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Multiple Nonnegative Solutions for a Class of Fourth-Order BVPs Via a New Topological Approach

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Abstract

In this paper, we propose extensions of Leray-Schauder boundary condition for a sum of two operators $T+S$ in the case when T is an expansive operator and $I-S$ is a completely continuous operator. As their applications, we investigate a class of fourth-order nonlinear boundary value problems with integral boundary conditions. We give conditions for the parameters of the considered boundary value problem that ensure existence of at least two non trivial bounded nonnegative classical solutions of the considered boundary value problem. The results in the paper are provided with a suitable example.

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1. Introduction

Since 1970, the interest for fourth order boundary value problems (BVPs for short) has risen due to their important applications in practical problems. For instance, the deformation of an elastic beam under an external force h supported at both ends is described by the linear boundary value problem

$$\begin{aligned} &x^{(4)}(t) = h(t), \quad t \in (0,1), \\ &x(0) = x(1) = x''(0) = x''(1) = 0, \end{aligned}$$

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where vanishing moments at the ends of the attached beam motivate the boundary conditions (see [9] for more details). The existence of solutions for nonlinear fourth-order BVPs has gained much interest in the last years (see, e.g., [2, 3, 4, 6, 10, 11, 12, 13, 15, 17]). Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point, and nonlocal boundary conditions as special cases.

In this paper, we investigate the existence of at least two nonnegative solutions to the fourth-order nonlinear boundary value problem with integral boundary conditions:

$$\begin{aligned} x^{(4)}(t) &= w(t)f(t,x(t),x''(t)), \quad t\in(0,1),\\ x(0) &= \int_0^1 h_1(s)x(s)ds, \quad x(1)=\int_0^1 k_1(s)x(s)ds,\\ x''(0) &= \int_0^1 h_2(s)x''(s)ds, \quad x''(1)=\int_0^1 k_2(s)x''(s)ds, \end{aligned} \tag{1.1}$$

where

(H1) $w\in L^1([0,1])$ is nonnegative and may be singular at $t=0$ and (or) $t=1$, $f\in\mathcal{C}([0,1]\times\mathbb{R}\times\mathbb{R})$,

$$|f(t,u,v)|\le a_1(t)|u|^{p_1}+a_2(t)|v|^{p_2}+a_3(t), \quad t\in[0,1], \quad u,v\in\mathbb{R},$$

$a_1,a_2,a_3\in\mathcal{C}([0,1])$ are given nonnegative functions, p_1,p_2 are given nonnegative constants.

(H2) h_1, h_2, k_1, $k_2\in L^1([0,1])$ with $m_1\nu_1+n_1\mu_1\neq 0$, $m_2\nu_2+n_2\mu_2\neq 0$,

for

$$\begin{aligned} m_1 &= \int_0^1 sh_1(s)ds, \quad m_2=\int_0^1 sh_2(s)ds,\\ n_1 &= 1-\int_0^1 sk_1(s)ds, \quad n_2=1-\int_0^1 sk_2(s)ds,\\ \mu_1 &= 1-\int_0^1 h_1(s)ds, \quad \mu_2=1-\int_0^1 h_2(s)ds,\\ \nu_1 &= 1-\int_0^1 k_1(s)ds, \quad \nu_2=1-\int_0^1 k_2(s)ds. \end{aligned}$$

In 2003 and 2004, the authors of [11, 18] studied the existence of solutions of Problem (1.1) for $h_1=h_2=k_1=k_2=0$, by using the Krasnosel'skii's fixed point theorem and fixed point index theory on cones of Banach spaces, respectively.

By using the Krasnosel'skii fixed point theorem of cone expansion and compression, in [15] is proved the existence of at least two positive solutions of BVP (1.1) when w may be singular at $t=0$ and (or) $t=1$, $w\in L^1([0,1])$, $f:[0,1]\times[0,\infty)\times(-\infty,0]\to[0,\infty)$ is continuous, h_1, h_2, k_1, $k_2\in L^1([0,1])$ are nonnegative with $\mu_1>0$, $\nu_1>0$, $\mu_2>0$, $\nu_2>0$.

The paper is organized as follows. Some background material and auxiliary results are provided in the next section, extensions of Leray-Schauder boundary condition are given in the case of completely continuous mappings as well as in the case of the sum $T+S$, where T is an expansive mapping and (I-S) is a completely continuous one. The main existence result of this paper is presented and proved in Section 3. It complements and improves similar results obtained in [15]. In Section 4, we discuss and compare our result with those obtained in [15]. We end the paper by giving in Section 5 an example of application with some numerical computations.

2. Auxiliary Results

Let E be a real Banach space.

Definition 2.1. A closed, convex set $\mathcal{P}$ in E is said to be cone if

1. $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,
2. $x, -x \in \mathcal{P}$ implies $x = 0$.

Every cone $\mathcal{P}$ defines a partial ordering $\leq$ in E defined by :

$$x \leq y \text{ if and only if } y - x \in \mathcal{P}.$$

Definition 2.2. A mapping $K : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

In the sequel, we give an extension of the Leray-Schauder boundary condition, which allows to increase the field of applications of this condition. First, we present our result for the completely continuous mappings. Next, we extend it to the case of the sum $T+S$, where T is an expansive mapping and $(I-S)$ is a completely continuous one.

Lemma 2.3. *Let X be a closed convex subset of a Banach space E and $U \subset X$ a bounded open subset with $0 \in U$. Assume $K : \overline{U} \to X$ is a completely continuous mapping without fixed point on the boundary ∂U with $\gamma = dist(0, (I-K)(\partial U))$ and there exists $\varepsilon > 0$ small enough such that*

$$Kx \neq \lambda x \text{ for all } x \in \partial U \text{ and } \lambda \geq 1 + \varepsilon. \tag{2.1}$$

Then the fixed point index $i\,(K, U, X) = 1$.

