Journal of New Results in Science 11(1) (2022) 1-12

Journal of New Results in Science

https://dergipark.org.tr/en/pub/jnrs

Research Article

Open Access

E-ISSN: 1304-7981 https://doi.org/10.54187/jnrs.978216

Inequalities for 3-convex functions and applications

Praphull Chhabra[1]

Keywords
Contra harmonic mean coefficient, Taylor's series with remainders, 3-Convex functions, Jain-Saraswat's functional coefficient, Chi-m coefficients

Abstract — In this article, we derived new information inequalities on Jain-Saraswat's functional coefficient of distance (2013) for 3-convex functions. Further, we evaluated some important relations among Relative Jensen Shannon coefficient of distance, Relative Arithmetic Geometric coefficient of distance, Triangular discrimination, Chi-square coefficient of distance and many more. Moreover, we explained the series version of this functional coefficient of distance by using the Taylor's series with both Lagrange's and Cauchy's form of remainders.

Subject Classification (2020): 94A17, 26D15.

1. Introduction

Coefficient of distances are used in measuring the distance or affinity among finite number of probability distributions (both discrete and continuous). Actually, these are for quantifying the dissimilarity among probability distributions. Some researchers, such as Csiszar, Bregman, Burbea-Rao, Lin-Wong and Jain-Saraswat's, took a deep study on generalized functional coefficient of distances. After putting a suitable function in these generalized coefficient of distance measures, some famous coefficient of distances can be obtained, like: Kullback Leibler coefficient of distance, J-coefficient of distance, Arithmetic geometric mean coefficient of distance, Jensen Shannon mean coefficient of distance, Bhattacharya Coefficient of distance and many more.

Definition 1.1. Convex function: A function $g(y)$ is said to be convex over an interval (a,b) if for every $y_1, y_2 \in (a,b)$ and $0 \le \mu \le 1$, we have

$$g\left[\mu y_1 + (1-\mu) y_2\right] \le \mu g(y_1) + (1-\mu) g(y_2) \tag{1.1}$$

and said to be strictly convex if equality does not hold only if $\mu \ne 0$ or $\mu \ne 1$.

[1]praphull.chhabra@uem.edu.in (Corresponding Author)
[1]Department of Mathematics and Statistics, University of Engineering and Management, Jaipur-303807 (Rajasthan), India
Article History: Received: 04 Aug 2021 - Accepted: 02 Feb 2022 - Published: 30 Apr 2022

Now, we first state the divided difference and then will define the n-convex functions.

Definition 1.2. Divided difference: The n^{th} order divided difference of a function $g:[a,b]\rightarrow R$ at mutually distinct points $y_0, y_1, y_2, ..., y_n \in [a,b]$ is defined by

$$g\left[y_0, y_1, y_2, ..., y_n\right] = \frac{g\left[y_1, y_2, ..., y_n\right] - g\left[y_0, y_1, y_2, ..., y_{n-1}\right]}{y_n - y_0} \tag{1.2}$$

where the value $g\left[y_0, y_1, y_2, ..., y_n\right]$ is independent of the order of the points $y_0, y_1, y_2, ..., y_n$.

So, 3^{rd} order divided difference can be easily defined as

$$g\left[y_0, y_1, y_2, y_3\right] = \frac{g\left[y_1, y_2, y_3\right] - g\left[y_0, y_1, y_2\right]}{y_3 - y_0} \tag{1.3}$$

Remark 1.3. Definition of the 3^{rd} order divided differences can be extended to include the following cases [1], in which some or all the points coincide:

R_1: If function g is differentiable on $[a,b]$, then we have

$$g\left[y, y, y_0, y_1\right] = \frac{g'(y)}{(y-y_0)(y-y_1)} + \frac{g(y)(y_0+y_1-2y)}{(y-y_0)^2(y-y_1)^2} + \frac{g(y_0)}{(y_0-y)^2(y_0-y_1)} + \frac{g(y_1)}{(y_1-y)^2(y_1-y_0)} \tag{1.4}$$

for three mutually distinct points $y, y_0, y_1 \in [a,b]$.

R_2: If function g is differentiable on $[a,b]$, we have

$$g\left[y, y, y_0, y_0\right] = \frac{(y_0-y)\left[g'(y_0)+g'(y)\right] + 2\left[g(y)-g(y_0)\right]}{(y_0-y)^3} \tag{1.5}$$

for two mutually distinct points $y, y_0 \in [a,b]$.

R_3: If function g is twice differentiable on $[a,b]$, we have

$$g\left[y, y, y, y_0\right] = \frac{\left[g(y_0) - g(y) - (y_0-y)g'(y) - \frac{(y_0-y)^2 g''(y)}{2!}\right]}{(y_0-y)^3} \tag{1.6}$$

for two mutually distinct points $y, y_0 \in [a,b]$.

