Süleyman Demirel Üniversitesi Fen Edebiyat Fakültesi Fen Dergisi
Süleyman Demirel University Faculty of Arts and Sciences Journal of Science
2022, 17(2): 247-260
DOI: 10.29233/sdufeffd.978260
Atıf için / For Citation: M. Düz, "Solution of Lane-Emden Equation with Fourier Decomposition Method", *Süleyman Demirel Üniversitesi Fen Edebiyat Fakültesi Fen Dergisi*, 17(2), 247-260, 2022.
Araştırma Makalesi

Solution of Lane-Emden Equation with Fourier Decomposition Method

Murat DÜZ*

[1]*Karabük Üniversitesi, Fen Fakültesi,Matematik Bölümü, 78450, Karabük,Türkiye*

**mduz@karabuk.edu.tr*

(Alınış / Received: 3.08.2021, Kabul / Accepted: 17.06.2022, Yayımlanma / Published: 25.11.2022)

Abstract: In this article, we tried to get the solution of a class of Lane Emden type equations by using the Fourier Decomposition Method. This method is obtained by using the Fourier transform and the Adomian Decomposition method (FADM) together.

Key words: Fourier transform, Lane-Emden equation, Adomian Decomposition method

Fourier Ayrıştırma Metodu ile Lane-Emden Denkleminin Çözümü

Öz: Bu makalede Fourier ayrıştırma metodunu kullanarak Lane-Emden tipi denklemlerin bir sınıfının çözümünü elde etmeyi denedik. Bu metot Fourier dönüşümü ve Adomian ayrıştırma metodu (FADM) birlikte kullanılarak elde edilir.

Anahtar Kelimeler: Fourier dönüşümü, Lane-Emden denklemi, Adomian Ayrıştırma metodu

1. Introduction

Most problems encountered in real life are not linear. Some of these have singular initial values. It is not easy to solve such problems due to their non-linearity and singular initial conditions. Lane-Emden equations are nonlinear equations with singular initial values that have important applications. The Lane-Emden equation with initial conditions is defined as follows.

$$\begin{gathered} y'' + \frac{2}{x}y' + f(y) = 0 \\ y(0) = a, y'(0) = 0, \end{gathered} \tag{1}$$

where $f(y)$ is a given function of y. As can be seen, this equation has singular initial values.

There are many studies about the solutions of Lane-Emden equations. These equations are solved with the differential transformation method, Adomian Decomposition method, Variational iteration method, Homotopy perturbation method, Rational Legendre approximation, Rational Chebyshev tau method, Pade approximation [1-7]. These equations are also solved by the combination of the Adomian Decomposition method and the spectral method[8], the combination of the Laplace transform and the

Murat DÜZ, mduz@karabuk.edu.tr, ORCID: https://orcid.org/0000-0003-2387-4045

Adomian decomposition method, and the combination of the modified homotopy perturbation method and the Fourier transform. F. Yin, J. Song, F. Lu, H. Leng [11] converted the Lane-Emden equation into an integral equation and gave an approximate solution using Legendre wavelets. TM. Elzaki applied Elzaki transform and differential transform methods for solving some nonlinear differential equations [13]. In [15] Nazari-Golshan, A. et al. used the modified homotopy perturbation method coupled with the Fourier transform to solve the Lane-Emden equations.

2. Material and Method

This section has been investigated under two sub-title. Fourier transform has been mentioned in the first sub-title and FADM has been mentioned in the second sub-title**.**

2.1. Fourier Transform

Definition 2.1.1 We represent the Fourier transform of $f(t)$ in non-unitary angular frequency form [19]

$$\mathcal{F}[f(t)] = F(w) = \int_{-\infty}^{\infty} f(t).\, e^{-iwt} dt, \tag{2}$$

and it's inverse transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w).\, e^{iwt} dt. \tag{3}$$

Theorem 2.1.1 [11-12]Let $f(t)$ be continuous or partly continuous in the interval $(-\infty, \infty)$ and $f(t), f'(t), f''(t), \ldots, f^{(n-1)}(t) \to 0$ for$|t| \to \infty$.