Proof. Consider the homotopic deformation $H : [0,1] \times \overline{U} \to X$ defined by

$$H(t,x) = \frac{1}{\varepsilon + 1}\, tKx.$$

The operator H is completely continuous and has no fixed point on ∂U, $\forall t \in [0,1]$; otherwise, we may distinguish between two cases:
- If $t = 0$, there exists some $x_0 \in \partial U$ such that $x_0 = 0$, contradicting $0 \in U$.
- If $t \in (0,1]$, there exists some $x_0 \in \partial U$ such that $\frac{1}{\varepsilon+1}\, tKx_0 = x_0$; then

$$Kx_0 = \frac{1+\varepsilon}{t}\, x_0 \text{ with } \frac{1+\varepsilon}{t} \geq 1 + \varepsilon,$$

leading to a contradiction with the hypothesis (2.1).
From the invariance under homotopy and the normalization properties of the index (see [8, Theorem 2.3.1]), we deduce

$$i\,(\frac{1}{\varepsilon+1}\, K, U, X) = i\,(0, U, X) = 1.$$

Now, we show that

$$i\,(K, U, X) = i\,(\frac{1}{\varepsilon+1}\, K, U, X).$$

Since K has no fixed point in ∂U and $(I-K)(\partial U)$ is a closed set (see [14, Lemma 1]), we get $0 \notin \overline{(I-K)(\partial U)}$. Hence,

$$\inf_{x \in \partial U} \|x - Kx\| = \gamma > 0.$$

Let ε be sufficiently small so that $\|\frac{\varepsilon}{\varepsilon+1}Kx\| < \frac{\gamma}{2}$. Hence

$$\|Kx - \frac{1}{\varepsilon+1}Kx\| = \|Kx - Kx + \frac{\varepsilon}{\varepsilon+1}Kx\| = \|\frac{\varepsilon}{\varepsilon+1}Kx\| < \frac{\gamma}{2},\ \forall x \in \partial U.$$

Define the convex deformation $G : [0,1] \times \overline{U} \to X$ by

$$G(t,x) = tKx + (1-t)\frac{1}{\varepsilon+1}Kx.$$

The operator G is completely continuous and has no fixed point on ∂U, $\forall t \in [0,1]$. In fact, for all $x \in \partial U$ and $t \in [0,1]$, we have

$$\begin{array}{rcl} \|x - G(t,x)\| & = & \|x - tKx - (1-t)\frac{1}{\varepsilon+1}Kx\| \\ & \geq & \|x - \frac{1}{\varepsilon+1}Kx\| - t\|Kx - \frac{1}{\varepsilon+1}Kx\| \\ & \geq & \|x - Kx\| - \|\frac{\varepsilon}{\varepsilon+1}Kx\| - t\|Kx - \frac{1}{\varepsilon+1}Kx\| \\ & > & \gamma - \frac{\gamma}{2} - \frac{\gamma}{2} = 0. \end{array}$$

Then our claim follows from the homotopy invariance property of the index. □

Now, we recall the definition of an expansive mapping.

Definition 2.4. Let X and Y be real Banach spaces. A mapping $T : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|Tx - Ty\|_Y \geq h\|x - y\|_X, \quad \text{for any } x, y \in X.$$

The extension of the fixed point index for $T + S$, where T is an expansive mapping and $I - S$ is a completely continuous one, is based on the following result.

Lemma 2.5. *[16] Let $(X, \|.\|)$ be a linear normed space and $D \subset X$. Assume that the mapping $T : D \to X$ is expansive with constant $h > 1$. Then the inverse of $T : D \to T(D)$ exists and*

$$\|T^{-1}x - T^{-1}y\| \leq \frac{1}{h}\|x - y\|, \quad \forall x, y \in T(D).$$

In the sequel, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$, Ω is a subset of $\mathcal{P}$, and U is a bounded open subset of $\mathcal{P}$, and $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$.

Assume that $I - S : \overline{U} \to E$ is a completely continuous mapping and $T : \Omega \to E$ is an expansive one with constant $h > 1$. By Lemma 2.5, the operator T^{-1} is h^{-1}-Lipschitzian on $T(\Omega)$. Suppose that

$$(I - S)(\overline{U}) \subset T(\Omega), \tag{2.2}$$

and

$$x \neq Tx + Sx, \quad \text{for all } x \in \partial U \cap \Omega. \tag{2.3}$$

Then $x \neq T^{-1}(I - S)x$, for all $x \in \partial U$ and the mapping $T^{-1}(I - S) : \overline{U} \to \mathcal{P}$ is completely continuous. From [8, Theorem 2.3.1], the fixed point index $i\,(T^{-1}(I - S), U, \mathcal{P})$ is well defined. Thus we put

$$i_*\,(T + S, U \cap \Omega, \mathcal{P}) = \begin{cases} i\,(T^{-1}(I - S), U, \mathcal{P}), & \text{if } U \cap \Omega \neq \emptyset \\ 0, & \text{if } U \cap \Omega = \emptyset. \end{cases} \tag{2.4}$$

Using the main properties of the fixed point index for strict set contractions (in particular completely continuous mapping), Djebali, Benslimane and Mebarki in [5], have discussed the properties of the generalized fixed point index i_*. The following lemma gives the computation of this index.

Lemma 2.6. *Assume that $T : \Omega \to E$ is an expansive mapping, $I - S : \overline{U} \to E$ is a completely continuous mapping, and $(I - S)(\overline{U}) \subset T(\Omega)$. Suppose that $T + S$ has no fixed point on $\partial U \cap \Omega$.*
Then we have the following results:

(1) *If $0 \in U$ and there exists $\varepsilon > 0$ small enough such that*

$$(I - S)x \neq T(\lambda x) \text{ for all } \lambda \geq 1 + \varepsilon,\ x \in \partial U \text{ and } \lambda x \in \Omega,$$

then the fixed point index $i_* (T + S, U \cap \Omega, \mathcal{P}) = 1$.

(2) *If there exists $u_0 \in \mathcal{P}^*$ such that*

$$(I - S)x \neq T(x - \lambda u_0), \text{ for all } \lambda > 0 \text{ and } x \in \partial U \cap (\Omega + \lambda u_0),$$

then the fixed point index $i_* (T + S, U \cap \Omega, \mathcal{P}) = 0$.

