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A new generalization of the differential transform method for solving boundary value problems

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Abstract − In this article, we propose a new generalization of the differential transformation method (DTM), i.e., α-Parameterized Differential Transform Method (α-PDTM), for finding approximate solutions to the boundary value problems. We then apply the proposed method to two boundary value problems for different values of the parameter α. Afterwards, we compare its solutions with DTM and exact solutions. Moreover, we present several visual illustrations.

Subject Classification (2020): 34B15, 65K05

1. Introduction

Recently, different semi-analytical or numerical methods, such as the finite difference method, the Adomian decomposition method, the shooting method, the homotopy perturbation method, the differential transformation method, the predictor correctors method, are of increasing interest due to the complexity of searching for analytical solutions to many initial or boundary value problems (BVP's) for various type differential equations (for more details, see [1-5]).

A semi-analytical method, called differential transformation method (DTM), was firstly proposed and applied by Zhou [6] in 1986 for solving initial and boundary value problems for differential equations arising in the analysis of an electric circuit.

Although DTM is based on the Taylor series method (TSM), this method differs from the classical TSM, which requires the symbolic computation of the high order derivatives of the data function [7]. Moreover, unlike many approximate methods, DTM does not require linearization of the differential equations, calculation of auxiliary parameters, determination of auxiliary functions which can require massive numerical computation, etc.

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DTM consists of the following simple steps:

- The considered differential equations are converted into a system of linear algebraic equations

- The systems of linear algebraic equations are solved by using various types of numerical or analytical methods

- The inverse differential transformation is applied to find the solution to the original problem.

Therefore, its capabilities have attracted many authors to use DTM for solving not only regular initial or BVP's but also similar problems with various singularities (for more details, see [8-12])

This study develops a new generalization of DTM, referred to as α- parameterized differential transform method (α-PDTM), which differs from the classical DTM in calculating the coefficients of a differential transformation. To justify and illustrate the reliability and efficiency of the presented method, two boundary value problems are solved by α-PDTM for different values of the parameter α. Then, the α-PDTM solutions are compared with DTM and exact solutions.

2. The General Framework of the Proposed Method

This section has developed a new generalization of the well-known DTM for the approximate solution of boundary value problems.

Throughout this paper, let $-\infty < a < b < \infty, f: [a,b] \to \mathbb{R}$ be an analytic function, $Y_k(f, x_0)$ denote the differential transform of f at the $x_0 \in [a,b]$, i.e.,

$$Y_k(f, x_0) = \frac{f^{(k)}(x_0)}{k!}, k = 0,1,2 \ldots,$$

and let, for $\alpha \in [0,1]$,

$$D(f, \alpha; k) := \alpha Y_k(f, a) + (1-\alpha) Y_k(f, b), k = 0,1,2, \ldots$$

Definition 2.1. The sequence

$$D_\alpha(f) := (D(f, \alpha; 0), D(f, \alpha; 1), \ldots)$$

is called the α-parametrized differential transform (α-PDT) of f.

Let $C = (c_0, c_1, c_2, \ldots)$ be a real sequence. Then,

$$\mathbb{Z}_\alpha(C, x) := \sum_{k=0}^{\infty} c_k (x - (\alpha a + (1-\alpha) b))^k$$

if the series is convergent in the whole real axis.

Definition 2.2. The series

$$\mathbb{Z}_\alpha(D_\alpha(f), x) = \sum_{k=0}^{\infty} D(f, \alpha; k)\,(x - (\alpha a + (1-\alpha) b))^k$$

is called the α-parametrized differential inverse transform (α-PDIT) of f, if the series is convergent in the whole real axis.

Remark 2.1. In the special cases $\alpha = 0$ and $\alpha = 1$, the equalities $\mathbb{Z}_0(D_0(f), x) = f(x)$ and $\mathbb{Z}_1(D_1(f), x) = f(x)$ are hold, respectively, for any analytic function f. That is, $\mathbb{Z}_\alpha = D_\alpha^{-1}$.

Remark 2.2. If $\alpha = 0$ and $\alpha = 1$, then α-PDT is reduced to DT at the endpoints $x = b$ and $x = a$, respectively. Similarly, in these cases, α-PDIT is reduced to DIT. Therefore, α-PDTM is a generalization of DTM.