If $f(t), f'(t), f''(t), \ldots, f^{(n-1)}(t)$ are absolutely integrable in the interval $(-\infty, \infty)$, then

$$\mathcal{F}\left[f^{(n)}(t)\right] = (iw)^n \mathcal{F}[f(t)]. \tag{4}$$

Definition 2.1.2 [20] The Dirac delta distribution can be rigorously thought of as a distribution on a real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(t) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0. \end{cases}$$

The Dirac delta has properties, that

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\int_{-\infty}^{\infty} f(t).\delta(t-t_0)dt = f(t_0), \quad (5)$$

$$\int_{-\infty}^{\infty} f(t).\delta^{(n)}(t-t_0)dt = (-1)^n.f^{(n)}(t_0), \quad (6)$$

$$t.\delta'(t) = -\delta(t), \quad (7)$$

Theorem 2.1.2 [11-12] Fourier transforms of some functions are following:

$i)\mathcal{F}[\delta(t)] = 1.$

$ii)\mathcal{F}[1] = 2\pi.\delta(w)$

$iii)\ \mathcal{F}[t^n] = 2\pi.i^n.\delta^{(n)}(w)$

$iv)\mathcal{F}[t^n.f(t)] = i^n \dfrac{d^n\mathcal{F}[f(t)]}{dw^n}$

$v)\mathcal{F}\left[e^{iw_0t}\right] = 2\pi\delta(w-w_0)$

$vi) If \mathcal{F}[f(t)] = F(w), then\ \mathcal{F}\left[e^{iw_0t}.f(t)\right] = F(w-w_0)$

$vii)\mathcal{F}[e^{at}] = 2\pi\delta(w+ia)$

$viii) If \mathcal{F}[f(t)] = F(w), then\ \mathcal{F}[e^{a.t}.f(t)] = F(w+ia)$

Theorem 2.1.3 [12,13]

$$(w-w_o)^n.\delta^{(n)}(w-w_0) = (-1)^n.n!.\delta(w-w_0), \quad (8)$$

where $\delta(w-w_0)$ is defined as following

$$\delta(w-w_0) = \begin{cases} 0, & w \neq w_{0,} \\ \infty, & w = w_{0.} \end{cases}$$

Theorem 2.1.4 [12,13]

$$\int_{-\infty}^{\infty} \frac{\delta(w-w_0)f(w)}{(w-w_0)^n} dw = \frac{1}{n!}\frac{d^nf(w)}{dw^n}(w=w_0).$$

2.2 Basic Idea of FADM

We consider $F\big(y(x)\big) = g(x)$, where F represents a general differential operator involving both the linear and nonlinear terms. The linear term is decomposed into $L + R$, where L is the highest order differential operator and R is the remainder of the linear operator. Thus the equation can be written

$$Ly + Ry + Ny = g(x),$$

where Ny represents the nonlinear terms.

$$Ly = g(x) - Ry - Ny$$
$$\mathcal{F}[Ly] = \mathcal{F}[g(x)] - \mathcal{F}[R(y)] - \mathcal{F}[N(y)]$$
$$\mathcal{F}\left[L\left(\sum_{n=0}^{\infty} y_n\right)\right] = \mathcal{F}[g(x)] - \mathcal{F}\left[R\left(\sum_{n=0}^{\infty} y_n\right)\right] - \mathcal{F}\left[L\left(\sum_{n=0}^{\infty} A_n\right)\right]$$

where the components $y_0, y_1, y_2, \ldots$ are usually determined recursively by:

$$y_0 = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[g(x)]}{(iw)^n}\right],$$
$$y = \sum_{n=0}^{\infty} y_n,$$

where $A(n)$ adomian polynomial. If $N(y) = f(y)$ then $A(0) = f(y_0)$.

$$A_1 = y_1.\frac{df(y_0)}{dy_0}$$
$$A_2 = y_2.\frac{df(y_0)}{dy_0} + \frac{y_1^2}{2}\frac{d^2f(y_0)}{dy_0^2}$$
$$A_3 = y_3.\frac{df(y_0)}{dy_0} + y_1 y_2 \frac{d^2f(y_0)}{dy_0^2} + \frac{y_1^3}{3!}\frac{d^3f(y_0)}{dy_0^3}$$

2.3 Solution of Lane-Emden Equation with FADM

We let consider the following Lane-Emden equation.

$$y'' + \frac{2}{x}y' + f(y) = 0$$
$$x.y'' + 2y' + xf(y) = 0.$$

If we apply the Fourier transform to the above equation, then

$$\mathcal{F}(y'') + 2\mathcal{F}(y') + \mathcal{F}(xf(y)) = 0$$
$$i\frac{d\left(-w^2Y(w)\right)}{dw} + 2iwY + \mathcal{F}(xf(y)) = 0$$
$$-iw^2\frac{dY}{dw} + \mathcal{F}(x.f(y)) = 0$$
$$\frac{dY}{dw} = \frac{\mathcal{F}(x.f(y))}{iw^2}$$