Proof. **(1)** The mapping $T^{-1}(I - S) : \overline{U} \to \mathcal{P}$ is completely continuous without fixed point on ∂U, and our hypothesis implies

$$T^{-1}(I - S)x \neq \lambda x \quad \text{for all} \quad x \in \partial U \quad \text{and} \quad \lambda \geq 1 + \varepsilon.$$

Then, our claim follows from (2.4) and Lemma 2.3.

(2) See the proof of [5, Proposition 3.13].

□

Remark 2.7. The result (1) in Lemma 2.6 is an extension of [5, Corollary 3.7] in the case where $I - S$ is a 0-set contraction.

Now, we combine the results (1) and (2) of Lemma 2.6 to establish the following multiplicity result. This result will be used to prove our main result.

Theorem 2.8. *Let U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping, $I - S : \overline{U}_3 \to E$ is a completely continuous one and $(I - S)(\overline{U}_3) \subset T(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) $(I - S)x \neq T(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,

(ii) there exists $\varepsilon > 0$ *small enough such that* $(I - S)x \neq T(\lambda x)$, *for all* $\lambda \geq 1 + \varepsilon$, $x \in \partial U_2$, *and* $\lambda x \in \Omega$,

(iii) $(I - S)x \neq T(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof. If $(I - S)x = Tx$ for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_2 \cap \Omega$ of the operator $T + S$. Suppose that $(I - S)x \neq Tx$ for any $x \in \partial U_2 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_1 \cap \Omega$ and $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$, otherwise the result is obvious. By Lemma 2.6, we have

$$i_* (T + S, U_1 \cap \Omega, \mathcal{P}) = i_* (T + S, U_3 \cap \Omega, \mathcal{P}) = 0 \text{ and } i_* (T + S, U_2 \cap \Omega, \mathcal{P}) = 1.$$

The additivity property of the index i_* yields

$$i_* (T + S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{P}) = 1 \text{ and } i_* (T + S, (U_3 \setminus \overline{U}_2) \cap \Omega, \mathcal{P}) = -1.$$

Consequently, by the existence property of the index i_*, $T + S$ has at least two fixed points $x_1 \in (U_2 \setminus U_1) \cap \Omega$ and $x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$. □

Let

$$
\begin{aligned}
G(t,s) &= \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases} \\
H_1(t,s) &= G(t,s) + \frac{m_1+\mu_1 t}{m_1\nu_1+n_1\mu_1}\int_0^1 k_1(\nu)G(t,\nu)d\nu \\
&\quad + \frac{n_1-\nu_1 t}{m_1\nu_1+n_1\mu_1}\int_0^1 h_1(\nu)G(t,\nu)d\nu, \\
H_2(t,s) &= G(t,s) + \frac{m_2+\mu_2 t}{m_2\nu_2+n_2\mu_2}\int_0^1 k_2(\nu)G(t,\nu)d\nu \\
&\quad + \frac{n_2-\nu_2 t}{m_2\nu_2+n_2\mu_2}\int_0^1 h_2(\nu)G(t,\nu)d\nu, \\
H(t,s) &= \int_0^1 H_1(t,\nu)H_2(\nu,s)d\nu, \quad t,s\in[0,1],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{K}_1 &= \int_0^1 |k_1(\nu)|d\nu, \quad \mathbb{K}_2 = \int_0^1 |k_2(\nu)|d\nu, \\
\mathbb{H}_1 &= \int_0^1 |h_1(\nu)|d\nu, \quad \mathbb{H}_2 = \int_0^1 |h_2(\nu)|d\nu, \\
A_1 &= 1 + \frac{|m_1|+|\mu_1|}{|m_1\nu_1+n_1\mu_1|}\mathbb{K}_1 + \frac{|n_1|+|\nu_1|}{|m_1\nu_1+n_1\mu_1|}\mathbb{H}_1, \\
A_2 &= 1 + \frac{|m_2|+|\mu_2|}{|m_2\nu_2+n_2\mu_2|}\mathbb{K}_2 + \frac{|n_2|+|\nu_2|}{|m_2\nu_2+n_2\mu_2|}\mathbb{H}_2, \\
A_3 &= \int_0^1 w(s)a_1(s)ds, \\
A_4 &= \int_0^1 w(s)a_2(s)ds, \\
A_5 &= \int_0^1 w(s)a_3(s)ds.
\end{aligned}
$$

Then

$$0 \le G(t,s) \le 1, \quad t,s\in[0,1],$$

and

$$
\begin{aligned}
|H_1(t,s)| &\le G(t,s) + \frac{|m_1|+|\mu_1|}{|m_1\nu_1+n_1\mu_1|}\int_0^1 |k_1(\nu)|G(t,\nu)d\nu \\
&\quad + \frac{|n_1|+|\nu_1|}{|m_1\nu_1+n_1\mu_1|}\int_0^1 |h_1(\nu)|G(t,\nu)d\nu \\
&\le 1 + \frac{|m_1|+|\mu_1|}{|m_1\nu_1+n_1\mu_1|}\mathbb{K}_1 + \frac{|n_1|+|\nu_1|}{|m_1\nu_1+n_1\mu_1|}\mathbb{H}_1 = A_1,
\end{aligned}
$$

$$\begin{aligned}
|H_2(t,s)| &\leq G(t,s) + \frac{|m_2|+|\mu_2|}{|m_2\nu_2+n_2\mu_2|}\int_0^1 |k_2(\nu)|G(t,\nu)d\nu \\
&\quad + \frac{|n_2|+|\nu_2|}{|m_2\nu_2+n_2\mu_2|}\int_0^1 |h_2(\nu)|G(t,\nu)d\nu \\
&\leq 1 + \frac{|m_2|+|\mu_2|}{|m_2\nu_2+n_2\mu_2|}\mathbb{K}_2 + \frac{|n_2|+|\nu_2|}{|m_2\nu_2+n_2\mu_2|}\mathbb{H}_2 = A_2, \\
|H(t,s)| &= \left|\int_0^1 H_1(t,\nu)H_2(\nu,s)d\nu\right| \\
&\leq \int_0^1 |H_1(t,\nu)||H_2(t,\nu)|d\nu \\
&\leq A_1A_2, \quad t,s\in[0,1].
\end{aligned}$$

