 3 and since $|\varphi'|$ is exponentially s–convexity and $|\varphi'(z)| \leq \mathbb{M}$, we have

$$\left| \left(\frac{(z-\delta_1)^\alpha + (\delta_2 - z)^\alpha}{\delta_2 - \delta_1} \right) \varphi(z) - \frac{\Gamma(\alpha+1)}{\delta_2 - \delta_1} \{ J^\alpha_{z^-} \varphi(\delta_1) + J^\alpha_{z^+} \varphi(\delta_2) \} \right|$$
$$\leq \frac{(z-\delta_1)^{\alpha+1}}{\delta_2 - \delta_1} \int_0^1 \lambda^\alpha |\varphi'(\lambda z + (1-\lambda)\delta_1)| \, d\lambda \quad \frac{(\delta_2 - z)^{\alpha+1}}{\delta_2 - \delta_1} \int_0^1 \lambda^\alpha |\varphi'(\lambda z + (1-\lambda)\delta_2)| \, d\lambda.$$

$$\leq \frac{(z-\delta_1)^{\alpha+1}}{\delta_2 - \delta_1} \int_0^1 \lambda^\alpha \Big\{ \left(e^{s\lambda} - 1\right) |\varphi'(z)| + \left(e^{s(1-\lambda)} - 1\right) |\varphi'(\delta_1)| \Big\} d\lambda$$
$$+ \frac{(\delta_2 - z)^{\alpha+1}}{\delta_2 - \delta_1} \int_0^1 \lambda^\alpha \Big\{ \left(e^{s\lambda} - 1\right) |\varphi'(z)| + \left(e^{s(1-\lambda)} - 1\right) |\varphi'(\delta_1)| \Big\} d\lambda$$
$$\leq \frac{(z-\delta_1)^{\alpha+1}}{\delta_2 - \delta_1} \Big\{ |\varphi'(z)| \int_0^1 \lambda^\alpha \left(e^{s\lambda} - 1\right) d\lambda + |\varphi'(\delta_1)| \int_0^1 \lambda^\alpha \left(e^{s(1-\lambda)} - 1\right) d\lambda \Big\}$$
$$+ \frac{(\delta_2 - z)^{\alpha+1}}{\delta_2 - \delta_1} \Big\{ |\varphi'(z)| \int_0^1 \lambda^\alpha \left(e^{s\lambda} - 1\right) d\lambda + |\varphi'(\delta_2)| \int_0^1 \lambda^\alpha \left(e^{s(1-\lambda)} - 1\right) d\lambda \Big\}$$
$$\leq \frac{\mathbb{M}}{(\delta_2 - \delta_1)} \times (z-\delta_1)^{\alpha+1} \Bigg\{ \left(\frac{\gamma(\alpha+1,-s) - \Gamma(\alpha+1)}{(-s)^\alpha s} - \frac{1}{\alpha+1} \right)$$
$$- \left(\frac{(\gamma(\alpha+1,s) - \Gamma(\alpha+1)) e^s}{s^{\alpha+1}} + \frac{1}{\alpha+1} \right) \Bigg\}$$
$$+ \frac{\mathbb{M}}{(\delta_2 - \delta_1)} \times (\delta_2 - z)^{\alpha+1} \Bigg\{ \left(\frac{\gamma(\alpha+1,-s) - \Gamma(\alpha+1)}{(-s)^\alpha s} - \frac{1}{\alpha+1} \right)$$
$$- \left(\frac{(\gamma(\alpha+1,s) - \Gamma(\alpha+1)) e^s}{s^{\alpha+1}} + \frac{1}{\alpha+1} \right) \Bigg\}.$$