Now let's apply the inverse Fourier transform to the above equation

$$\mathcal{F}^{-1}\left[\frac{dY}{dw}\right] = \mathcal{F}^{-1}\left[\frac{\mathcal{F}(xf(y))}{iw^2}\right]$$
$$-i.x.y_{n+1} = \mathcal{F}^{-1}\left[\frac{\mathcal{F}(xA_n))}{iw^2}\right]$$
$$y_{n+1} = \frac{1}{x}\mathcal{F}^{-1}\left[\frac{\mathcal{F}(xA_n))}{w^2}\right].$$

2.4 Some examples

Example 1 Now let us choose $f(y) = y^n$ which the nonlinear part of the equation which has been studied in many studies and try to solve the equation accordingly [2].

$$x.y'' + 2y' + xy^n = 0. \tag{9}$$

From initial conditions $y_0 = a$.

$$A_0 = f(y_0) = f(a) = a^n$$

$$y_1 = x^{-1}\mathcal{F}^{-1}\left[\frac{\mathcal{F}(xA_0))}{w^2}\right] = x^{-1}\mathcal{F}^{-1}\left[\frac{\mathcal{F}(xa^n))}{w^2}\right] = x^{-1}\mathcal{F}^{-1}\left[\frac{2\pi i a^n \delta'(w)}{w^2}\right]$$

$$= \frac{ia^n}{x}\int_{-\infty}^{\infty}\frac{\delta'(w)}{w^2}e^{iwx}dw = -\frac{ia^n}{x}\int_{-\infty}^{\infty}\frac{\delta(w)}{w^3}e^{iwx}dw = -\frac{ia^n}{x}\frac{(ix)^3}{6} = -\frac{x^2}{6}a^n$$

$$A_1 = y_1 f'(y_0) = -na^{2n-1}\frac{x^2}{6}$$

$$y_2 = x^{-1}\mathcal{F}^{-1}\left[\frac{\mathcal{F}(xA_1))}{w^2}\right] = \frac{1}{x}\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(-na^{2n-1}\frac{x^3}{6})\right)}{w^2}\right] = -\frac{na^{2n-1}}{6x}\mathcal{F}^{-1}\left[\frac{2\pi i^3\delta'''(w)}{w^2}\right]$$

$$= \frac{nia^{2n-1}}{6x}\int_{-\infty}^{\infty}\frac{\delta'''(w)}{w^2}e^{iwx}dw = -\frac{nia^{2n-1}}{x}\int_{-\infty}^{\infty}\frac{\delta(w)}{w^5}e^{iwx}dw$$

$$= -\frac{nia^{2n-1}}{x}\frac{(ix)^5}{5!} = \frac{n.a^{2n-1}x^4}{120}$$

$$A_2 = y_2 f'(y_0) + \frac{y_1^2}{2}f''(y_0) = \frac{n.a^{2n-1}x^4}{120}na^{n-1} + \frac{x^4}{72}a^{2n}(n^2-n)a^{n-2}$$

$$= \frac{a^{3n-2}x^4(8n^2-5n)}{360}$$

$$y_3 = \frac{1}{x}\mathcal{F}^{-1}\left[\frac{\mathcal{F}(xA_2))}{w^2}\right] = \frac{1}{x}\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(\frac{a^{3n-2}x^5(8n^2-5n)}{360})\right)}{w^2}\right]$$

$$= \frac{a^{3n-2}(8n^2-5n)}{360x}\mathcal{F}^{-1}\left[\frac{2\pi i^5\delta^{(5)}(w)}{w^2}\right]$$

$$= -i\frac{a^{3n-2}(8n^2-5n)}{360x}\int_{-\infty}^{\infty}\frac{5!\,\delta(w)}{w^7}e^{iwx}dw$$

$$= -i\frac{a^{3n-2}(8n^2-5n)}{360x}\frac{(ix)^7}{42} = -\frac{a^{3n-2}(8n^2-5n)x^6}{15120}$$

$$y = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$y = a - \frac{x^2}{6}a^n + \frac{n.a^{2n-1}x^4}{120} - \frac{a^{3n-2}(8n^2-5n)x^6}{15120} + \cdots$$

Case 1 If we choose $n = 1$, then the equation (9) takes the form

$$y'' + \frac{2}{x}y' + y = 0$$

$$y(0) = a, y'(0) = 0$$

and the solution to this problem is as follows:

$$\begin{aligned} y &= a - \frac{x^2}{6}a + \frac{a.x^4}{120} - \frac{3.a.x^6}{15120} + \cdots \\ &= \frac{a}{x}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) \\ &= a.\frac{sinx}{x} \end{aligned}$$

