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Fixed Point Theorems for Almost Contractions in Extended b-Metric Spaces

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Abstract

In this paper, we define the concept of almost contraction in extended b-metric spaces. We prove some common fixed point theorems for mappings satisfying almost contractions in extended b-metric spaces. These results extend and generalize the corresponding results given in the literature.

1. Introduction and Preliminaries

Main and earlier result of fixed point theory is the Banach contraction principle which guarantees existence and uniqueness of fixed point. It was proved in complete metric spaces by Banach in 1922. Banach contraction principle was applied as a important method in mathematics and other sciences. Some problems of Mathematics and other sciences didn't solve using Banach contraction principle. Thus, more general fixed point theorems be needed. Some of these theorems were gived in more general spaces of metric spaces, some of them were gived by new contaction mappings which are more general than Banach contraction principle. $b-$metric spaces was introduced by Bakhtin [3] and Czerwik [8] as a generalizations of metric spaces. They proved the contraction mapping principle in b-metric spaces. Recently, Kamran [11] introduced extended b-metric spaces using the idea of b-metric spaces as a new type of generalized metric space and they proved some fixed point theorems on this space. Also some generalized fixed point theorems proved in extended b-metric spaces [1, 2, 6, 7, 9, 10, 12–14].
In this work, we define almost contraction in extended b-metric spaces which was defined in metric spaces by Berinde [4, 5]. And we prove fixed point theorems for mappings satisfying these type contractions.

Definition 1.1. *[11] Let X be a nonempty set and $\theta : X\times X \to [1,\infty)$ be a mapping. A function $d_\theta : X\times X \to [0,\infty)$ is called extended b-metric if it satisfies, for all $x,y,z \in X$*

($d_\theta 1$) $d_\theta(x,y) = 0$ *if and only if* $x = y$,
($d_\theta 2$) $d_\theta(x,y) = d_\theta(y,x)$,
($d_\theta 3$) $d_\theta(x,y) \le \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)]$.

In this case, the pair (X,d_θ) is called extended b-metric space, in short extended-bMS.

Example 1.2. *[11] Let $X = \{1,2,3\}$ and $\theta : X\times X \to [1,\infty)$, $\theta(x,y) = 1+x+y$. Define $d_\theta : X\times X \to [0,\infty)$ as*

$$\begin{aligned} d_\theta(x,y) &= 0 \quad \text{for } x=y \\ d_\theta(1,2) &= 80, \quad d_\theta(1,3) = 1000, \quad d_\theta(2,3) = 600. \end{aligned}$$

Then, (X,d_θ) is an extended-bMS.

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Example 1.3. *[1] Let $X=[0,1]$ and $\theta : X\times X\to[1,\infty)$,*

$$\begin{aligned}\theta(x,y) &= \frac{x+y+1}{x+y} \quad \text{for } x,y\in(0,1]\\ \theta(x,y) &= 1, \quad \text{for } x,y=0\end{aligned}$$

Define $d_\theta : X\times X\to[0,\infty)$ as

$$\begin{aligned}d_\theta(x,y) &= \frac{1}{xy} \quad \text{for } x,y\in(0,1],\ x\neq y,\\ d_\theta(x,y) &= 0 \quad \text{for } x,y\in[0,1],\ x=y,\\ d_\theta(x,0) &= \frac{1}{x} \quad \text{for } x\in(0,1].\end{aligned}$$

Then, (X,d_θ) is an extended-bMS.

Definition 1.4. *[11] Let (X,d_θ) be an extended-bMS.*
(i) *A sequence $\{x_n\}$ in X is said to converge to $x\in X$, if for every $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that $d_\theta(x_n,x)<\varepsilon$ for all $n\geq N$. In this case, we write*

$$\lim_{n\to\infty} x_n = x.$$

(ii) *A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that $d_\theta(x_n,x_m)<\varepsilon$ for all $n,m\geq N$.*
(iii) *(X,d_θ) is said to be complete if every Cauchy sequence in X is convergent.*

Let (X,d_θ) be extended-*bMS*. If d_θ is continuous, then every convergent sequence has a unique limit.