R_4: If function g is thrice differentiable on $[a,b]$ and $y \in [a,b]$, then we have

$$g\left[y, y, y, y\right] = \frac{g'''(y)}{3!} \tag{1.7}$$

Definition 1.4. ***n*-Convex function**: A function $g:[a,b]\rightarrow R$ is designated as n-Convex function if and only if $\forall$ choices of $n+1$ distinct points $y_0, y_1, y_2, ..., y_n \in [a,b]$, we have n^{th} order divided difference positive including zero, i.e., $g\left[y_0, y_1, y_2, ..., y_n\right] \geq 0$.

So, a function will be 3-convex if its 3^{rd} order divided difference is positive including zero or 3-convex functions are characterized by the third order divided difference.

Remark 1.5. Definition of the $3-$convex functions can be extended to include the following cases [1], in which some or all the points coincide:

R_5: Differentiable function g is 3-convex if and only if $g\left[y, y, y_0, y_1\right] \geq 0$ for all three mutually different points $y, y_0, y_1 \in [a, b]$.

R_6: Differentiable function g is 3-convex if and only if $g\left[y, y, y_0, y_0\right] \geq 0$ for all two mutually different points $y, y_0 \in [a, b]$.

R_7: Twice differentiable function g is 3-convex if and only if $g\left[y, y, y, y_0\right] \geq 0$ for all two mutually different points $y, y_0 \in [a, b]$.

R_8: Thrice differentiable function g is 3-convex if and only if $g\left[y, y, y, y\right] \geq 0$ for each $y, \in [a, b]$.

Convex functions have wide applications in pure and applied mathematics. Recently many generalizations and extensions have been made for the convexity, like: preinvexity [2], GA-convexity [3], strong convexity [4], s-convexity [5], and others. Also some standard inequalities have been defined for different type of convex functions, such as: for PP-convex functions [6], for harmonically convex and harmonically quasi convex functions [7], for interval-valued convex functions [8], for s-convex functions on fractal sets [9], for products of co-ordinated convex interval-valued functions [10], for (α, m)-convex mappings [11], for modified (p, h)-convex functions [12], for generalized harmonically convex functions [13] and many more.

This work is dealing with the discrete probability distributions. The outcomes are valid for the continuous distributions as well. So let $\Theta_n = \{U = (u_1, u_2, u_3, ..., u_n) : u_i > 0, \sum_{i=1}^{n} u_i = 1\}$, $n \geq 2$ be the set of all complete finite discrete probability distributions. If we take $u_i \geq 0$ for some $i = 1, 2, 3..., n$, then we have to suppose that $0g(0) = 0g\left(\frac{0}{0}\right) = 0$.

Jain and Saraswat (2013, [14]) introduced the following functional coefficient of distance;

$$S_\phi(U:W) = \sum_{i=1}^{n} w_i \phi\left(\frac{u_i + w_i}{2w_i}\right) \tag{1.8}$$

Here $\phi : (0, \infty) \to R$ (set of real no.) is real, differentiable function (preferably a convex function) and $U = (u_1, u_2, ..., u_n), W = (w_1, w_2, ..., w_n) \in \Theta_n$, where u_i and w_i are probability mass functions. Several properties, information inequalities and their applications on $S_\phi(U:W)$ can be seen in the articles [15] and [16] to [17]. Also several well known coefficient of distances can be obtained by appropriately defining a function in $S_\phi(U:W)$, like: Triangular discrimination, Chi-square coefficient of distance, Relative Jensen Shannon coefficient of distance, Relative Arithmetic geometric coefficient of distance, Variational distance, Relative J-coefficient of distance and many more (Details can bee seen Section 3).

$S_\phi(U:W)$ is a natural distance measure from a true probability distribution U to an arbitrary probability distribution W. Typically U represents a precise calculated probability distribution, whereas W represents an approximation of U. The concept of coefficient of distance measure is working efficiently to resolve different problems related to probability theory.

Now before moving to the main results, we need to define some terminologies and definitions.

Let S be a non-empty set and V be a vector space of real valued functions $g : S \to R$ (set of real numbers) having the following properties:

P_1: If $g_1, g_2 \in V$, then $\left(ag_1 + bg_2\right) \in V \;\forall\; a, b \in R$;

P_2: If $1 \in V$, i.e., if $g_1\left(y\right) = 1$ for each $y \in S$, then $g_1 \in V$.

Also, let $G : V \to R$ be a positive linear functional, having the following properties:

P_3: $G(ag_1 + bg_2) = aG(g_1) + bG(g_2)\ \forall\ g_1, g_2 \in V$ and $a, b \in R$;

P_4: $g_1 \in V, g_1(y) \geq 0$ for each $y \in S \Rightarrow G(g_1) \geq 0$.