Case 2 If we choose $n = 2$, then the equation (9) takes the form

$$y'' + \frac{2}{x}y' + y^2 = 0$$
$$y(0) = a, y'(0) = 0$$

and the solution to this problem is as follows:

$$y = a - \frac{x^2}{6}a^2 + \frac{a^3x^4}{60} - \frac{11.a^4x^6}{7560} + \cdots$$

Example 2 : Let's find the solution to the following isothermal gas spheres [1-2]

$$y'' + \frac{2}{x}y' + e^y = 0 \qquad (10)$$

with the initial conditions

$$y(0) = y'(0) = 0.$$

Solution: We can write equation (10) as follows

$$x.y'' + 2y' + x.e^y = 0$$

Now we will use the Fourier transform of the above equation.

$$\mathcal{F}[x.y''] + 2.\mathcal{F}[y'] + \mathcal{F}[x.e^y] = 0$$

$$i\frac{d}{dw}[(iw)^2Y] + 2iwY + \mathcal{F}[x.e^y] = 0$$

$$-i\left(2wY + w^2\frac{dY}{dw}\right) + 2iwY = -\mathcal{F}[x.e^y]$$

$$\frac{dY}{dw} = \frac{\mathcal{F}[x.e^y]}{iw^2}$$

By taking inverse Fourier transform on both sides, we get

$$\mathcal{F}^{-1}\left[\frac{dY}{dw}\right] = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[x.e^{y}]}{iw^2}\right]$$

$$-ixy_{n+1} = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[x.A_n]}{iw^2}\right]$$

$$A_0 = e^{y_0} = 1$$

$$y_1 = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[x.A_0]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[x]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{2\pi\delta'^{(w)}}{w^2}\right]$$
$$= -\frac{1}{ix}\int_{-\infty}^{\infty}\frac{\delta'^{(w)}}{w^2}e^{iwx}dw = \frac{1}{ix}\int_{-\infty}^{\infty}\frac{\delta(w)}{w^3}e^{iwx}dw = -\frac{x^2}{6}$$

$$A_1 = y_1.e^{y_0} = -\frac{x^2}{6}$$

$$y_2 = \frac{1}{-ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[x.A_1]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left[-\frac{x^3}{6}\right]}{iw^2}\right]$$

$$= \frac{1}{-ix}\mathcal{F}^{-1}\left[\frac{-2\pi i^3\delta'''(w)}{6iw^2}\right] = \frac{1}{-2ix}\int_{-\infty}^{\infty}\frac{\delta'''}{3w^2}e^{iwx}dw = \frac{1}{ix}\int_{-\infty}^{\infty}\frac{\delta}{w^5}e^{iwx}dw = \frac{1}{ix}\frac{(ix)^5}{5!}$$
$$= \frac{x^4}{5!}$$

$$A_2 = y_2.e^{y_0} + \frac{y_1^2}{2}.e^{y_0} = \frac{x^4}{45}$$

$$y_3 = \frac{1}{-ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[x.A(2)]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left[\frac{x^5}{45}\right]}{iw^2}\right] = \frac{1}{45x}\mathcal{F}^{-1}\left[2\pi i^5.\frac{\delta^{(5)}}{w^2}\right]$$

$$= \frac{i}{45x}\int_{-\infty}^{\infty}-\frac{5!}{w^7}\delta.e^{iwx}dw = \frac{-i}{45x}\frac{5!}{7!}(ix)^7 = -\frac{x^6}{45.6.7}.$$

so that the solution in a series form is given by

$$y = y_0 + y_1 + y_2 + \cdots = -\frac{x^2}{6} + \frac{x^4}{5!} - \frac{x^6}{45.6.7} + \cdots$$

The solution is compatible with [1].

Example 3: Let's find the solution to the following differential equation [3]

$$y'' + \frac{2}{x}y' + 4(2e^{y} + e^{\frac{y}{2}}) = 0 \qquad (11)$$

with the initial conditions

$$y(0) = y'(0) = 0.$$

Solution: We can write equation (11) as follows:

$$x.y'' + 2y' + 8x.e^{y} + 4xe^{\frac{y}{2}} = 0.$$

Now we will use the Fourier transform of the above equation.