In [15, Lemma 5], it is proved that if $x\in\mathcal{C}^2([0,1])$ is a solution to the integral equation

$$x(t) = \int_0^1 H(t,s)w(s)f(s,x(s),x''(s))ds,$$

then $x\in\mathcal{C}^2([0,1])\cap\mathcal{C}^4((0,1))$ and it satisfies the BVP (1.1).
Let $g\in\mathcal{C}([0,1])$ be a positive function such that

$$\int_0^1 \left((1-s)^2+2(1-s)+2\right)g(s)ds \leq A \tag{2.5}$$

for some positive constant A. For $x\in\mathcal{C}^2([0,1])$, define the operator

$$Fx(t) = \int_0^t (t-s)^2 g(s)\left(-x(s)+\int_0^1 H(s,s_1)w(s_1)f\left(s_1,x(s_1),x''(s_1)\right)ds_1\right)ds, \quad t\in[0,1], \tag{2.6}$$

and the norm

$$\|x\| = \max\{\max_{t\in[0,1]}|x(t)|, \max_{t\in[0,1]}|x'(t)|, \max_{t\in[0,1]}|x''(t)|\}.$$

Lemma 2.9. *Suppose* $(H1)$ *and* $(H2)$*. If* $x\in\mathcal{C}^2([0,1])$ *is a solution to the equation*

$$0 = \frac{L_1}{5} + Fx(t), \quad t\in[0,1], \tag{2.7}$$

where L_1 *is an arbitrary constant, then* $x\in\mathcal{C}^2([0,1])\cap\mathcal{C}^4((0,1))$ *is a solution to the BVP* (1.1).

Proof. Let $x\in\mathcal{C}^2([0,1])$ is a solution to the integral equation (2.7). We differentiate three times with respect to t the integral equation (2.7) and we get

$$0 = g(t)\left(-x(t)+\int_0^1 H(t,s_1)w(s_1)f\left(s_1,x(s_1),x''(s_1)\right)ds_1\right), \quad t\in[0,1],$$

whereupon

$$x(t) = \int_0^1 H(t,s_1)w(s_1)f\left(s_1,x(s_1),x''(s_1)\right)ds_1, \quad t\in[0,1].$$

Then $x\in\mathcal{C}^2([0,1])\cap\mathcal{C}^4((0,1))$ is a solution to the BVP (1.1). This completes the proof. □

Lemma 2.10. *Assume (H1) and (H2). Let $x \in \mathcal{C}^2([0,1])$ and $\|x\| \le c$ for some positive constant c. Then we have*

$$\|Fx\| \le A\left(c + A_1A_2\left(A_3c^{p_1} + A_4c^{p_2} + A_5\right)\right).$$

Proof. Let $x \in \mathcal{C}^2([0,1])$ and $\|x\| \le c$. Then

$$\begin{aligned}
|Fx(t)| &= \left|\int_0^t (t-s)^2 g(s)\left(-x(s) + \int_0^1 H(s,s_1)w(s_1)f(s_1,x(s_1),x''(s_1))ds_1\right)ds\right| \\
&\le \int_0^t (t-s)^2 g(s)\left(|x(s)| + \int_0^1 |H(s,s_1)|w(s_1)|f(s_1,x(s_1),x''(s_1))|ds_1\right)ds \\
&\le \int_0^1 (1-s)^2 g(s)\left(c + A_1A_2\int_0^1 w(s_1)\left(a_1(s_1)|x(s_1)|^{p_1} + a_2(s_1)|x''(s_1)|^{p_2} + a_3(s_1)\right)ds_1\right)ds \\
&\le \int_0^1 (1-s)^2 g(s)\Bigg(c + A_1A_2\Bigg(c^{p_1}\int_0^1 w(s_1)a_1(s_1)ds_1 + c^{p_2}\int_0^1 w(s_1)a_2(s_1)ds_1 \\
&\quad + \int_0^1 w(s_1)a_3(s_1)ds_1\Bigg)\Bigg)ds \\
&\le \left(c + A_1A_2\left(c^{p_1}A_3 + c^{p_2}A_4 + A_5\right)\right)\int_0^1 (1-s)^2 g(s)ds \\
&\le A\left(c + A_1A_2\left(c^{p_1}A_3 + c^{p_2}A_4 + A_5\right)\right), \quad t \in [0,1],
\end{aligned}$$

and

$$\begin{aligned}
\left|(Fx)'(t)\right| &= \left|2\int_0^t (t-s)g(s)\left(-x(s) + \int_0^1 H(s,s_1)w(s_1)f(s_1,x(s_1),x''(s_1))ds_1\right)ds\right| \\
&\le 2\int_0^1 (1-s)g(s)\left(c + A_1A_2\int_0^1 w(s_1)\left(a_1(s_1)|x(s_1)|^{p_1} + a_2(s_1)|x''(s_1)|^{p_2} + a_3(s_1)\right)ds_1\right)ds \\
&\le A\left(c + A_1A_2\left(c^{p_1}A_3 + c^{p_2}A_4 + A_5\right)\right), \quad t \in [0,1],
\end{aligned}$$

and

$$\begin{aligned}
\left|(Fx)''(t)\right| &= \left|2\int_0^t g(s)\left(-x(s) + \int_0^1 H(s,s_1)w(s_1)f(s_1,x(s_1),x''(s_1))ds_1\right)ds\right| \\
&\le 2\int_0^1 g(s)\left(c + A_1A_2\int_0^1 w(s_1)\left(a_1(s_1)|x(s_1)|^{p_1} + a_2(s_1)|x''(s_1)|^{p_2} + a_3(s_1)\right)ds_1\right)ds \\
&\le A\left(c + A_1A_2\left(c^{p_1}A_3 + c^{p_2}A_4 + A_5\right)\right), \quad t \in [0,1].
\end{aligned}$$