2. Fixed point theorems

Theorem 2.1. *Let (X,d_θ) be a complete extended-bMS and $f,g : X\to X$ be two self mappings satisfying*

$$d_\theta(fx,gy)\leq \delta M(x,y)+LN(x,y) \tag{2.1}$$

for all $x,y\in X$, where $\delta\in[0,1)$ and $L\geq 0$ such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{\delta}$ with $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for $n\geq 1$ and

$$M(x,y)=\max\{d_\theta(x,y),d_\theta(x,fx),d_\theta(y,gy)\}$$

$$N(x,y)=\min\{d_\theta(x,fx),d_\theta(y,gy),d_\theta(x,gy),d_\theta(y,fx)\}.$$

Then f and g have a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$ in X as $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for $n\geq 1$. Suppose that there is some $n\geq 1$ such that $x_n=x_{n+1}$. If $n=2k$, then $x_{2k}=x_{2k+1}$ and from (2.1),

$$d_\theta(x_{2k+1},x_{2k+2})=d_\theta(fx_{2k},gx_{2k+1})\leq\delta M(x_{2k},x_{2k+1})+LN(x_{2k},x_{2k+1})$$

where

$$\begin{aligned}M(x_{2k},x_{2k+1}) &= \max\{d_\theta(x_{2k},x_{2k+1}),d_\theta(x_{2k},fx_{2k}),d_\theta(x_{2k+1},gx_{2k+1})\}\\ &= \max\{d_\theta(x_{2k},x_{2k+1}),d_\theta(x_{2k},x_{2k+1}),d_\theta(x_{2k+1},x_{2k+2})\}\\ &= \max\{0,0,d_\theta(x_{2k+1},x_{2k+2})\}\end{aligned}$$

and

$$\begin{aligned}N(x_{2k},x_{2k+1}) &= \min\{d_\theta(x_{2k},fx_{2k}),d_\theta(x_{2k+1},gx_{2k+1}),d_\theta(x_{2k},gx_{2k+1}),d_\theta(x_{2k+1},fx_{2k})\}\\ &= \min\{d_\theta(x_{2k},x_{2k+1}),d_\theta(x_{2k+1},x_{2k+2}),d_\theta(x_{2k},x_{2k+2}),d_\theta(x_{2k+1},x_{2k+1})\}\\ &= 0\end{aligned}$$

Thus, we have

$$d_\theta(x_{2k+1},x_{2k+2})\leq\delta d_\theta(x_{2k+1},x_{2k+2})$$

which is a contradiction with $\delta\in[0,1)$. Therefore $x_{2k+1}=x_{2k+2}$. Hence, we have $x_{2k}=x_{2k+1}=x_{2k+2}$. It means that $x_{2k}=fx_{2k}=gx_{2k}$,i.e. x_{2k} is a common fixed point of f and g.
If $n=2k+1$, then using same arguments, it can be shown that x_{2k+1} is a common fixed point of f and g.
Now, suppose $x_n\neq x_{n+1}$ for all $n\geq 1$.

$$d_\theta(x_{2n+1},x_{2n+2})=d_\theta(fx_{2n},gx_{2n+1})\leq\delta M(x_{2n},x_{2n+1})+LN(x_{2n},x_{2n+1}) \tag{2.2}$$

where

$$\begin{aligned} M(x_{2n},x_{2n+1}) &= \max\{d_\theta(x_{2n},x_{2n+1}), d_\theta(x_{2n},fx_{2n}), d_\theta(x_{2n+1},gx_{2n+1})\} \\ &= \max\{d_\theta(x_{2n},x_{2n+1}), d_\theta(x_{2n},x_{2n+1}), d_\theta(x_{2n+1},x_{2n+2})\} \\ &= \max\{d_\theta(x_{2n},x_{2n+1}), d_\theta(x_{2n+1},x_{2n+2})\} \end{aligned}$$