$$\mathcal{F}[x.y''] + 2.\mathcal{F}[y'] + \mathcal{F}[8x.e^{y}] + \mathcal{F}\left[4xe^{\frac{y}{2}}\right] = 0$$

$$i\frac{d}{dw}[(iw)^2Y] + 2iwY + \mathcal{F}[8x.e^{y}] + \mathcal{F}\left[4xe^{\frac{y}{2}}\right] = 0$$

$$-i\left(2wY + w^2\frac{dY}{dw}\right) + 2iwY = -\mathcal{F}[8x.e^{y}] - \mathcal{F}\left[4xe^{\frac{y}{2}}\right]$$

$$\frac{dY}{dw} = \frac{\mathcal{F}[8x.e^{y}] + \mathcal{F}\left[4xe^{\frac{y}{2}}\right]}{iw^2}.$$

By taking inverse Fourier transform on both sides, we get

$$\mathcal{F}^{-1}\left[\frac{dY}{dw}\right] = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[8x.e^{y}] + \mathcal{F}\left[4xe^{\frac{y}{2}}\right]}{iw^2}\right]$$

$$-ixy_{n+1} = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[A_n + B_n]}{iw^2}\right].$$

From initial conditions is $y_0 = 0$.

$$A_0 + B_0 = 8x.e^{y_0} + 4xe^{\frac{y_0}{2}} = 12x$$

$$y_1 = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[A_0 + B_0]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[12x]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{12.2\pi i\delta'(w)}{iw^2}\right]$$
$$= -\frac{1}{ix}\int_{-\infty}^{\infty} 12\frac{\delta'(w)}{w^2}e^{iwx}dw = \frac{12}{ix}\int_{-\infty}^{\infty}\frac{\delta(w)}{w^3}e^{iwx}dw = -2x^2$$

$$A_1 + B_1 = 8xy_1e^{y_0} + 4xy_1\frac{1}{2}e^{\frac{y_0}{2}} = -20x^3$$

$$y_2 = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[A_1 + B_1]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[-20x^3]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{-20.2\pi i^3\delta'''(w)}{iw^2}\right]$$
$$= \frac{1}{ix}\int_{-\infty}^{\infty}\frac{20.6.\delta(w)}{w^5}e^{iwx}dw = \frac{120}{ix}\int_{-\infty}^{\infty}\frac{\delta(w)}{w^5}e^{iwx}dw = x^4$$

$$A_2 + B_2 = 8xy_2e^{y_0} + \frac{y_1^2}{2}e^{y_0} + 4xy_2\frac{1}{2}e^{\frac{y_0}{2}}$$

$$y_3 = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[A_2 + B_2]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{\mathcal{F}[28x^5]}{iw^2}\right] = -\frac{1}{ix}\mathcal{F}^{-1}\left[\frac{-56\pi i^5\delta^{(5)}}{iw^2}\right]$$
$$= -\frac{2}{3}x^6$$

So the approximate solution to this problem is

$$y = y_0 + y_1 + y_2 + \cdots = -2x^2 + x^4 - \tfrac{2}{3}x^6.$$

This is the first three terms of $y = -2\ln(1 + x^2)$ which the solution of the equation.

Table 1. The exact and numerical solutions of applying FADM for equation (11).

x	*Exact solution* $y(x) = -2ln(1+x^2)$	*Numerical solution* $(y_0 + y_1)_{app}$	*Numerical solution* $(y_0 + y_1 + y_2)_{app}$	*Numerical solution* $(y_0 + y_1 + y_2 + y_3)_{app}$	*Absolute error* $\lvert y - (y_0 + y_1)_{app}\rvert$	*Absolute error* $\lvert y - (y_0 + y_1 + y_2)_{app}\rvert$	*Absolute error* $\lvert y - (y_0 + y_1 + y_2 + y_3)_{app}\rvert$
0.1	0.019900661706336184	−0.020000000000000004	−0.019900000000000004	−0.01990066666666667	0.00009933829366381985	$6.617063361 \times 10^{-7}$	$4.96033048663 \times 10^{-9}$
0.2	−0.07844142630656266	−0.08000000000000002	−0.0784000000000001	−0.07844266666666667	0.0015585736934373567	0.0000414260656264745	0.000001240360
0.3	−0.17235539248210482	−0.18000000000000005	−0.17190000000000005	−0.17238600000000007	0.007644607517895224	0.0004553924821047717	0.00003060751789524252
0.4	−0.2968400102365468	−0.32000000000000006	−0.29440000000000005	−0.2971306666666667	0.023159989763453237	0.0024400102365467746	0.0002906564301198844
0.5	−0.44628710262841953	−0.5	−0.4375	−0.4479166666666667	0.05371289737158047	0.00878710262841953	0.0016295640382471555
0.6	−0.6149693994959211	−0.72	−0.5904	−0.6215040000000001	0.10503060050407886	0.024569399495921074	0.006534600504078947
0.7	−0.7975522399147358	−0.9800000000000002	−0.7399000000000001	−0.8183326666666668	0.18244776008526442	0.05765223991473567	0.020780426751931036
0.8	−0.9893924836722142	−1.2800000000000002	−0.8704000000000001	−1.0451626666666667	0.29060751632778603	0.11899248367221416	0.055770182994452466
0.9	−1.1866536905554688	−1.62	−0.9639	−1.318194	0.43334630944453134	0.2227536905554688	0.13154030944453132
1.	−1.3862943	−2.	−1.	−1.6666666666666665	0.6137056388801094	0.3862943611198906	0.28037230554677595