Consequently

$$\|Fx\| \le A\left(c + A_1A_2\left(c^{p_1}A_3 + c^{p_2}A_4 + A_5\right)\right).$$

□

3. Main Result

Theorem 3.1. *Under the assumptions (H1) and (H2), the BVP (1.1) has at least two non trivial bounded nonnegative classical solutions x_1, x_2 in $\mathcal{C}^2([0,1]) \cap \mathcal{C}^4((0,1))$.*

Proof. Consider the Banach space $E = \mathcal{C}^2([0,1])$ endowed with the norm

$$\|x\| = \max\{\max_{t\in[0,1]} |x(t)|, \max_{t\in[0,1]} |x'(t)|, \max_{t\in[0,1]} |x''(t)|\},$$

and the positive cone

$$\mathcal{P} = \{x \in E : x \geq 0 \quad \text{on} \quad [0,1]\}.$$

For $x \in E$, define the operators

$$\begin{aligned} Tx(t) &= (1+m\varepsilon)x(t) - \varepsilon\frac{L_1}{10}, \\ Sx(t) &= -\varepsilon Fx(t) - m\varepsilon x(t) - \varepsilon\frac{L_1}{10}, \ t \in [0,1], \end{aligned}$$

where ε, L_1 are positive constants, m > 0 is large enough and the operator F is given by formula (2.6). Note that any fixed point $x \in E$ of the operator $T + S$ is a solution to the BVP (1.1).
Let r_1 and R_1 be positive constants that satisfy the following conditions

$$r_1 < L_1 < \frac{R_1}{\frac{2}{5m} + 1},$$

$$A\left(R_1 + A_1A_2\left(R_1^{p_1}A_3 + R_1^{p_2}A_4 + A_5\right)\right) \leq \frac{L_1}{5}, \tag{3.1}$$

where A is the constant which appears in (2.5). Define

$$\begin{aligned} \mathcal{P}_{r_1} &= \{v \in \mathcal{P} : \|v\| < r_1\}, \\ \mathcal{P}_{L_1} &= \{v \in \mathcal{P} : \|v\| < L_1\}, \\ \mathcal{P}_{R_1} &= \{v \in \mathcal{P} : \|v\| < R_1\}, \\ R_2 &= \frac{(1+m\varepsilon)R_1 + \varepsilon A\left(R_1 + A_1A_2\left(R_1^{p_1}A_3 + R_1^{p_2}A_4 + A_5\right)\right) + \varepsilon\frac{L_1}{5}}{1+m\varepsilon}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \leq R_2\}. \end{aligned}$$

The proof of our result is based on Theorem 2.8 and it is divided into 5 steps.

Step 1. For $x_1, x_2 \in \Omega$, we have

$$\|Tx_1 - Tx_2\| = (1+m\varepsilon)\|x_1 - x_2\|,$$

whereupon $T : \Omega \to E$ is an expansive operator with a constant $1 + m\varepsilon > 1$.

Step 2 We prove that $I - S$ is completely continuous operator.

1. $I - S$ is continuous. Indeed, let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in E. We have

$$|(I-S)x_n(t) - (I-S)x(t)| \leq \varepsilon|Fx_n(t) - Fx(t)| + (1+m\varepsilon)|x_n(t) - x(t)|, \quad \forall t \in [0,1]. \tag{3.2}$$

Note that $f(\cdot,\cdot,\cdot)$ is uniformly continuous on $[0,1]\times[0,M]\times[0,M]$ for any positive constant M. Take $\varepsilon>0$. Then there is an $N\in\mathbb{N}$ so that

$$
\begin{aligned}
|x_n(s)-x(s)| &< \varepsilon,\\
|f(s,x_n(s),x_n''(s))-f(s,x(s),x''(s)| &< \varepsilon
\end{aligned}
$$

for any $s\in[0,1]$ and for any $n\geq N$, $n\in\mathbb{N}$. Hence,

$$
\begin{aligned}
&|Fx_n(t)-Fx(t)|\\
\leq\ & \int_0^t (t-s)^2 g(s)\left(|x_n(s)-x(s)|\right.\\
+\ & \left.\int_0^1 |H(s,s_1)|w(s_1)|f\left(s_1,x_n(s_1),x_n''(s_1)\right)-f\left(s_1,x(s_1),x''(s_1)\right)|ds_1\right)ds\\
<\ & \varepsilon\left(\int_0^1 g(s)\left(1+A_1A_2\int_0^1 w(s_1)ds_1\right)ds\right)\\
=\ & \varepsilon\left(1+A_1A_2\int_0^1 w(s_1)ds_1\right)\left(\int_0^1 g(s)ds\right),\quad t\in[0,1],\quad n\geq N.
\end{aligned}
$$

So, $|Fx_n(t)-Fx(t)|\to0$, as $n\to\infty$. Thus $|(I-S)x_n(t)-(I-S)x(t)|\to0$, as $n\to\infty$. In the same way we prove that $|((I-S)x_n)'(t)-((I-S)x)'(t)|\to0$ and $|((I-S)x_n)''(t)-((I-S)x)''(t)|\to0$, as $n\to\infty$, and then conclude that $Sx_n\to Sx$, as $n\to\infty$ in E, which ends the proof.