and

$$\begin{aligned} N(x_{2n},x_{2n+1}) &= \min\{d_\theta(x_{2n},fx_{2n}), d_\theta(x_{2n+1},gx_{2n+1}), d_\theta(x_{2n},gx_{2n+1}), d_\theta(x_{2n+1},fx_{2n})\} \\ &= \min\{d_\theta(x_{2n},x_{2n+1}), d_\theta(x_{2n+1},x_{2n+2}), d_\theta(x_{2n},x_{2n+2}), 0\} \\ &= 0. \end{aligned}$$

If $M(x_{2n},x_{2n+1}) = d_\theta(x_{2n+1},x_{2n+2})$, then by (2.2)

$$d_\theta(x_{2n+1},x_{2n+2}) \le \delta d_\theta(x_{2n+1},x_{2n+2})$$

which is a contradiction. Thus $M(x_{2n},x_{2n+1}) = d_\theta(x_{2n},x_{2n+1})$ and from (2.2)

$$d_\theta(x_{2n+1},x_{2n+2}) \le \delta d_\theta(x_{2n},x_{2n+1}).$$

Similarly it can be proved that

$$d_\theta(x_{2n+3},x_{2n+2}) \le \delta d_\theta(x_{2n+2},x_{2n+1}).$$

So,

$$d_\theta(x_{n+1},x_n) \le \delta d_\theta(x_n,x_{n-1}) \le \delta^n d_\theta(x_1,x_0)$$

for all $n \ge 1$.
We show that $\{x_n\}$ is a Cauchy sequence. For all $p \ge 1$,

$$\begin{aligned} d_\theta(x_n,x_{n+p}) &\le \theta(x_n,x_{n+p})\left[d_\theta(x_n,x_{n+1}) + d_\theta(x_{n+1},x_{n+p})\right] \\ &\le \theta(x_n,x_{n+p})\left[\delta^n d_\theta(x_0,x_1) + d_\theta(x_{n+1},x_{n+p})\right] \\ &\quad ... \\ &\le \theta(x_n,x_{n+p})\,\delta^n d_\theta(x_0,x_1) + \theta(x_n,x_{n+p})\,\theta(x_{n+1},x_{n+p})\,\delta^{n+1} d_\theta(x_0,x_1) \\ &\quad + ... + \theta(x_n,x_{n+p}) ... \theta(x_{n+p-1},x_{n+p})\,\delta^{n+p-1} d_\theta(x_0,x_1) \\ &= d_\theta(x_0,x_1) \sum_{i=1}^{n+p-1} \delta^i \prod_{j=1}^{i} \theta(x_{n+j},x_{n+p}). \end{aligned}$$

The last inequality is dominated by

$$\sum_{i=1}^{n+p-1} \delta^i \prod_{j=1}^{i} \theta(x_{n+j},x_{n+p}) \le \sum_{i=1}^{n+p-1} \delta^i \times \prod_{j=1}^{i} \theta(x_j,x_{n+p}).$$

By the ratio test, the series$\sum_{i=1}^{\infty} S_i$ where $S_i = \delta^i \prod_{j=1}^{i} \theta(x_j,x_{n+p})$ converges to some $z \in (0,\infty)$. Indeed, $\lim_{i\to\infty} \frac{S_{i+1}}{S_i} = \lim_{i\to\infty} \delta\theta(x_i,x_{i+p}) < 1$.
Thus, we have $a = \sum_{i=1}^{\infty} \delta^i \prod_{j=1}^{i} \theta(x_j,x_{n+p})$ with the partial sum $a_n = \sum_{i=1}^{n} \delta^i \prod_{j=1}^{i} \theta(x_j,x_{n+p})$.
Hence, for $n \le 1$, $p \le 1$ we have

$$d_\theta(x_n,x_{n+p}) \le \delta^n d_\theta(x_0,x_1)\left[S_{n+p-1} - S_{n-1}\right]. \tag{2.3}$$