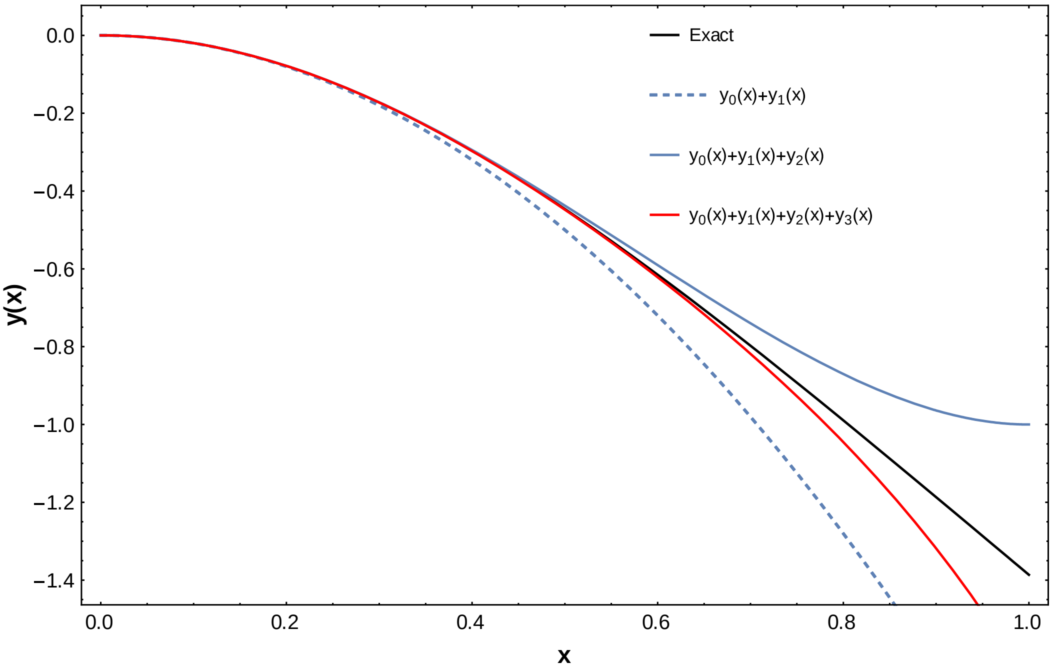

Figure 1. The exact and numerical solutions of $(y_0 + y_1)$, $(y_0 + y_1 + y_2)$,$(y_0 + y_1 + y_2 + y_3)$ using **FADM** for equation (11).

In the previous examples, the homogeneous linear Lane-Emden equation with constant coefficients was solved using the Fourier Decomposition method.

In the following example, we will try to solve the non-homogeneous linear Lane-Emden equation using only the Fourier transform method.

Example 4 : We let consider the following isothermal gas spheres [11]

$$y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3. \qquad (12)$$

Solution: This equation is linear and non-homogeneous. We can write equation (12) as follows:

$$x.y'' + 2y' + x.y = 6x + 12x^2 + x^3 + x^4.$$

Now we will use the Fourier transform of the above equation.

$$\mathcal{F}[x.y''] + 2.\mathcal{F}[y'] + \mathcal{F}[x.y] = \mathcal{F}[6x + 12x^2 + x^3 + x^4]$$

$$i\frac{d}{dw}[-w^2Y] + 2iwY + i\frac{dY}{dw} = 6.2\pi i\delta' + 12.2\pi i^2\delta'' + 2\pi i^3\delta''' + 2\pi i^4\delta^{(4)}$$

$$-2iwY - iw^2\frac{dY}{dw} + 2iwY + i\frac{dY}{dw} = 12\pi i\delta' + 24\pi i^2\delta'' + 2\pi i^3\delta''' + 2\pi i^4\delta^{(4)}$$

$$i(1-w^2)\frac{dY}{dw}=12\pi i\delta'+24\pi i^2\delta''+2\pi i^3\delta'''+2\pi i^4\delta^{(4)}$$

$$\frac{\mathrm{dY}}{\mathrm{dw}}=\frac{12\pi\delta'}{1-\mathrm{w}^2}+\frac{24\pi\mathrm{i}\delta''}{1-\mathrm{w}^2}+\frac{2\pi\mathrm{i}^2\delta'''}{1-\mathrm{w}^2}+\frac{2\pi\mathrm{i}^3\delta^{(4)}}{1-\mathrm{w}^2}$$