Definition 2.3. Let $n \in \mathbb{N}$. Then, the series

$$D_{\alpha,n}(f) := (D(f,\alpha;0), D(f,\alpha;1), \dots, D(f,\alpha;n), 0, 0, \dots)$$

is called n-term α-PDT of f.

Remark 2.3. It is convenient to use n-term α-PDT instead of α-PDT in practical applications.

Definition 2.4. Let $n \in \mathbb{N}$. Then,

$$\tilde{f}_{\alpha,n}(x) := \mathbb{Z}_\alpha(D_{\alpha,n}(f), x) = \sum_{k=0}^{n} D(f,\alpha;k)(x - x_\alpha)^k$$

is called n-term α-parametrized approximation of f where $x_\alpha := \alpha a + (1-\alpha)b$.

Theorem 2.1. If $f(x) = \text{const}$, then $\mathbb{Z}_\alpha(D_\alpha(f), x) = f(x)$.

Theorem 2.2. If $f(x) = \text{const}$, then $\mathbb{Z}_\alpha\left(D_{\alpha,n}(f), x\right) = f(x),\ \ n = 0,1,2, \dots$

Theorem 2.3. Let $\beta \in \mathbb{R}$. Then,

a) $D_\alpha(\beta f) = \beta D_\alpha(f)$

b) $\mathbb{Z}_\alpha(D_\alpha(\beta f), x) = \beta \mathbb{Z}_\alpha(D_\alpha(f), x)$

c) $\mathbb{Z}_\alpha\left(D_{\alpha,n}(\beta f), x\right) = \beta \mathbb{Z}_\alpha\left(D_{\alpha,n}(f), x\right), n = 0,1,2, \dots$

Theorem 2.4. The following equalities are true.

a) $D_\alpha(f \pm g) = D_\alpha(f) \pm D_\alpha(g)$

b) $\mathbb{Z}_\alpha(D_\alpha(f \pm g), x) = \mathbb{Z}_\alpha(D_\alpha(f), x) \pm \mathbb{Z}_\alpha(D_\alpha(g), x)$

c) $\mathbb{Z}_\alpha\left(D_{\alpha,n}(f \pm g), x\right) = \mathbb{Z}_\alpha\left(D_{\alpha,n}(f), x\right) \pm \mathbb{Z}_\alpha\left(D_{\alpha,n}(g), x\right), n = 0,1,2, \dots$

Theorem 2.5. Let $g(x) = \frac{d^m f(x)}{dx^m}, n \in \mathbb{N}$. Then,

a) $D(g,\alpha;k) = \frac{(k+m)!}{k!} D(f,\alpha;k+m)$

b) $\frac{d^m}{dx^m} \tilde{f}_{\alpha,n}(x) = \sum_{k=0}^{n} \frac{(k+m)!}{k!} D(f,\alpha;k+m)(x - x_\alpha)^k$, where $x_\alpha = \alpha a + (1-\alpha)b$.

Theorem 2.6. If $f(x) = g(x)h(x)$, then

$$D(f,\alpha;k) = \sum_{m=0}^{k} [\alpha Y_m(g;a) Y_{k-m}(h;a) + (1-\alpha) Y_m(g;b) Y_{k-m}(h;b)]$$

3. Using α-PDT Method to Solve Boundary Value Problems

Example 3.1: Let us consider the following homogeneous differential equation

$$y''(x) + 2y(x) = 0, x \in [-1,0] \tag{3.1}$$

with the nonhomogeneous boundary conditions

$$y(-1) = 0 \text{ and } y'(0) = 1 \tag{3.2}$$

Then, the exact solution for this problem and its graph (Fig. 1) are as follows:

$$y(x) = \frac{1}{2}(\sqrt{2}\sin(\sqrt{2}\,x) + \sqrt{2}\cos(\sqrt{2}\,x)\tan(\sqrt{2})) \quad (3.3)$$