Now, the following theorem and its proof can be seen in the literature [18].

Theorem 1.6. Let ϕ be a 3-convex function defined on an interval $I \subset R$ whose interior contains the interval $[\rho,\sigma]$ with $-\infty < \rho < \sigma < \infty$ and differentiable on $[\rho,\sigma]$, then we have

$$[G(g)-\rho]\left[\frac{\phi(\sigma)-\phi(\rho)}{\sigma-\rho}-\frac{\phi'_+(\rho)}{2}\right]-\frac{G[(g-\rho\times 1)\phi'(g)]}{2} \leq \frac{\sigma-G(g)}{\sigma-\rho}\phi(\rho)+\frac{G(g)-\rho}{\sigma-\rho}\phi(\sigma)-G[\phi(g)]$$

$$\leq \frac{G[(\sigma\times 1-g)\phi'(g)]}{2}-[\sigma-G(g)]\left[\frac{\phi(\sigma)-\phi(\rho)}{\sigma-\rho}-\frac{\phi'_-(\sigma)}{2}\right] \tag{1.9}$$

and

$$[\sigma-G(g)]\left[\phi'_-(\sigma)-\frac{\phi(\sigma)-\phi(\rho)}{\sigma-\rho}\right]-\frac{\phi''_-(\sigma)G\left[(\sigma\times 1-g)^2\right]}{2} \leq \frac{\sigma-G(g)}{\sigma-\rho}\phi(\rho)+\frac{G(g)-\rho}{\sigma-\rho}\phi(\sigma)-G[\phi(g)]$$

$$\leq [G(g)-\rho]\left[\frac{\phi(\sigma)-\phi(\rho)}{\sigma-\rho}-\phi'_+(\rho)\right]-\frac{\phi''_+(\rho)G\left[(g-\rho\times 1)^2\right]}{2} \tag{1.10}$$

where G be any positive linear functional on V with $G(1) = 1$ and V satisfy conditions P_1 and P_2 on a non empty set S. Also $g \in V$ such that $\phi o g \in V$ and $\rho \leq g(y) \leq \sigma$ for $y \in S$.

Note 1.7. N_1: If the function $-\phi$ is 3-convex then the inequalities (1.9) and (1.10) are reversed.

N_2: The inequalities (1.9) and (1.10) had been proved in literature [18] by considering remarks R_6 and R_7 respectively.

N_3: Inequality based on remark R_5 had also been proved in the same literature.

2. New Inequalities on Jain-Saraswat's Functional Coefficient of Distance

In this section, we find the new information inequalities on well known Jain-Saraswat functional coefficient of distance $S_\phi(U:W)$ for 3-convex functions defined in the interval $\left(\frac{1}{2},\infty\right]$ by using the inequalities (1.9) and (1.10).

Theorem 2.1. Let $\phi : [\rho,\sigma] \subset (0,\infty) \to R$ be a 3-convex function defined and differentiable in the interval $[\rho,\sigma]$ with $\frac{1}{2} < \rho \leq 1 \leq \sigma < \infty$. Also $U, W \in \Theta_n$ such that $\frac{u_i+w_i}{2w_i} \in [\rho,\sigma]$ for each $i = 1, 2, ..., n$. Then we have

$$(1-\rho)\left[\phi'(c)-\frac{\phi'_+(\rho)}{2}\right]-\frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{u_i+w_i}{2}-\rho w_i\right)\phi'\left(\frac{u_i+w_i}{2w_i}\right)\right] \leq \phi'(c)+D(\rho,\sigma)-S_\phi(U:W)$$

$$\leq \frac{1}{2}\sum_{i=1}^{n}\left[\left(\sigma w_i-\frac{u_i+w_i}{2}\right)\phi'\left(\frac{u_i+w_i}{2w_i}\right)\right]-(\sigma-1)\left[\phi'(c)-\frac{\phi'_-(\sigma)}{2}\right] \tag{2.1}$$

and

$$(\sigma-1)\left[\phi'_-(\sigma)-\phi'(c)\right]-\frac{\phi''_-(\sigma)}{2}\left[(\sigma-1)^2+\frac{1}{4}\chi^2(U:W)\right] \leq \phi'(c)+D(\rho,\sigma)-S_\phi(U:W)$$

$$\leq (1-\rho)\left[\phi'(c)-\phi'_+(\rho)\right]-\frac{\phi''_+(\rho)}{2}\left[(\rho-1)^2+\frac{1}{4}\chi^2(U:W)\right] \tag{2.2}$$

where $\phi'(c) = \frac{\phi(\sigma)-\phi(\rho)}{\sigma-\rho}$; $c \in [\rho,\sigma]$, $D(\rho,\sigma) = \frac{\sigma\phi(\rho)-\rho\phi(\sigma)}{\sigma-\rho}$, $S_\phi(U:W)$ is given by (1.8), and

$$\chi^2(U:W) = \sum_{i=1}^{n} \frac{(u_i - w_i)^2}{w_i} \tag{2.3}$$

is the famous Chi-square coefficient of distance or Pearson coefficient of distance [19]. If the function $-\phi$ is 3-convex then the inequalities (2.1) and (2.2) are reversed.