After simplification, we get(26). The proof is completed. □

Corollary 1. *Under the similar consideration in Theorem 7, by choosing $s = 1$, we obtain*

$$\left| \left(\frac{(z-\delta_1)^\alpha + (\delta_2 - z)^\alpha}{\delta_2 - \delta_1} \right) \varphi(z) - \frac{\Gamma(\alpha+1)}{\delta_2 - \delta_1} \{ J^\alpha_{z^-} \varphi(\delta_1) + J^\alpha_{z^+} \varphi(\delta_2) \} \right|$$

$$\leq \frac{\mathbb{M}}{(\delta_2-\delta_1)}$$
$$\times\Bigg[(z-\delta_1)^{\alpha+1}\Bigg\{\left(\frac{\gamma(\alpha+1,-1)-\Gamma(\alpha+1)}{(-1)^{\alpha}}-\frac{1}{\alpha+1}\right)$$
$$-\left((\gamma(\alpha+1,1)-\Gamma(\alpha+1))e+\frac{1}{\alpha+1}\right)\Bigg\}$$
$$+(\delta_2-z)^{\alpha+1}\Bigg\{\left(\frac{\gamma(\alpha+1,-1)-\Gamma(\alpha+1)}{(-1)^{\alpha}}-\frac{1}{\alpha+1}\right)$$
$$-\left((\gamma(\alpha+1,1)-\Gamma(\alpha+1))e+\frac{1}{\alpha+1}\right)\Bigg\}\Bigg].$$

Theorem 8. *Suppose a mapping $\varphi:\mathbb{A}\subseteq\mathbb{R}\to\mathbb{R}$ is differentiable on $\mathbb{A}^o$, where $\delta_1,\delta_2\in\mathbb{A}$ with $\delta_1<\delta_2$. If $|\varphi'|^q$ is exponentially s–convex on $[\delta_1,\delta_2]$ for some $s\in[\ln 2.4,1]$, $q>1$, $q^{-1}=1-p^{-1}$, $\varphi'\in L[\delta_1,\delta_2]$ and $|\varphi'(z)|\leq\mathbb{M}$, for all $z\in[\delta_1,\delta_2]$, with $\alpha>0$, then the following inequality holds:*

$$\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J_{z^-}^{\alpha}\,\varphi(\delta_1)\;+\;J_{z^+}^{\alpha}\,\varphi(\delta_2)\}\right|$$
$$\leq\frac{2^{\frac{1}{q}}\mathbb{M}}{(\delta_2-\delta_1)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}$$
$$\times\left[(z-\delta_1)^{\alpha+1}\left\{\left(\frac{e^s-s-1}{s}\right)\right\}^{\frac{1}{q}}+(\delta_2-z)^{\alpha+1}\left\{\left(\frac{e^s-s-1}{s}\right)\right\}^{\frac{1}{q}}\right].\tag{27}$$

Proof. From Lemma 3 and famous Hölder's inequality, we have

$$\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J_{z^-}^{\alpha}\,\varphi(\delta_1)\;+\;J_{z^+}^{\alpha}\,\varphi(\delta_2)\}\right|$$
$$\leq\frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1\lambda^{\alpha}\,|\varphi'(\lambda z+(1-\lambda)\delta_1)|\,d\lambda$$
$$+\frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1\lambda^{\alpha}\,|\varphi'(\lambda z+(1-\lambda)\delta_2)|\,d\lambda$$
$$\leq\frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\left(\int_0^1\lambda^{\alpha p}d\lambda\right)^{\frac{1}{p}}\left(\int_0^1|\varphi'(\lambda z+(1-\lambda)\delta_1)|^q\,d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\left(\int_0^1\lambda^{\alpha p}d\lambda\right)^{\frac{1}{p}}\left(\int_0^1|\varphi'(\lambda z+(1-\lambda)\delta_2)|^q\,d\lambda\right)^{\frac{1}{q}}.\tag{28}$$

Since $|\varphi'|^q$ is exponentially s–convexity and $|\varphi'(z)|\leq\mathbb{M}$, we obtain

$$\int_0^1|\varphi'(\lambda z+(1-\lambda)\delta_1)|^q\,d\lambda=\int_0^1\left\{\left(e^{s\lambda}-1\right)|\varphi'(z)|^q+\left(e^{s(1-\lambda)}-1\right)|\varphi'(\delta_1)|^q\right\}d\lambda$$

$$\leq \mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right) + \mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right)$$
$$\leq 2\mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right) \tag{29}$$

and

$$\int_0^1 \left|\varphi'\left(\lambda z + (1-\lambda)\,\delta_2\right)\right|^q d\lambda = \int_0^1 \left\{ \left(e^{s\lambda} - 1\right)\left|\varphi'(z)\right|^q + \left(e^{s(1-\lambda)} - 1\right)\left|\varphi'(\delta_2)\right|^q \right\} d\lambda$$
$$\leq \mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right) + \mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right)$$
$$\leq 2\mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right). \tag{30}$$