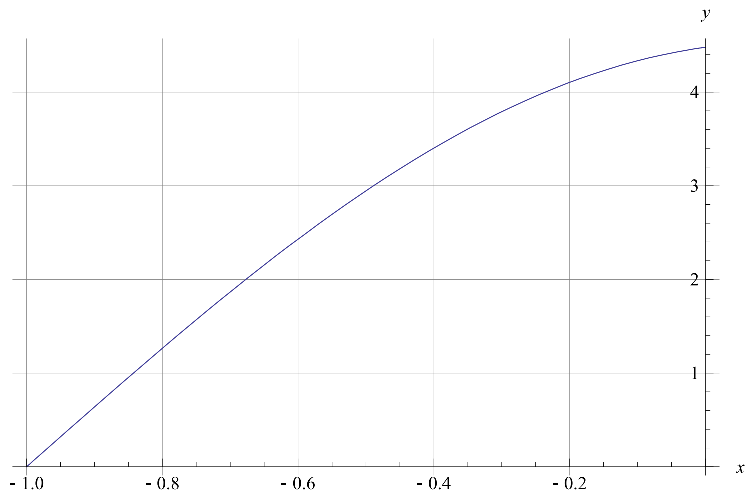

Figure 1. Graph of the exact solution of the problem (3.1)-(3.3)

If it is applied α-PDT to both sides of (3.1), then

$$D(y'' + 2y, \alpha; k) = (k+1)(k+2)D(y, \alpha; k+2) + 2D(y, \alpha; k) = 0 \quad (3.4)$$

Therefore, from the definition of α-PDT,

$$y_\alpha(x) = \sum_{k=0}^{\infty} D(y, \alpha; k)(x - x_\alpha)^k$$

and

$$y'_\alpha(x) = \sum_{k=0}^{\infty} kD(y, \alpha; k)(x - x_\alpha)^{k-1}$$

Moreover, for the boundary conditions $\mathrm{y}(-1) = 0$ and $y'(0) = 1$,

$$y_\alpha(-1) = \sum_{k=0}^{N} D(y, \alpha; k)(\alpha - 1)^k = 0 \quad (3.5)$$

and

$$y'_\alpha(0) = \sum_{k=0}^{N} kD(y, \alpha; k)(\alpha - 1)^{k-1} = 1 \quad (3.6)$$

respectively. Here, denoting $D(y, \alpha; 0) = A$ and $D(y, \alpha; 1) = B$ and then substituting in the recursive relation (3.4), $D(y, \alpha; 2) = -A$ is obtained. Now proceeding the iteration using (3.4), the other terms of the α-parametrized sequence $D(y, \alpha; n)$ can be calculated as

$$D(y, \alpha; 3) = \frac{-B}{3}, \quad D(y, \alpha; 4) = \frac{A}{6}, \quad D(y, \alpha; 5) = \frac{B}{30}, \quad D(y, \alpha; 6) = \frac{-A}{90}, \ldots.$$

Hence, the α-parametrized series solution $y(x, \alpha)$ is evaluated up to $N = 6$:

$$
\begin{aligned}
y(x,\alpha) &= \sum_{k=0}^{6} D(y,\alpha;k)(x-x_\alpha)^k \\
&= A+(x+\alpha)B-(x+\alpha)^2A-(x+\alpha)^3\frac{B}{3}+(x+\alpha)^4\frac{A}{6}+(x+\alpha)^5\frac{B}{30}-(x+\alpha)^6\frac{A}{90}
\end{aligned} \tag{3.7}
$$

where $x_\alpha = -\alpha$ and $D(y,\alpha;0) = A$, $D(y,\alpha;1) = B$. Thus, from (3.7),

$$y(-1,\alpha) = A+(\alpha-1)B-(\alpha-1)^2A-(\alpha-1)^3\frac{B}{3}+(\alpha-1)^4\frac{A}{6}+(\alpha-1)^5\frac{B}{30}-(\alpha-1)^6\frac{A}{90}=0$$

and

$$y'(0,\alpha) = B-2A\alpha-B\alpha^2-\frac{2A}{3}\alpha^3+\frac{B}{6}\alpha^4-\frac{A}{15}\alpha^5=1$$

Furthermore, the numbers A and B are evaluated from the boundary conditions (3.2) as follows:

$$A = -\frac{90(-21+5\alpha+20\alpha^2-5\alpha^4+\alpha^5)}{420+30\alpha^2-120\alpha^3+220\alpha^4-216\alpha^5+105\alpha^6+40\alpha^7-30\alpha^8+\alpha^{10}}$$

and

$$B = -\frac{30(-14-126\alpha+15\alpha^2+40\alpha^3-6\alpha^5+\alpha^6)}{420+30\alpha^2-120\alpha^3+220\alpha^4-216\alpha^5+105\alpha^6+40\alpha^7-30\alpha^8+\alpha^{10}}$$