2. $(I-S)(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Indeed, For $x\in\overline{\mathcal{P}_{R_1}}$, we get

$$
\begin{aligned}
\|(I-S)x\| &\leq \varepsilon\|Fx\|+(1+m\varepsilon)\|x\|+\varepsilon\frac{L_1}{10}\\
&\leq \varepsilon A\left(R_1+A_1A_2\left(R_1^{p_1}A_3+R_1^{p_2}A_4+A_5\right)\right)+(1+m\varepsilon)R_1+\varepsilon\frac{L_1}{10}.
\end{aligned}
$$

3. $(I-S)(\overline{\mathcal{P}_{R_1}})$ is equicontinuous in E. Indeed, let $t_1,t_2\in[0,1], t_1<t_2$ and $x\in\overline{\mathcal{P}_{R_1}}$. Then, we deduce

$$
\begin{aligned}
&|Fx(t_1)-Fx(t_2)|\\
=\ & \Big|\int_0^{t_1}(t_1-s)^2g(s)\left(-x(s)+\int_0^1 H(s,s_1)w(s_1)f(s_1,x(s_1),x''(s_1))ds_1\right)ds\\
&-\int_0^{t_2}(t_2-s)^2g(s)\left(-x(s)+\int_0^1 H(s,s_1)w(s_1)f(s_1,x(s_1),x''(s_1))ds_1\right)ds\Big|\\
\leq\ & \int_0^{t_1}\left((t_1-s)^2-(t_2-s)^2\right)g(s)\left(|x(s)|+\int_0^1 |H(s,s_1)|w(s_1)|f(s_1,x(s_1),x''(s_1))|ds_1\right)ds\\
&+\int_{t_1}^{t_2}(t_2-s)^2g(s)\left(|x(s)|+\int_0^1 |H(s,s_1)|w(s_1)|f(s_1,x(s_1),x''(s_1))|ds_1\right)ds\\
\leq\ & \int_0^1\left((t_1-s)^2-(t_2-s)^2\right)g(s)\left(|x(s)|+\int_0^1 |H(s,s_1)|w(s_1)|f(s_1,x(s_1),x''(s_1))|ds_1\right)ds\\
&+\int_{t_1}^{t_2}(1-s)^2g(s)\left(|x(s)|+\int_0^1 |H(s,s_1)|w(s_1)|f(s_1,x(s_1),x''(s_1))|ds_1\right)ds\\
&\to0,\quad \text{as } |t_1-t_2|\to0.
\end{aligned}
$$

Similarly,

$$|(Fx)'(t_2) - (Fx)'(t_1)| \to 0, \quad \text{as } |t_1 - t_2| \to 0,$$

and

$$|(Fx)''(t_2) - (Fx)''(t_1)| \to 0, \quad \text{as } |t_1 - t_2| \to 0.$$

Consequently,

$$\begin{aligned} & |(I-S)x(t_2) - (I-S)x(t_1)| \\ \leq\ & \varepsilon|Fx(t_2) - Fx(t_1)| + (1+\varepsilon m)|x(t_2) - x(t_1)| \to 0, \quad \text{as } |t_1 - t_2| \to 0, \end{aligned}$$

$$\begin{aligned} & |((I-S)x)'(t_2) - ((I-S)x)'(t_1)| \\ \leq\ & \varepsilon|(Fx)'(t_2) - (Fx)'(t_1)| + (1+\varepsilon m)|x'(t_2) - x'(t_1)| \to 0, \quad \text{as } |t_1 - t_2| \to 0, \end{aligned}$$

$$\begin{aligned} & |((I-S)x)''(t_2) - ((I-S)x)''(t_1)| \\ \leq\ & \varepsilon|(Fx)''(t_2) - (Fx)''(t_1)| + (1+\varepsilon m)|x''(t_2) - x''(t_1)| \to 0, \quad \text{as } |t_1 - t_2| \to 0. \end{aligned}$$

Therefore, $(I-S)(\overline{\mathcal{P}_{R_1}})$ is equicontinuous.

According to the Arzelà-Ascoli compactness criterion, we conclude that the operator $(I-S) : \overline{\mathcal{P}_{R_1}} \to E$ is completely continuous.

Step 3. Let $u \in \overline{\mathcal{P}_{R_1}}$ be arbitrarily chosen. Then

$$\begin{aligned} (I-S)u &= u - Su \\ &= u + \varepsilon Fu + m\varepsilon u + \varepsilon \frac{L_1}{10} \\ &= (1+m\varepsilon)u + \varepsilon Fu + \varepsilon \frac{L_1}{10}. \end{aligned}$$

Set

$$v = \frac{(1+m\varepsilon)u + \varepsilon Fu + \varepsilon \frac{L_1}{5}}{1+m\varepsilon}.$$

By Lemma 2.10 and the condition (3.1), it follows

$$\begin{aligned} -\frac{L_1}{5} &\leq -A\left(R_1 + A_1 A_2 \left(R_1^{p_1} A_3 + R_1^{p_2} A_4 + A_5\right)\right) \\ &\leq Fu \\ &\leq A\left(R_1 + A_1 A_2 \left(R_1^{p_1} A_3 + R_1^{p_2} A_4 + A_5\right)\right) \\ &\leq \frac{L_1}{5}. \end{aligned}$$

Therefore $Fu + \frac{L_1}{5} \geq 0$ and $v \geq 0$. Moreover,

$$\begin{aligned} \|v\| &\leq \frac{(1+m\varepsilon)R_1 + \varepsilon A\left(R_1 + A_1 A_2 \left(R_1^{p_1} A_3 + R_1^{p_2} A_4 + A_5\right)\right) + \varepsilon \frac{L_1}{5}}{1+m\varepsilon} \\ &= R_2. \end{aligned}$$

Therefore $v \in \Omega$ and

$$\begin{aligned} Tv &= (1+m\varepsilon)v - \varepsilon\frac{L_1}{10} \\ &= (1+m\varepsilon)u + \varepsilon Fu + \varepsilon\frac{L_1}{10} \\ &= (I-S)u. \end{aligned}$$

Thus, $(I-S)(\overline{\mathcal{P}_{R_1}}) \subset T(\Omega)$.