By taking inverse Fourier transform on both sides, we get

$$\mathcal{F}^{-1}\left[\frac{dY}{dw}\right]=\mathcal{F}^{-1}\left[\frac{12\pi\delta'}{1-w^2}\right]+\mathcal{F}^{-1}\left[\frac{24\pi i\delta''}{1-w^2}\right]+\mathcal{F}^{-1}\left[\frac{2\pi i^2\delta'''}{1-w^2}\right]+\mathcal{F}^{-1}\left[\frac{2\pi i^3\delta^{(4)}}{1-w^2}\right]$$

$$\frac{xy}{i}=\int_{-\infty}^{\infty}\frac{6\delta'}{1-w^2}e^{iwx}dw+12i\int_{-\infty}^{\infty}\frac{\delta''}{1-w^2}e^{iwx}dw+\int_{-\infty}^{\infty}\frac{i^2\delta'''}{1-w^2}e^{iwx}dw$$
$$+\int_{-\infty}^{\infty}\frac{i^3\delta^{(4)}}{1-w^2}e^{iwx}dw$$

$$\frac{xy}{i}=-6\int_{-\infty}^{\infty}\frac{\delta}{w}.\frac{e^{iwx}}{1-w^2}dw+24i\int_{-\infty}^{\infty}\frac{\delta}{w^2}.\frac{e^{iwx}}{1-w^2}dw-6i^2\int_{-\infty}^{\infty}\frac{\delta}{w^3}.\frac{e^{iwx}}{1-w^2}dw$$
$$+4!\,i^3\int_{-\infty}^{\infty}\frac{\delta}{w^4}.\frac{e^{iwx}}{1-w^2}dw$$

$$\frac{xy}{i}=-6ix+\frac{24}{2}i(2-x^2)-i^2(6ix-ix^3)+i^3(24-12x^2+x^4)$$

$$xy=6x-24+12x^2-6x+x^3+24-12x^2+x^4$$

$$\mathrm{y}=\mathrm{x}^2+\mathrm{x}^3$$

which is the exact solution to equation (12).

Example 5 (see [1, 8]): Consider the following Lane-Emden equation

$$y''+\frac{2}{x}y'+(y^2-0.6)^{\frac{3}{2}}=0, \tag{13}$$

with initial conditions : $y(0)=1, y'(0)=0$.
Solution: We can write equation (13) as follows

$$x.y''+2y'+x(y^2-0.6)^{\frac{3}{2}}=0.$$

Now we will use the Fourier transform of the above equation.

$$\mathcal{F}[x.y'']+2.\mathcal{F}[y']+\mathcal{F}\left[x(y^2-0.6)^{\frac{3}{2}}\right]=0$$

$$i\frac{d}{dw}[-w^2Y]+2iwY+i\,\frac{dA_n}{dw}=0$$

$$\frac{\mathrm{dY}}{\mathrm{dw}} = \frac{1}{\mathrm{w}^2}\frac{\mathrm{dA_n}}{\mathrm{dw}}.$$

By taking inverse Fourier transform on both sides, we get

$$\mathcal{F}^{-1}\left[\frac{dY}{dw}\right] = \mathcal{F}^{-1}\left[\frac{1}{w^2}\frac{dA_n}{dw}\right]$$

$$\frac{xy_{n+1}}{i} = \mathcal{F}^{-1}\left[\frac{1}{w^2}\frac{dA_n}{dw}\right]$$

$$y_{n+1} = \frac{i}{x}\mathcal{F}^{-1}\left[\frac{1}{w^2}\frac{dA_n}{dw}\right].$$

From initial conditions $y_0 = 1$.