For $\alpha = \frac{1}{2}$, $\alpha = \frac{1}{3}$, and $\alpha = \frac{1}{4}$, the numerical α-PDT solutions are presented in Fig. 2-4 as follows:

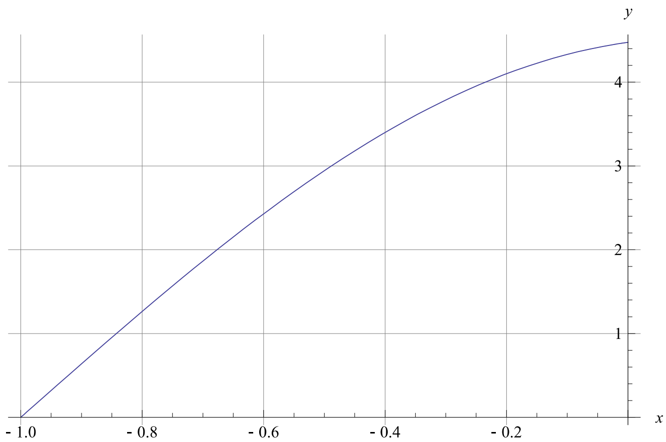

Figure 2. Graph of the numerical α-PDT solution for $\alpha = \frac{1}{2}$

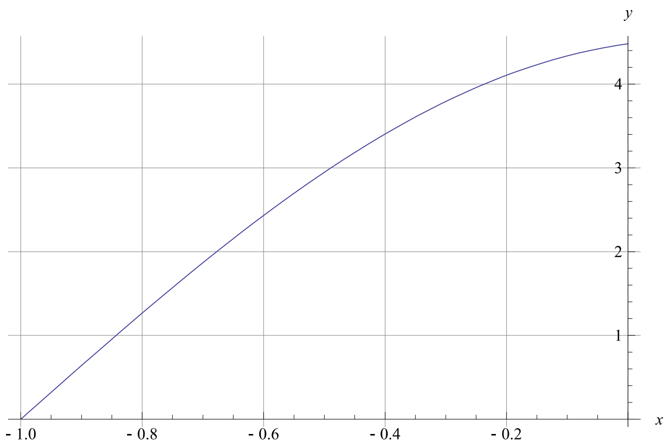

Figure 3. Graph of the numerical α-PDT solution for $\alpha = \frac{1}{3}$

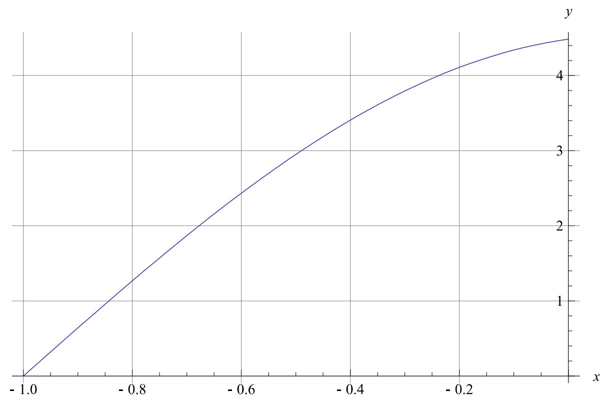

Figure 4. Graph of the numerical α-PDT solution for $\alpha = \frac{1}{4}$

Therefore, for $\alpha = \frac{1}{4}$, the exact, DTM, and α-PDT solutions are presented in Fig. 5 as follows:

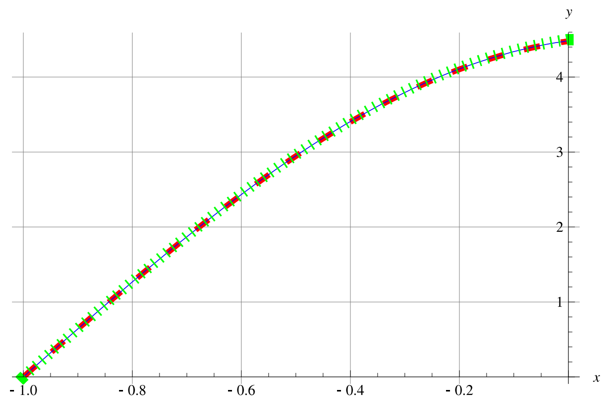

Figure 5. Comparison of the exact solution (red dashing) with DTM (green dotted) and α-PDT solutions for $\alpha = \frac{1}{4}$ (blue line).