Proof.
Let $t = (t_1, t_2, ..., t_n)$ such that $t_i \in [\rho,\sigma]$ for $i = 1,2,...,n$. Now, replace g with t in inequalities (1.9) and (1.10) and after that put $G(t) = \sum_{i=1}^{n} u_i t_i$, we obtain

$$\left[\sum_{i=1}^{n} u_i t_i - \rho\right]\left[\frac{\phi(\sigma)-\phi(\rho)}{\sigma-\rho} - \frac{\phi'_+(\rho)}{2}\right] - \frac{\sum_{i=1}^{n} u_i (t_i-\rho)\phi'(t_i)}{2} \le \frac{\sigma - \sum_{i=1}^{n} u_i t_i}{\sigma-\rho}\phi(\rho) + \frac{\sum_{i=1}^{n} u_i t_i - \rho}{\sigma-\rho}\phi(\sigma) - \sum_{i=1}^{n} u_i\phi(t_i)$$

$$\le \frac{\sum_{i=1}^{n} u_i(\sigma-t_i)\phi'(t_i)}{2} - \left[\sigma - \sum_{i=1}^{n} u_i t_i\right]\left[\frac{\phi(\sigma)-\phi(\rho)}{\sigma-\rho} - \frac{\phi'_-(\sigma)}{2}\right] \tag{2.4}$$

and

$$\left[\sigma - \sum_{i=1}^{n} u_i t_i\right]\left[\phi'_-(\sigma) - \frac{\phi(\sigma)-\phi(\rho)}{\sigma-\rho}\right] - \frac{\phi''_-(\sigma)\left[\sum_{i=1}^{n} u_i(\sigma-t_i)^2\right]}{2} \le \frac{\sigma - \sum_{i=1}^{n} u_i t_i}{\sigma-\rho}\phi(\rho) + \frac{\sum_{i=1}^{n} u_i t_i - \rho}{\sigma-\rho}\phi(\sigma) - \sum_{i=1}^{n} u_i\phi(t_i)$$

$$\le \left[\sum_{i=1}^{n} u_i t_i - \rho\right]\left[\frac{\phi(\sigma)-\phi(\rho)}{\sigma-\rho} - \phi'_+(\rho)\right] - \frac{\phi''_+(\rho)\left[\sum_{i=1}^{n} u_i(t_i-\rho)^2\right]}{2} \tag{2.5}$$

respectively. Now replace u_i with w_i and after that put $t_i = \frac{u_i+w_i}{2w_i}$ for $i = 1,2,...,n$ with $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} w_i = 1$, we get the desired results (2.1) and (2.2) respectively.

Here we are just evaluating the second term of the inequality (2.5) for the convenience purpose only:

$$\begin{aligned}
\textstyle\sum_{i=1}^{n} u_i(\sigma-t_i)^2 &= \textstyle\sum_{i=1}^{n} w_i\left(\sigma - \frac{u_i+w_i}{2w_i}\right)^2 \\
&= \textstyle\sum_{i=1}^{n}\left[w_i\sigma^2 - \sigma(u_i+w_i) + \frac{(u_i+w_i)^2}{4w_i}\right] \\
&= \sigma^2 - 2\sigma + \textstyle\sum_{i=1}^{n}\frac{u_i^2+w_i^2-2u_iw_i+4u_iw_i}{4w_i} \\
&= \sigma^2 - 2\sigma + \textstyle\sum_{i=1}^{n}\left[\frac{(u_i-w_i)^2}{4w_i} + u_i\right] \\
&= (\sigma-1)^2 + \frac{1}{4}\chi^2(U:W)
\end{aligned}$$

3. Important Outcomes and Bounds

In this section, we obtain some important results on different well known coefficient of distances by using the inequality (2.1) and also bounds of the different coefficient of distances in terms of the famous Chi-square coefficient of distance by using the inequality (2.2).

Since the above inequalities (2.1) and (2.2) are totally based on 3-convex functions, so we will consider the 3-convex functions only, i.e., a function with its third order derivative in the given domain is positive including zero.