By connecting (29) and (30) with (28), we get (27). The proof is completed. □

Corollary 2. *Under the similar consideration in Theorem 8, by choosing* $s = 1$, *we obtain*

$$\left| \left(\frac{(z-\delta_1)^\alpha + (\delta_2 - z)^\alpha}{\delta_2 - \delta_1}\right)\varphi(z) - \frac{\Gamma(\alpha+1)}{\delta_2 - \delta_1}\left\{J^\alpha_{z^-}\ \varphi(\delta_1) \ + \ J^\alpha_{z^+}\ \varphi(\delta_2)\right\} \right|$$
$$\leq \frac{2^{\frac{1}{q}}\mathbb{M}}{(\delta_2 - \delta_1)} \left(\frac{1}{\alpha p + 1}\right)^{\frac{1}{p}} \left[(z-\delta_1)^{\alpha+1} \left\{(e-2)\right\}^{\frac{1}{q}} + (\delta_2 - z)^{\alpha+1} \left\{(e-2)\right\}^{\frac{1}{q}} \right].$$

Theorem 9. *Suppose a mapping* $\varphi : \mathbb{A} \subseteq \mathbb{R} \to \mathbb{R}$ *is differentiable on* $\mathbb{A}^o$, *where* $\delta_1, \delta_2 \in \mathbb{A}$ *with* $\delta_1 < \delta_2$. *If* $|\varphi'|^q$ *is exponentially* s*–convex on* $[\delta_1, \delta_2]$ *for some* $s \in [\ln 2.4, 1]$, $q \geq 1$, $q^{-1} = 1 - p^{-1}$, $\varphi' \in L[\delta_1, \delta_2]$ *and* $|\varphi'(z)| \leq \mathbb{M}$, *for all* $z \in [\delta_1, \delta_2]$, *with* $\alpha > 0$, *then the following inequality holds:*

$$\left| \left(\frac{(z-\delta_1)^\alpha + (\delta_2 - z)^\alpha}{\delta_2 - \delta_1}\right)\varphi(z) - \frac{\Gamma(\alpha+1)}{\delta_2 - \delta_1}\left\{J^\alpha_{z^-}\ \varphi(\delta_1) \ + \ J^\alpha_{z^+}\ \varphi(\delta_2)\right\} \right|$$
$$\leq \frac{\mathbb{M}}{(\delta_2 - \delta_1)} \left(\frac{1}{\alpha + 1}\right)^{1-\frac{1}{q}}$$
$$\times \left[(z-\delta_1)^{\alpha+1} \left\{ \left(\frac{\gamma(\alpha+1, -s) - \Gamma(\alpha+1)}{(-s)^\alpha s} - \frac{1}{\alpha+1}\right) \right.\right.$$
$$\left. - \left((\gamma(\alpha+1, s) - \Gamma(\alpha+1))\, s^{-\alpha-1} e^s - \frac{1}{\alpha+1} \right) \right\}^{\frac{1}{q}}$$
$$+ \ (\delta_2 - z)^{\alpha+1} \left\{ \left(\frac{\gamma(\alpha+1, -s) - \Gamma(\alpha+1)}{(-s)^\alpha s} - \frac{1}{\alpha+1}\right) \right.$$
$$\left.\left. - \left((\gamma(\alpha+1, s) - \Gamma(\alpha+1))\, s^{-\alpha-1} e^s - \frac{1}{\alpha+1} \right) \right\}^{\frac{1}{q}} \right]. \tag{31}$$

Proof. From Lemma 3 and power mean inequality, we have

$$\begin{aligned}
&\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J_{z^-}^{\alpha}\ \varphi(\delta_1)\ +\ J_{z^+}^{\alpha}\ \varphi(\delta_2)\}\right| \\
&\leq \frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1 \lambda^{\alpha}\ \left|\varphi'(\lambda z+(1-\lambda)\delta_1)\right|\ d\lambda \\
&+\frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1 \lambda^{\alpha}\ \left|\varphi'(\lambda z+(1-\lambda)\delta_2)\right|\ d\lambda \\
&\leq \frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\left(\int_0^1 \lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 \lambda^{\alpha}\left|\varphi'(\lambda z+(1-\lambda)\delta_1)\right|^q d\lambda\right)^{\frac{1}{q}} \\
&+\frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\left(\int_0^1 \lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 \lambda^{\alpha}\left|\varphi'(\lambda z+(1-\lambda)\delta_2)\right|^q d\lambda\right)^{\frac{1}{q}} \qquad (32)
\end{aligned}$$