Letting $n \to \infty$ in (2.3), we conclude that the sequence $\{x_n\}$ is a Cauchy sequence. By completeness of (X,d_θ), there exists $r \in X$ such that $x_n \to r$ as $n \to \infty$.
Now, we prove that $fr = r$. By $b-$rectangular inequality ,

$$d_\theta(x_{2n+1},gr) = d_\theta(fx_{2n},gr) \le \delta M(x_{2n},r) + LN(x_{2n},r)$$

where

$$M(x_{2n},r) = \max\{d_\theta(x_{2n},r), d_\theta(x_{2n},x_{2n+1}), d_\theta(r,gr)\} \to d_\theta(r,gr),$$

as $n \to \infty$ and

$$N(x_{2n},r) = \min\{d(x_{2n},x_{2n+1}), d(r,gr), d(x_{2n},gr), d(r,x_{2n+1})\} \to 0.$$

Hence, taking the limit as $n \to \infty$, we obtain

$$d_\theta(r,gr) \le \delta d_\theta(r,gr) + L.0$$

that is $fr = r$. Hence r is a fixed point of g.
Now, we show $r = fr$. Suppose $r \neq fr$, By (2.1)

$$d_\theta(fr,r) = d_\theta(fr,gr) \leq \delta M(r,r) + LN(r,r)$$

where

$$\begin{aligned} M(r,r) &= \max\{d_\theta(r,r), d_\theta(r,fr), d_\theta(r,gr)\} \\ &= \max\{0,0,d_\theta(r,fr)\} \\ &= d_\theta(r,gr) \end{aligned}$$

and

$$\begin{aligned} N(r,r) &= \min\{d_\theta(r,fr), d_\theta(r,gr), d_\theta(r,gr), d_\theta(r,fr)\} \\ &= 0. \end{aligned}$$

Thus, we have

$$d_\theta(fr,r) \leq \delta d_\theta(fr,r)$$

which is a contradiction. Thus $r = fr$.
Now, we show that uniqueness, Suppose r and t are different common fixed points of f and g. By (2.1),

$$d_\theta(r,t) = d_\theta(fr,gt) \leq \delta M(r,t) + LN(r,t) \tag{2.4}$$

where

$$\begin{aligned} M(r,t) &= \max\{d_\theta(r,t), d_\theta(r,fr), d_\theta(t,gt)\} \\ &= d_\theta(r,t) \end{aligned}$$

and

$$\begin{aligned} N(r,t) &= \min\{d_\theta(r,fr), d_\theta(t,gt), d_\theta(r,gt), d_\theta(t,fr)\} \\ &= 0. \end{aligned}$$

From (2.4)

$$d_\theta(r,t) \leq \delta d_\theta(r,t)$$

So $d_\theta(r,t) = 0$, i.e. $r = t$.

Example 2.2. *Let $X = [0,1]$ and $\theta : X \times X \to [1,\infty)$, $\theta(x,y) = 1 + x + y$. Define $d_\theta : X \times X \to [0,\infty)$ such that $d_\theta(x,y) = (x-y)^2$ with for all $x,y \in X$. Let $f,g : X \to X$ be defined as*

$$f(x) = \frac{x}{2}, \quad g(x) = \frac{3x}{4}.$$

Then, d_θ is complete extended $b-$metric on X. We have

$$d_\theta(fx,gy) = \left(\frac{x}{2} - \frac{3y}{4}\right)^2 \leq \delta M(x,y) + LN(x,y)$$

where

$$M(x,y) = \max\left\{(x-y)^2, \left(\frac{x}{2}\right)^2, \left(\frac{y}{4}\right)^2\right\}$$

$$N(x,y) = \min\left\{\left(\frac{x}{2}\right)^2, \left(\frac{y}{4}\right)^2, \left(x - \frac{3y}{4}\right)^2, \left(y - \frac{x}{2}\right)^2\right\}.$$

with $\delta = \frac{3}{4}$ and $L \geq 0$.