Example 3.1. Let

$$\phi\left(y\right)=\frac{\left(y-1\right)^{2}}{y},y\in\left[\rho,\sigma\right]\subset\left(0,\infty\right),\phi'\left(y\right)=1-\frac{1}{y^{2}}$$

$$\phi''\left(y\right)=\frac{2}{y^{3}},\phi'''\left(y\right)=-\frac{6}{y^{4}}$$

Now for the function $\phi\left(y\right)$, we have

$$\begin{aligned}\sum_{i=1}^{n}\left(u_i+w_i\right)\phi'\left(\frac{u_i+w_i}{2w_i}\right) &= \sum_{i=1}^{n}\left(u_i+w_i\right)\left[1-\frac{4w_i^2}{(u_i+w_i)^2}\right]\\ &= \sum_{i=1}^{n}\left[u_i+w_i-\frac{4w_i^2}{u_i+w_i}\right]\\ &= \sum_{i=1}^{n}\frac{u_i^2+w_i^2+2u_iw_i-4w_i^2}{u_i+w_i}\\ &= C\left(U:W\right)+H\left(U:W\right)-4\zeta\left(U:W\right)\end{aligned}$$

where $C\left(U:W\right)=\sum_{i=1}^{n}\frac{u_i^2+w_i^2}{u_i+w_i}$ is the Contra harmonic mean coefficient of distance [20], $H\left(U:W\right)=\sum_{i=1}^{n}\frac{2u_iw_i}{u_i+w_i}$ is the Harmonic mean coefficient of distance [20] and $\zeta\left(U:W\right)=\sum_{i=1}^{n}\frac{w_i^2}{u_i+w_i}$.

Moreover,

$$\begin{aligned}\sum_{i=1}^{n}w_i\phi'\left(\frac{u_i+w_i}{2w_i}\right) &= \sum_{i=1}^{n}\left[1-\frac{4w_i^2}{(u_i+w_i)^2}\right]\\ &= \sum_{i=1}^{n}\frac{w_i(u_i+3w_i)(u_i-w_i)}{(u_i+w_i)^2}\end{aligned}$$

and

$$\begin{aligned}S_\phi\left(U:W\right) &= \frac{1}{2}\sum_{i=1}^{n}\frac{(u_i-w_i)^2}{u_i+w_i}\\ &= \frac{1}{2}\Delta\left(U:W\right)\end{aligned}$$

is the famous Triangular discrimination [21].

By putting the above calculated data in the inequalities (2.1) and (2.2), we have an interesting relation among the Contra Harmonic Mean Coefficient of distance, the Harmonic Mean Coefficient of distance and the Triangular Discrimination. Also we obtain the bounds of the Triangular Discrimination in terms of the Chi-square Coefficient of distance.

Since the function $\phi\left(y\right)$ is not 3-convex but the function $-\phi\left(y\right)$ is 3-convex as for the function $-\phi\left(y\right)$ the third order derivative is positive including zero, i.e., $-\phi'''\left(y\right)=\frac{6}{y^4}>0$, so naturally the outcomes will be reversed in sign.

Example 3.2. Let

$$\phi\left(y\right)=-\log y,y\in\left[\rho,\sigma\right]\subset\left(0,\infty\right),\phi'\left(y\right)=-\frac{1}{y}$$

$$\phi''\left(y\right)=\frac{1}{y^{2}},\phi'''\left(y\right)=-\frac{2}{y^{3}}$$

Now for the function $\phi\left(y\right)$, we have

$$\begin{aligned}\sum_{i=1}^{n}\left(u_i+w_i\right)\phi'\left(\frac{u_i+w_i}{2w_i}\right) &= -\sum_{i=1}^{n}\left(u_i+w_i\right)\left(\frac{2w_i}{u_i+w_i}\right)\\ &= -\sum_{i=1}^{n}2w_i\\ &= -2\times 1\\ &= -2A\left(U:W\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n} w_i\phi'\left(\frac{u_i+w_i}{2w_i}\right) &= -\sum_{i=1}^{n} w_i \frac{2w_i}{u_i+w_i}\\ &= -2\zeta(U:W)\end{aligned}$$

where $A(U:W)=\sum_{i=1}^{n}\left(\frac{u_i+w_i}{2}\right)$ is the Arithmetic mean coefficient of distance [20].

Furthermore,

$$\begin{aligned}S_\phi(U:W) &= \sum_{i=1}^{n} w_i \log\left(\frac{2w_i}{u_i+w_i}\right)\\ &= F(W:U)\end{aligned}$$

is the ad-joint of the Relative Jensen Shannon coefficient of distance $F(U:W)$ [22].

By putting the above calculated data in the inequalities (2.1) and (2.2), we get a outcome between the Relative Jensen Shannon coefficient of distance and the Arithmetic mean coefficient of distance. Also we obtain the bounds of the Relative Jensen Shannon coefficient of distance in terms of the Chi-square Coefficient of distance.

Since the function $\phi(y)$ is not 3-convex but the function $-\phi(y)$ is 3-convex as $-\phi'''(y)=\frac{2}{y^3}>0$, so naturally the outcomes will be reversed in sign.