$$A_0 = (0.4)^{\frac{3}{2}}$$

$$y_1 = \frac{i}{x}\mathcal{F}^{-1}\left[\frac{1}{w^2}\frac{dA_0}{dw}\right] = \frac{i}{x}\mathcal{F}^{-1}\left[\frac{(0.4)^{\frac{3}{2}}.2\pi\delta'^{(w)}}{w^2}\right] = \frac{i}{x}\mathcal{F}^{-1}\left[\frac{-(0.4)^{\frac{3}{2}}.2\pi\delta(w)}{w^3}\right]$$

$$= -\frac{(0.4)^{\frac{3}{2}}.i}{x}\int_{-\infty}^{\infty}\frac{\delta(w)}{w^3}e^{iwx}dw = -\frac{(0.4)^{\frac{3}{2}}x^2}{6}$$

$$A_1 = -\frac{(0.4)^2x^2}{2}$$

$$y_2 = \frac{i}{x}\mathcal{F}^{-1}\left[\frac{1}{w^2}\frac{dA_1}{dw}\right] = -\frac{(0.4)^2}{2x}\mathcal{F}^{-1}\left[\frac{-2\pi i\delta'''(w)}{w^2}\right] =$$

$$= \frac{-3!\,(0.4)^2 i}{2x}\int_{-\infty}^{\infty}\frac{\delta(w)}{w^5}e^{iwx}dw = \frac{(0.4)^2x^4}{40}$$

$$A_2 = \left(\frac{7(0.4)^{\frac{5}{2}}}{60} + \frac{(0.4)^{\frac{7}{2}}}{24}\right)x^4$$

$$y_3 = \frac{i}{x}\mathcal{F}^{-1}\left[\frac{1}{w^2}\frac{dA_2}{dw}\right] = \frac{\left(\frac{7(0.4)^{\frac{5}{2}}}{60} + \frac{(0.4)^{\frac{7}{2}}}{24}\right)}{x}\mathcal{F}^{-1}\left[\frac{2\pi i\delta^{(5)}(w)}{w^2}\right] =$$

$$= -\frac{\left(\frac{7(0.4)^{\frac{5}{2}}}{60} + \frac{(0.4)^{\frac{7}{2}}}{24}\right)i5!}{x}\int_{-\infty}^{\infty}\frac{\delta(w)}{w^7}e^{iwx}dw = -\frac{2(0.4)^{\frac{5}{2}}x^6}{630}$$

An approximate solution obtained using Adomian Decomposition method by Wazwaz is

$$y(x) = 1 - \left(\frac{1}{6}\right) r^3 x^2 + \left(\frac{1}{40}\right) r^4 x^4 - \left(\frac{1}{7!}\right) r^5 (5r^2 + 14) x^6 + \left(\frac{1}{3.9!}\right) r^6 (339r^2 + 280) x^8 + \left(\frac{1}{5.11!}\right) r^7 (1425r^4 + 11436r^2 + 4256) x^{10} + O(11),$$

where $r^2 = 0.4$. This example is solved by the FTADM with $n = 12$.
In this example, the numerical example is preseted to show the validity of a proposed method. In addition, the numerical results are compared with approximate solution using SADM [8].

Table 2. Comparison of the numerical result of example 5 by proposed method and SADM ($n = 12$).

x	$FTADM$	$SADM$ [8]	$Residual\ error$
0.1	0.99957876	0.99957876	1.33×10^{-22}
0.2	0.99831983	0.99831983	0.92×10^{-21}
0.3	0.99623743	0.99623743	4.26×10^{-20}
0.4	0.99335491	0.99335491	$2.11568744 \times 10^{-18}$
0.5	0.98970415	0.98970415	$4.58620014 \times 10^{-16}$
0.6	0.98532489	0.98532488	$3.85214460 \times 10^{-15}$
0.7	0.98026377	0.98026375	$9.23068545 \times 10^{-14}$
0.8	0.97457341	0.97457333	$7.58776235 \times 10^{-12}$
0.9	0.96831125	0.96831099	$4.55683214 \times 10^{-9}$
1	0.96153880	0.96153777	$2.52786054 \times 10^{-6}$

3. Conclusion and Comment

In this paper, solutions of Lane-Emden type differential equations have been studied by using Fourier transform and the Adomian decomposition method. The results showed that the FADM is an efficient. But we don't say that FTADM is more effective than other methods. As can be seen from the examples, the terms for the approximate solution obtained by this method are generally identical with the terms obtained by other methods.The FADM is seen that an alternative for solving non linear equations.

The fourth example is a linear equation and applied only Fourier transform for solving thıs equation and we obtained a special solution of this equation which satisfing

$$y(0) = y'(0) = 0.$$

Author Statement

Murat DÜZ: Methodology, Validation, Investigation,Review and Editing

Acknowledgement

As the authors of this study, we declare that we do not have any support and thank you statement.

Conflict of Interest

As the author of this study, I declare that I don't have any conflict of interest statement

Ethics Committee Approval and Informed Consent

As the author of this study, I declare that I do not have any ethics committee approval and/or informed consent statement.

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