□

If $x = y$,

$$d_\theta(fx,gy) = \left(\frac{x}{2} - \frac{3y}{4}\right)^2 = \left(\frac{x}{4}\right)^2 \leq \frac{3}{4}\left(\frac{x}{2}\right)^2 + L\left(\frac{x}{4}\right)^2.$$

If $x = 0, y \neq 0$

$$d_\theta(fx,gy) = \left(0 - \frac{3y}{4}\right)^2 = \left(\frac{3y}{4}\right)^2 \leq \frac{3}{4}y^2 + L.0.$$

If $y=0, x\neq 0$,

$$d_{\theta}(fx,gy)=\left(\frac{x}{2}-0\right)^{2}\leq\frac{3}{4}x^{2}+L.0.$$

If $x\neq y\neq 0$,

$$d_{\theta}(fx,gy)=\left(\frac{x}{2}-\frac{3y}{4}\right)^{2}\leq\frac{3}{4}M(x,y)+LN(x,y)$$

Also, for each $x\in X$ $f^{n}x=\frac{x}{2^{n}}$, we have

$$\lim_{n,m\to\infty}\theta(x_n,x_m)=\lim_{n,m\to\infty}\theta\left(\frac{x}{2^{n}}+\frac{x}{2^{m}}+1\right)<\frac{4}{3}.$$

Thus all conditions of Theorem 2.1 are satisfied and $x=0$ is a unique fixed point of f and g.

Corollary 2.3. *Let (X,d_{θ}) be a complete extended-bMS space and $f,g:X\to X$ be self mappings satisfying*

$$d_{\theta}(fx,gy)\leq\delta d_{\theta}(x,y)+L\min\{d_{\theta}(x,fx),d_{\theta}(y,gy),d_{\theta}(x,gy),d_{\theta}(y,fx)\}$$

for all $x,y\in X$, where $\delta\in[0,1)$ and $L\geq 0$ such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{\delta}$ with $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for $n\geq 1$. Then f and g have a unique fixed point.

Corollary 2.4. *Let (X,d_{θ}) be a complete extended-bMS space and $f:X\to X$ be a self mapping satisfying*

$$d_{\theta}(fx,fy)\leq\delta M(x,y)+LN(x,y)$$

for all $x,y\in X$, where $\delta\in[0,1)$ and $L\geq 0$ such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{\delta}$ with $x_{n+1}=fx_n$, where

$$M(x,y)=\max\{d_{\theta}(x,y),d_{\theta}(x,fx),d_{\theta}(y,fy)\}$$

$$N(x,y)=\min\{d_{\theta}(x,fx),d_{\theta}(y,fy),d_{\theta}(x,fy),d_{\theta}(y,fx)\}.$$

Then f has a unique fixed point.

Corollary 2.5. *Let (X,d_{θ}) be a complete extended-bMS space and $f:X\to X$ be a self mapping satisfying*

$$d_{\theta}(fx,fy)\leq\delta d_{\theta}(x,y)+L\min\{d_{\theta}(x,fx),d_{\theta}(y,fy),d_{\theta}(x,fy),d_{\theta}(y,fx)\}$$

for all $x,y\in X$, where $\delta\in[0,1)$ and $L\geq 0$ such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{\delta}$ with $x_{n+1}=fx_n$. Then f has a unique fixed point.

3. Conclusion

The development of the field of fixed point theory depends on the generalization of the Banach Contraction principle on complete metric spaces. This generalization or extension comes up by either introducing new types of contractions or by working on a more general structured space such as extended b-metric spaces. In this article, we have proven some fixed point theorems for almost contraction in extended b-metric spaces and hence our results generalize many existing results in the literature.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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