Example 3.3. Let

$$\phi(y)=y\log y,\ y\in[\rho,\sigma]\subset(0,\infty),\phi'(y)=1+\log y$$

$$\phi''(y)=\frac{1}{y},\phi'''(y)=-\frac{1}{y^2}$$

Now for the function $\phi(y)$, we have

$$\begin{aligned}\sum_{i=1}^{n}(u_i+w_i)\phi'\left(\frac{u_i+w_i}{2w_i}\right) &= \sum_{i=1}^{n}(u_i+w_i)\left[1+\log\frac{u_i+w_i}{2w_i}\right]\\ &= \sum_{i=1}^{n}(u_i+w_i)+\sum_{i=1}^{n}\left[(u_i+w_i)\log\frac{u_i+w_i}{2w_i}\right]\\ &= 2+2G(W:U)\\ &= 2[1+G(W:U)]\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} w_i\phi'\left(\frac{u_i+w_i}{2w_i}\right) &= \sum_{i=1}^{n} w_i\left[1+\log\frac{u_i+w_i}{2w_i}\right]\\ &= 1-\sum_{i=1}^{n} w_i\log\frac{2w_i}{u_i+w_i}\\ &= \sum_{i=1}^{n}\frac{u_i+w_i}{2}-F(W:U)\\ &= A(U:W)-F(W:U)\end{aligned}$$

and

$$\begin{aligned}S_\phi(U:W) &= \sum_{i=1}^{n}\frac{u_i+w_i}{2}\log\left(\frac{u_i+w_i}{2w_i}\right)\\ &= G(W:U)\end{aligned}$$

where $G(W:U)$ is the ad-joint of the Relative Arithmetic geometric coefficient of distance $G(U:W)$ [23].

By putting the above calculated data in the inequalities (2.1) and (2.2), we obtain a relation among the Relative Jensen Shannon coefficient of distance, the Relative Arithmetic geometric coefficient of distance and the Arithmetic mean coefficient of distance. Also we obtain the bounds of the Relative Arithmetic geometric coefficient of distance in terms of the Chi-square Coefficient of distance.

Since the function $\phi(y)$ is not 3-convex but the function $-\phi(y)$ is 3-convex as $-\phi'''(y) = \frac{1}{y^2} > 0$, so naturally the outcomes will be reversed in sign.

Example 3.4. Let

$$\phi(y) = (y-1)\log y, y \in [\rho,\sigma] \subset (0,\infty), \phi'(y) = 1 - \frac{1}{y} + \log y$$

$$\phi''(y) = \frac{1}{y} + \frac{1}{y^2}, \phi'''(y) = -\frac{1}{y^2} - \frac{2}{y^3}$$

Now for the function $\phi(y)$, we have

$$\begin{aligned} \sum_{i=1}^{n} (u_i + w_i)\phi'\left(\frac{u_i+w_i}{2w_i}\right) &= \sum_{i=1}^{n} (u_i + w_i)\left[1 - \frac{2w_i}{u_i+w_i} + \log\frac{u_i+w_i}{2w_i}\right] \\ &= \sum_{i=1}^{n} (u_i + w_i) - 2\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} (u_i + w_i)\log\frac{u_i+w_i}{2w_i} \\ &= 2 - 2 + 2G(W:U) \\ &= 2G(W:U) \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{n} w_i\phi'\left(\frac{u_i+w_i}{2w_i}\right) &= \sum_{i=1}^{n} w_i\left[1 - \frac{2w_i}{u_i+w_i} + \log\frac{u_i+w_i}{2w_i}\right] \\ &= A(U:W) - 2\zeta(U:W) - F(W:U) \end{aligned}$$

and

$$\begin{aligned} S_\phi(U:W) &= \frac{1}{2}\sum_{i=1}^{n} (u_i - w_i)\log\left(\frac{u_i+w_i}{2w_i}\right) \\ &= \frac{1}{2}J_R(U:W) \end{aligned}$$

where $J_R(U:W)$ is known as the Relative J-coefficient of distance [24].

By putting the above calculated data in the inequalities (2.1) and (2.2), we obtain an interesting relation among the Relative Jensen Shannon coefficient of distance, the Relative Arithmetic geometric coefficient of distance, the Relative J-coefficient of distance and the Arithmetic mean coefficient of distance. Also we obtain the bounds of the Relative J-coefficient of distance in terms of the Chi-square Coefficient of distance.

Moreover, the function $\phi(y)$ is not 3-convex but the function $-\phi(y)$ is 3-convex as $-\phi'''(y) = \frac{1}{y^2} + \frac{2}{y^3} > 0$, so naturally the outcomes will be reversed in sign.