Since $|\varphi'|^q$ is exponentially s–convexity and $|\varphi'(z)|\leq\mathbb{M}$, we obtain

$$\begin{aligned}
&\int_0^1 \lambda^{\alpha}\ \left|\varphi'(\lambda z+(1-\lambda)\delta_1)\right|^q\ d\lambda \\
&=\ \int_0^1 \lambda^{\alpha}\ \left\{\left(e^{s\lambda}-1\right)\left|\varphi'(z)\right|^q+\left(e^{s(1-\lambda)}-1\right)\left|\varphi'(\delta_1)\right|^q\right\}d\lambda \\
&\leq\ \mathbb{M}^q\left\{\left(\frac{\gamma(\alpha+1,-s)-\Gamma(\alpha+1)}{(-s)^{\alpha}s}-\frac{1}{\alpha+1}\right)\right. \\
&\left.-\left((\gamma(\alpha+1,s)-\Gamma(\alpha+1))\,s^{-\alpha-1}e^{s}-\frac{1}{\alpha+1}\right)\right\} \qquad (33)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \lambda^{\alpha}\ \left|\varphi'(\lambda z+(1-\lambda)\delta_2)\right|^q\ d\lambda \\
&=\ \int_0^1 \lambda^{\alpha}\ \left(e^{s\lambda}-1\right)\left|\varphi'(z)\right|^q+\left(e^{s(1-\lambda)}-1\right)\left|\varphi'(\delta_2)\right|^q\Big\}d\lambda \\
&\leq\ \mathbb{M}^q\left\{\left(\frac{\gamma(\alpha+1,-s)-\Gamma(\alpha+1)}{(-s)^{\alpha}s}-\frac{1}{\alpha+1}\right)\right. \\
&\left.-\left((\gamma(\alpha+1,s)-\Gamma(\alpha+1))\,s^{-\alpha-1}e^{s}-\frac{1}{\alpha+1}\right)\right\}. \qquad (34)
\end{aligned}$$

By connecting (33) and (34) with (32), we get (31). The proof is completed. □

Corollary 3. *Under the similar consideration in Theorem 9, by choosing $s=1$, we obtain*

$$\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J_{z^-}^{\alpha}\ \varphi(\delta_1)\ +\ J_{z^+}^{\alpha}\ \varphi(\delta_2)\}\right|$$

$$\leq \frac{\mathbb{M}}{(\delta_2-\delta_1)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}$$
$$\times\left[(z-\delta_1)^{\alpha+1}\ \left\{\left(\frac{\gamma(\alpha+1,-1)-\Gamma(\alpha+1)}{(-1)^{\alpha}}-\frac{1}{\alpha+1}\right)\right.\right.$$
$$\left.-\left((\gamma(\alpha+1,1)-\Gamma(\alpha+1))e-\frac{1}{\alpha+1}\right)\right\}^{\frac{1}{q}}$$
$$+\ (\delta_2-z)^{\alpha+1}\ \left\{\left(\frac{\gamma(\alpha+1,-1)-\Gamma(\alpha+1)}{(-1)^{\alpha}}-\frac{1}{\alpha+1}\right)\right.$$
$$\left.\left.-\left((\gamma(\alpha+1,1)-\Gamma(\alpha+1))e-\frac{1}{\alpha+1}\right)\right\}^{\frac{1}{q}}\right].$$

7. Conclusion

In this article, the authors showed the new class of exponentially s–convex functions, derive several new versions of the Hermite-Hadamard inequality using the class of exponentially s–convex functions in the frame of classical and fractional calculus. We have obtained some refinement of Hermite–Hadamard inequality. Finally, we have attained some Ostrowski type inequalities for exponentially s–convexity via fractional integral. We hope the consequences and techniques of this article will energize and inspire the researchers to explore a more interesting sequel in this area.

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