Step 4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda_0 > 0$ and $x_0 \in \partial\mathcal{P}_{r_1} \cap (\Omega + \lambda_0 u_0)$ or $x_0 \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda_0 u_0)$ such that

$$(I-S)x_0 = T(x_0 - \lambda_0 u_0).$$

Then

$$\varepsilon Fx_0(t) + (1+\varepsilon m)x_0(t) + \varepsilon\frac{L_1}{10} = (1+\varepsilon m)(x_0(t) - \lambda_0 u_0(t)) - \varepsilon\frac{L_1}{10},\ t \in [0,1].$$

Whereupon,

$$Fx_0(t) = -\lambda_0\frac{1+\varepsilon m}{\varepsilon}u_0(t) - \frac{L_1}{5},\ t \in [0,1].$$

So,

$$\|Fx_0\| = \|\lambda_0\frac{1+\varepsilon m}{\varepsilon}u_0 + \frac{L_1}{5}\| > \frac{L_1}{5},$$

which contradicts Lemma 2.10 and the inequality (3.1).

Step 5. Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $\lambda_1 \geq \varepsilon_1 + 1$ and $x_1 \in \partial\mathcal{P}_{L_1}$, $\lambda_1 x_1 \in \overline{\mathcal{P}_{R_2}}$ such that

$$(I-S)x_1 = T(\lambda_1 x_1). \tag{3.3}$$

Note that $x_1 \in \partial\mathcal{P}_{L_1}$ and $\lambda_1 x_1 \in \overline{\mathcal{P}_{R_2}}$ imply

$$\left(\frac{2}{5m} + 1\right) L_1 \leq \lambda_1 L_1 = \lambda_1\|x_1\| \leq R_2.$$

Then, using the equation (3.3) and the definitions for the operators T and S, we get

$$\varepsilon Fx_1 + (1+m\varepsilon)x_1 + \varepsilon\frac{L_1}{10} = \lambda_1(1+m\varepsilon)x_1 - \varepsilon\frac{L_1}{10},$$

or

$$\varepsilon(Fx_1 + \frac{L_1}{5}) = (\lambda_1 - 1)(1+m\varepsilon)x_1.$$

Hence,

$$2\frac{L_1}{5}\varepsilon \geq \varepsilon\|Fx_1 + \frac{L_1}{5}\| = (\lambda_1 - 1)(1+m\varepsilon)\|x_1\| = (\lambda_1 - 1)(1+m\varepsilon)L_1,$$

or

$$\lambda_1 \leq \frac{\frac{2}{5}\varepsilon}{1+m\varepsilon} + 1 < \frac{\frac{2}{5}\varepsilon}{m\varepsilon} + 1 = \frac{2}{5m} + 1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.8 hold for $U_1 = \mathcal{P}_{r_1}$, $U_2 = \mathcal{P}_{L_1}$ and $U_3 = \mathcal{P}_{R_1}$. Hence, the BVP (1.1) has at least two solutions x_1 and x_2 such that $x_1 \in (\mathcal{P}_{L_1} \backslash \mathcal{P}_{r_1}) \cap \Omega$, $x_2 \in \left(\overline{\mathcal{P}_{R_1}} \backslash \overline{\mathcal{P}_{L_1}}\right) \cap \Omega$ and

$$r_1 \leq \|x_1\| < L_1 < \|x_2\| \leq R_1.$$

□

4. Concluding remarks

In [15], the BVP (1.1) is investigated in the case when

(A1) w may be singular at $t = 0$ and (or) $t = 1$, $w \in L^1([0,1])$, $f : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$ is continuous, h_1, h_2, k_1, $k_2 \in L^1([0,1])$ are nonnegative with $\mu_1 > 0$, $\nu_1 > 0$, $\mu_2 > 0$, $\nu_2 > 0$.

If $(A1)$ holds and $Nf_0 > 1$, $Nf_\infty > 1$, and there exists $b > 0$ such that $\max\limits_{t\in[0,1], 0<|x|+|y|\le b} f(t,x,y) < \frac{b}{L}$, where

$$f_\beta = \liminf_{|x|+|y|\to\beta} \min_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|}, \quad \beta = 0, \quad \beta = \infty,$$

and

$$\begin{aligned}
L &= \left(\frac{\eta_1\eta_2}{16} + \frac{\eta_2}{4}\right) \int_0^1 w(s)ds, \\
N &= \left(\frac{\rho_1\rho_2}{120} + \frac{\rho_2}{4}\right) \delta^2 \int_\delta^{1-\delta} e(s)w(s)ds, \\
\eta_1 &= \frac{m_1 + n_1 + \mu_1(1-\nu_1)}{m_1\nu_1 + n_1\mu_1}, \quad \eta_2 = \frac{m_2 + n_2 + \mu_2(1-\nu_2)}{m_2\nu_2 + n_2\mu_2}, \\
\rho_1 &= \frac{1}{m_1\nu_1 + n_1\mu_1} \left(\mu_1 \int_0^1 e(\tau)k_1(\tau)d\tau + \nu_1 \int_0^1 e(\tau)h_1(\tau)d\tau\right), \\
\rho_2 &= \frac{1}{m_2\nu_2 + \nu_2\mu_2} \left(\mu_2 \int_0^1 e(\tau)k_2(\tau)d\tau + \nu_2 \int_0^1 e(\tau)h_2(\tau)d\tau\right), \\
e(t) &= t(1-t), \quad t \in [0,1],
\end{aligned}$$

in [15], it is proved that the BVP (1.1) has at least two positive solutions.
Moreover, if $(A1)$ holds and $Lf^0 < 1$, $Lf^\infty < 1$, and there exist $\delta \in \left(0, \frac{1}{2}\right)$ and $B > 0$ such that $f(t,x,y) > \frac{\delta^2 B}{N}$ for all $t \in J_\delta$, $x \in [\delta^2 B, B]$, $y \in [-B, -\delta^2 B]$, where $J_\delta = [\delta, 1-\delta]$,

$$f^\beta = \limsup_{|x|+|y|\to\beta} \max_{t\in[0,1]} \frac{f(t,x,y)}{|x|+|y|}, \quad \beta = 0, \quad \beta = \infty,$$

in [15], it is proved that the BVP (1.1) has at least two positive solutions.

When $\mu_1 < 0$ or $\nu_1 < 0$, or $\mu_2 < 0$, or $\nu_2 < 0$, then we can not apply the results in [15] and we can apply our main result. Thus, our main result and the results in [15] are complementary.