Example 3.2: Let us consider the following nonhomogeneous differential equation

$$y''(x) + 2y'(x) + y(x) = e^{-x}, x \in [-1,0] \tag{3.8}$$

with the nonhomogeneous boundary condition

$$y(-1) = 1 \text{ and } y(0) = 0 \tag{3.9}$$

The exact solution for this problem and its graph (Fig. 6) are as follows:

$$y(x) = \frac{1}{2} e^{-1-x} x(-2 + e + ex) \tag{3.10}$$

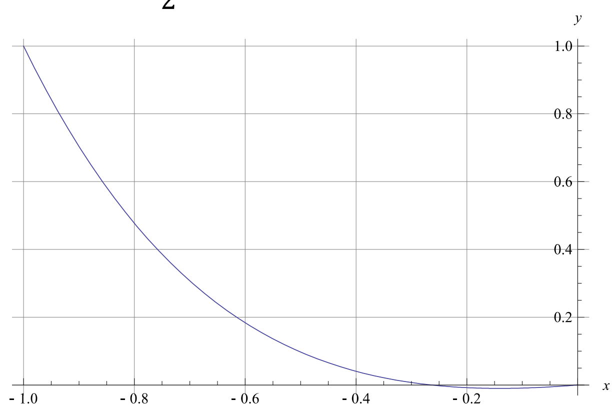

Figure 6. Graph of the exact solution of the problem (3.8)-(3.10)

If it is applied α-PDT to both sides of (3.8), then

$$(k+1)(k+2)D(y,\alpha;k+2) = -2(k+1)D(y,\alpha;k+1) - D(y,\alpha;k) + \frac{(-1)^k}{k!} \tag{3.11}$$

Therefore, from the definition of α-PDT,

$$y_\alpha(x) = \sum_{k=0}^{\infty} D(y,\alpha;k)(x-x_\alpha)^k$$

Moreover, for the boundary conditions $y(-1) = 1$ and $y(0) = 0$,

$$y_\alpha(-1) = \sum_{k=0}^{N} D(y,\alpha;k)(\alpha-1)^k = 1 \tag{3.12}$$

and

$$y_\alpha(0) = \sum_{k=0}^{N} D(y,\alpha;k)(\alpha)^k = 0 \tag{3.13}$$

respectively. Here, denoting $D(y,\alpha;0) = A$ and $D(y,\alpha;1) = B$ and then substituting in the recursive relation (3.11), $D(y,\alpha;2) = \frac{1}{2}[-2B - A + 1]$ is obtained. Now proceeding the iteration using (3.11), we can calculate the other terms of the α-parametrized sequence $D(y,\alpha;n)$ as

$$\mathrm{D}(\mathrm{y},\alpha;3) = \frac{1}{3!}[3\mathrm{B} + 2\mathrm{A} - 3],$$

$$\mathrm{D}(\mathrm{y},\alpha;4) = \frac{1}{12}\left[-2\mathrm{B} - \frac{3}{2}\mathrm{A} + 3\right],$$

$$\mathrm{D}(\mathrm{y},\alpha;5) = \frac{1}{20}\left[\frac{5}{6}\mathrm{B} + \frac{2}{3}\mathrm{A} - \frac{10}{6}\right],$$

$$\mathrm{D}(\mathrm{y},\alpha;6) = \frac{1}{30}\left[-\frac{3}{12}\mathrm{B} - \frac{5}{24}\mathrm{A} + \frac{15}{24}\right],$$

$$\vdots$$

Hence, the α-PDT series solution $y(x,\alpha)$ is evaluated up to $N = 6$:

$$\begin{aligned} y(x,\alpha) &= \sum_{k=0}^{6} D(y,\alpha;k)(x-x_\alpha)^k \\ &= A + (x+\alpha)B + (x+\alpha)^2\frac{1}{2}[-2B - A + 1] + (x+\alpha)^3\frac{1}{3!}[3B + 2A - 3] \\ &\quad +(x+\alpha)^4\frac{1}{12}\left[-2\mathrm{B} - \frac{3}{2}\mathrm{A} + 3\right] + (x+\alpha)^5\frac{1}{20}\left[\frac{5}{6}\mathrm{B} + \frac{2}{3}\mathrm{A} - \frac{10}{6}\right] \\ &\quad +(x+\alpha)^6\frac{1}{30}\left[-\frac{3}{12}\mathrm{B} - \frac{5}{24}\mathrm{A} + \frac{15}{24}\right] \end{aligned} \tag{3.14}$$