Example 3.5. Let

$$\phi(y) = |y-1| = \begin{cases} y-1 & \text{if } y \geq 1 \\ 1-y & \text{if } 0 < y < 1 \end{cases}, y \in [\rho,\sigma] \subset (0,\infty), \phi'(y) = \begin{cases} 1 & \text{if } y \geq 1 \\ -1 & \text{if } 0 < y < 1 \end{cases}$$

$$\phi''(y) = 0, \phi'''(y) = 0$$

Now for the function $\phi(y)$, we have

$$\sum_{i=1}^{n} (u_i + w_i)\phi'\left(\frac{u_i+w_i}{2w_i}\right) = \begin{cases} 2 & \text{if } y \geq 1 \\ -2 & \text{if } 0 < y < 1 \end{cases}, \quad \sum_{i=1}^{n} w_i\phi'\left(\frac{u_i+w_i}{2w_i}\right) = \begin{cases} 1 & \text{if } y \geq 1 \\ -1 & \text{if } 0 < y < 1 \end{cases}$$

and

$$S_\phi(U:W) = \frac{1}{2}\sum_{i=1}^{n} |u_i - w_i| = \frac{1}{2}V(U:W)$$

where $V(U:W)$ is the Variational distance [25].

Therefore, by putting the above evaluated data in the inequalities (2.1) and (2.2), we obtain a relation in terms of the Variational distance. Also we obtain the bounds of the Variational distance in terms of the Chi-square Coefficient of distance.

Remark 3.6. Similarly, we can take some more functions defined in the interval $\left(\frac{1}{2},\infty\right)$ and can get the several important relations among several coefficient of distances and bounds as well. It is very important to understand that the following functions are not 3-convex but negation is 3-convex in the domain $\left(\frac{1}{2},\infty\right)$ only. We will not go through the proofs of the results as these can be easily evaluated by the same procedure as we followed in the above examples. The functions and corresponding coefficient of distances are as follows:

1. $\phi_1(y) = \left(y-\frac{1}{2}\right)\log(2y-1) - y\log y$, $\phi_1'''(y) = -\frac{4y-1}{y^2(2y-1)^2}$, $-\phi_1'''(y) = \frac{4y-1}{y^2(2y-1)^2} > 0 \quad \forall\ y \in \left(\frac{1}{2},\infty\right) \Rightarrow$ $S_{\phi_1}(U:W) = \frac{1}{2}\left[\sum_{i=1}^{n} u_i \log\left(\frac{2u_i}{u_i+w_i}\right) + \sum_{i=1}^{n} w_i \log\left(\frac{2w_i}{u_i+w_i}\right)\right]$ = JS coefficient of distance [22, 26].
2. $\phi_2(y) = \frac{\left(1-\sqrt{2y-1}\right)^2}{2}$, $\phi_2'''(y) = -\frac{3}{(2y-1)^{\frac{5}{2}}}$, $-\phi_2'''(y) = \frac{3}{(2y-1)^{\frac{5}{2}}} > 0 \quad \forall\ y \in \left(\frac{1}{2},\infty\right) \Rightarrow S_{\phi_2}(U:W) = \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{u_i}-\sqrt{w_i}\right)^2$ = Hellinger discrimination [27].
3. $\phi_3(y) = \frac{8y(y-1)^2}{2y-1}$, $\phi_3'''(y) = -\frac{48}{(2y-1)^4}$, $-\phi_3'''(y) = \frac{48}{(2y-1)^4} > 0 \ \forall\ y \in R - \left(\frac{1}{2}\right) \Rightarrow S_{\phi_3}(U:W) = \sum_{i=1}^{n}\frac{(u_i-w_i)^2(u_i+w_i)}{u_i w_i}$ = Symmetric chi-square coefficient of distance [28].
4. $\phi_4(y) = 2(y-1)\log(2y-1)$, $\phi_4'''(y) = -\frac{8(2y+1)}{(2y-1)^3}$, $-\phi_4'''(y) = \frac{8(2y+1)}{(2y-1)^3} > 0 \ \forall\ y \in \left(\frac{1}{2},\infty\right) \Rightarrow S_{\phi_4}(U:W) = \sum_{i=1}^{n}(u_i-w_i)\log\frac{u_i}{w_i}$ = JKL coefficient of distance [29, 30].
5. $\phi_5(y) = (2y-1)\log(2y-1)$, $\phi_5'''(y) = -\frac{8}{(2y-1)^2}$, $-\phi_5'''(y) = \frac{8}{(2y-1)^2} > 0 \ \forall\ y \in R - \left(\frac{1}{2}\right) \Rightarrow S_{\phi_5}(U:W) = \sum_{i=1}^{n} u_i \log\frac{u_i}{w_i}$ = KL coefficient of distance [30].