5. Example

Let

$$\begin{aligned}
r_1 &= 1, \quad L_1 = 10, \quad R_1 = 20, \\
p_1 &= 2, \quad p_2 = 4, \quad m = 1000, \quad A = \frac{1}{10^{10}}.
\end{aligned}$$

Let also,

$$h_1(s) = h_2(s) = k_1(s) = k_2(s) = 4s, \quad a_1(s) = a_2(s) = a_3(s) = \frac{1}{3}, \quad w(s) = \frac{1}{\sqrt{s}}, \quad s \in [0,1].$$

Then

$$
\begin{aligned}
m_1 &= m_2 = 4\int_0^1 s^2 ds = \frac{4}{3}, \\
n_1 &= n_2 = 1 - \frac{4}{3} = -\frac{1}{3}, \\
\mu_1 &= \mu_2 = \nu_1 = \nu_2 = 1 - 4\int_0^1 s\, ds = -1 < 0, \\
\mathbb{K}_1 &= \mathbb{K}_2 = \mathbb{H}_1 = \mathbb{H}_2 = 4\int_0^1 s\, ds = 2, \\
A_1 &= A_2 = 1 + \left(\frac{4}{3} + 1\right)\cdot 2 + \left(\frac{1}{3} + 1\right)\cdot 2 = \frac{25}{3}, \\
A_3 &= A_4 = A_5 = \frac{1}{3}\int_0^1 \frac{ds}{\sqrt{s}} = \frac{2}{3}.
\end{aligned}
$$

Then

$$
\begin{aligned}
A\left(R_1 + A_1 A_2 \left(R_1^{p_1} A_3 + R_1^{p_2} A_4 + A_5\right)\right) &= \frac{1}{10^{10}}\left(20 + \frac{625}{9}\cdot\frac{2}{3}\cdot\left(20^2 + 20^4 + 1\right)\right) \\
&= \frac{1}{10} \\
&< 2 = \frac{L_1}{5}.
\end{aligned}
$$

$$
\frac{R_1}{L_1} = 2 > \frac{2}{5000} + 1 = \frac{2}{5m} + 1.
$$

Let $g(s) = \frac{1}{10^3}$, $s \in [0,1]$. Then

$$
\int_0^1 ((1-s)^2 + 2(1-s) + 2) g(s) ds = \frac{1}{10^3}\int_0^1 (s^2 - 4s + 5) ds = \frac{1}{3\cdot 10^2} < A.
$$

Consequently the BVP

$$
\begin{aligned}
x^{(4)}(t) &= \frac{1}{\sqrt{t}}\left(\frac{e^{-5t}\cos t (x(t))^2}{60\left(1 + (x''(t))^2 + 2(x(t))^4 + 3(x(t))^6 (x''(t))^8\right)} + \frac{(x''(t))^4}{30\left(1 + (x''(t))^8\right)}\right), \quad t \in (0,1), \\
x(0) &= x(1) = 4\int_0^1 s x(s) ds, \quad x''(0) = x''(1) = 4x'(1),
\end{aligned}
$$

has at least two nonnegative solutions.

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References

[1] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970), 75-88.

[2] R. P. Agarwal, On fourth order boundary value problems arising in beam analysis, Differential and integral equations 2 (1989), 91–110.

[3] R. P. Agarwal, S. Kelevedjiev, On the solvability of fourth-order two-point boundary value problems, Mathematics 2020, 8, 603.

[4] K. Bachouche, A. Benmezai, S. Djebali, Positive solutions to semi-positone fourth-order ϕ-Laplacian BVPs, Positivity 21 (2017), 193–212.

[5] S. Benslimane, S. Djebali, K. Mebarki, On the fixed point index for sums of operators, Fixed Point Theory, 23(2022), no. 1, 143–162.

[6] S. Djebali, T. Moussaoui, R. Precup, Fourth order p-laplacian nonlinear systems via the vector version of the Krasnosel'skii's fixed point theorem, Mediterr. J. Math 6 (2009), no 4, 447–460.

[7] S. Djebali, K. Mebarki, Fixed point index theory for perturbation of expansive mappings by k-set contraction, Top. Meth. Nonli. Anal., 54 (2019), no 2A, 613–640.

[8] D. Guo, V. Lakshmikantham, Nonlinear problems in abstract cones, Academic Press, Boston, Mass, USA, vol. 5, (1988).

[9] C. Gupta, Existence and uniqueness results for the bending of an elastic beam equation at resonnance, Journal of Mathematical Analysis and Applications, 135(1988), 208–225.

[10] L. Lin, Y. Liu, D. Zhao, Multiple solutions for a class of nonliner fourth-order boundary value problems, Symmetry 2020, 12, 1989.

[11] B. Liu, Positive solutions of the fourth-order two point boundary value problems, Appl. Math. Comput, 148 (2004), no. 2, 407–420.

[12] Y. Liu, D. O'Regan, Multiplicity results for a class of fourth order semipositone m-point boundary value problems, Appl. Anal. 91(2012), 911–921.

[13] R. Ma, H. Wang, On the existence of positive solutions of fourth-order ordinary differential equation, Anal. Appl. 59(1-4)(1995), 225–231.

[14] S. Reich, Fixed points of condensing functions, J. Math. Anal. Appl. 41 (1973) 460–467.

[15] Q. Wang, Y. Guo, Y. Ji, Positive solutions for fourth–order nonlinear differential equation with integral boundary conditions, Discrete Dynamics in Nature and Society, Vol. 2013, Article ID 684962, 10 pages.

[16] T. Xiang, R. Yuan, A class of expansive-type Krasnosel'skii fixed point theorems, Nonlinear Anal. 71 (2009), no. 7-8, 3229–3239.

[17] C. Zhai, C. Hiang, Existence of nontrivial solutions for a nonlinear fourth-order boundary value problem via iterative method, J. Nonlinear Sci. Appl. 9 (2016), 4295–4304.

[18] Y. Zhu, P. Weng, Multiple positive solutions for a fourth-order boundary value problem, Bol. Soc. Parana. Mat, 21(2003), 9–19.