where $x_\alpha = -\alpha$ and $D(y,\alpha;0) = A, D(y,\alpha;1) = B$. Thus, from (3.14),

$$\begin{aligned} y(-1,\alpha) = A + B(-1+\alpha) + \frac{1}{2}(1 - A - 2B)(-1+\alpha)^2 + \frac{1}{6}(-3 + 2A + 3B)(-1+\alpha)^3 + \frac{1}{12}(3 - \frac{3A}{2} \\ - 2B)(-1+\alpha)^4 + \frac{1}{20}(-\frac{5}{3} + \frac{2A}{3} + \frac{5B}{6})(-1+\alpha)^5 + \frac{1}{30}(\frac{5}{8} + \frac{5A}{24} - \frac{B}{4})(-1+\alpha)^6 = 1 \end{aligned}$$

and

$$y(0,\alpha) = A + B\alpha + \frac{1}{2}(1 - A - 2B)\alpha^2 + \frac{1}{6}(-3 + 2A + 3B)\alpha^3 + \frac{1}{12}(3 - \frac{3A}{2} - 2B)\alpha^4 + \frac{1}{20}(-\frac{5}{3} + \frac{2A}{3} + \frac{5B}{6})\alpha^5 + \frac{1}{30}(\frac{5}{8} + \frac{5A}{24} - \frac{B}{4})\alpha^6 = 0$$

Furthermore, the numbers A and B are evaluated from the boundary conditions (3.9) as follows:

$$A = -(15(-2040\alpha + 17496\alpha^2 - 27612\alpha^3 + 23404\alpha^4 - 13029\alpha^5 + 4977\alpha^6 - 1274\alpha^7 + 210\alpha^8 - 21\alpha^9 + \alpha^{10}))/(-234720 + 467400\alpha - 457320\alpha^2 + 278940\alpha^3 - 99200\alpha^4 - 649\alpha^5 + 23905\alpha^6 - 13250\alpha^7 + 3790\alpha^8 - 625\alpha^9 + 49\alpha^{10}$$

and

$$B = (5(-6120 + 69840\alpha - 77088\alpha^2 + 33108\alpha^3 + 3231\alpha^4 - 14236\alpha^5 + 11276\alpha^6 - 5068\alpha^7 + 1471\alpha^8 - 260\alpha^9 + 22\alpha^{10}))/(-234720 + 467400\alpha - 457320\alpha^2 + 278940\alpha^3 - 99200\alpha^4 - 649\alpha^5 + 23905\alpha^6 - 13250\alpha^7 + 3790\alpha^8 - 625\alpha^9 + 49\alpha^{10})$$

For $\alpha = \frac{1}{4}$ and $\alpha = \frac{1}{20}$, the numerical α-PDT solutions are presented in Fig. 7 and 8 as follows:

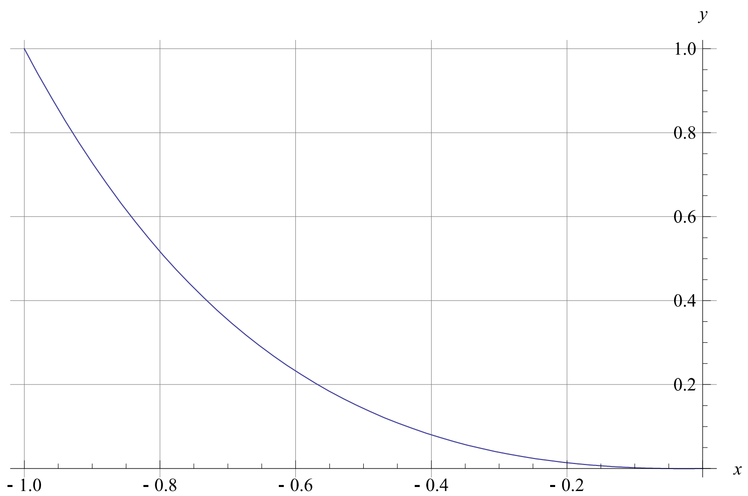

Figure 7. Graph of the numerical α-PDT solution for $\alpha = \frac{1}{4}$

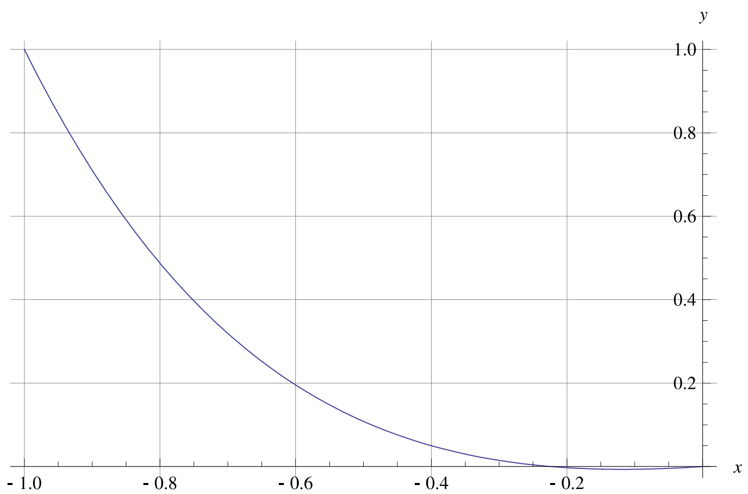

Figure 8. Graph of the numerical α-PDT solution for $\alpha = \frac{1}{20}$

Therefore, for $\alpha = \frac{1}{20}$, the exact and α-PDT solutions are presented in Fig. 9 as follows:

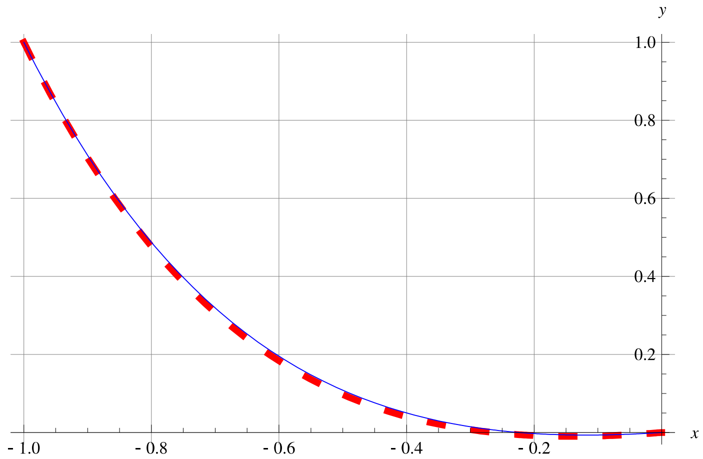

Figure 9. Comparison of the exact solution (red dashing) with the α-PDT solution for $\alpha = \frac{1}{20}$ (blue line).

Finally, the exact and DTM solutions are presented in Fig. 10 as follows:

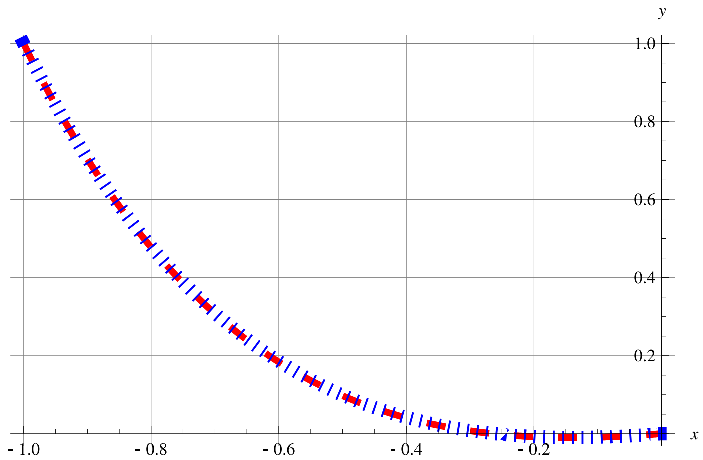

Figure 10. Comparison of the exact solution (red dashing) with the DTM solution (blue dotted).

4. Conclusion

In this study, we have proposed a new generalization of the classical differential transform method, the so-called α-Parameterized Differential Transform Method (α-PDTM), to find approximate solutions to boundary value problems for differential equations. We then compared the obtained approximate α-PDT solutions with the DTM solutions and exact solutions. The results show that α-PDTM is an efficient and reliable method.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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