4. The Infinite Series Version of the $S_\phi(U:W)$ with Remainders

In this section, we express $S_\phi(U:W)$ in infinite series form by using Taylor's series expansions with Lagrange's and Cauchy's form of remainders, respectively. Actually, this form also represents a relation between $S_\phi(U:W)$ and Chi-m coefficient of distance.

Theorem 4.1. If $\phi:(0,\infty)\to R$ be a real and differentiable function, also normalized, i.e., $\phi(1)=0$. Then for $U:W \in \Theta_n$, we have

$$S_\phi(U:W) = \frac{\phi''(1)}{2!(2)^2}\chi^2(U:W) + \frac{\phi'''(1)}{3!(2)^3}\chi^3(U:W) + \ldots + \frac{\phi^{(m)}(1)}{m!(2)^m}\chi^m(U:W) + (RM)_{m+1}(U:W) \tag{4.1}$$

or

$$S_\phi(U:W) = \sum_{j=1}^{m-1}\frac{\phi^{(j+1)}(1)}{(j+1)!(2)^{j+1}}\chi^{j+1}(U:W) + (RM)_{m+1}(U:W)$$

where

$$\chi^m(U:W) = \sum_{i=1}^{n}\frac{(u_i-w_i)^m}{w_i^{m-1}} \tag{4.2}$$

is the well known Chi-m coefficient of distance, [31]. Pearson coefficient of distance or Chi-square coefficient of distance (2.3) is the special case of the Chi-m coefficient of distance at $m=2$ and $(RM)_{m+1}(U:W)$ is the remainder in the probability distribution's sense.

If $(RM)_{m+1}(U:W)$ is the Resultant of the Lagrange's form of remainder then

$$(RM)_{m+1}(U:W)=\frac{1}{(m+1)!\,(2)^{m+1}}\sum_{i=1}^{n}\frac{(u_i-w_i)^{m+1}}{w_i^m}\phi^{(m+1)}\left[1+\frac{\omega}{2}\left(\frac{u_i-w_i}{w_i}\right)\right],0<\omega<1$$

If $(RM)_{m+1}(U:W)$ is the resultant of the Cauchy's's form of remainder then

$$(RM)_{m+1}(U:W)=\frac{(1-\omega)^m}{(m)!\,(2)^{m+1}}\sum_{i=1}^{n}\frac{(u_i-w_i)^{m+1}}{w_i^m}\phi^{(m+1)}\left[1+\frac{\omega}{2}\left(\frac{u_i-w_i}{w_i}\right)\right],0<\omega<1$$

Proof.
For the given function, Taylor's series expansion at a point $y=1$ is defined as:

$$\phi(y)=\phi(1)+(y-1)\phi'(1)+\frac{(y-1)^2}{2!}\phi''(1)+\frac{(y-1)^3}{3!}\phi'''(1)+\ldots+\frac{(y-1)^m}{m!}\phi^{(m)}(1)+(RM)_{m+1} \tag{4.3}$$

where

$$(RM)_{m+1}=\begin{cases}\frac{(y-1)^{m+1}}{(m+1)!}\phi^{(m+1)}\left[1+\omega(y-1)\right],0<\omega<1 & \text{Lagrange's form of remainder}\\ \frac{(y-1)^{m+1}(1-\omega)^m}{m!}\phi^{(m+1)}\left[1+\omega(y-1)\right],0<\omega<1 & \text{Cauchy's's form of remainder}\end{cases} \tag{4.4}$$

We simply obtain the result (4.1) by putting $y=\frac{u_i+w_i}{2w_i}$ in the equation (4.3) followed by multiplying with w_i and then summing over all from $i=1$ to $i=n$ with keep in the consideration that the given function is normalized, i.e., $\phi(1)=0$.

Remark 4.2. By using the result (4.1), we have the following relations on some special coefficient of distances in terms of the Chi-m coefficient of distance in series form, for the different functions (already discussed in section 3):

$$\phi(y)=\begin{cases}\frac{(y-1)^2}{y} & \Rightarrow \Delta(U:W)=\sum_{j=1}^{m-1}\frac{(-1)^{j+1}}{2^j}\chi^{j+1}(U:W)+\text{Appropriate remainder}\\ -\log y & \Rightarrow F(W:U)=\sum_{j=1}^{m-1}\frac{1}{j+1}\left(\frac{-1}{2}\right)^{j+1}\chi^{j+1}(U:W)+\text{Appropriate remainder}\\ y\log y & \Rightarrow G(W:U)=\sum_{j=1}^{2}\frac{(-1)^{j+1}\chi^{j+1}(U:W)}{(j+1)!2^{j+1}}+\sum_{j=3}^{m-1}\frac{(-1)^{j+1}\chi^{j+1}(U:W)}{j(j+1)2^{j+1}}+\text{Appropriate remainder}\end{cases}$$

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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