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Research

APPROXIMATION PROPERTIES OF MODIFIED SZASZ-SCHURER BASKAKOV TYPE OPERATORS

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ABSTRACT

In the present paper, we study some approximation properties of modified Szász-Schurer-Baskakov type operators. We estimate the moments for these operators using the Hypergeometric series, which are related to Laguerre polynomials. We give approximation properties of derivatives of these operators. Finally, we obtain the Voronovskaya type theorem for derivatives of these operators.

Keywords: Modified Szász-Schurer-Baskakov operator; Szász operator; point-wise convergence; approximation of derivatives; Modulus of continuity; Voronovskaya type theorem.

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1. INTRODUCTION

In 1962, Schurer defined the Bernstein-Schurer operators for any $n \in \mathbb{N},, f \in C[0,1+p]$ and non-negative integers p using well-known Bernstein operators[7]. He defined Bernstein-Schurer operators for $B_{n,p}: C[0,1+p] \to C[0,1]$as follows

$$B_{n,p}(f)(x) = \sum_{k=0}^{n+p} f\left(\frac{k}{n}\right)\binom{n+p}{k} x^k (1-x)^{n+p-k}, x \in [0,1]$$

and studied the approximation properties of these operators. It should be noted that the special case of $p = 0$ here gives classical Bernstein operators.

Later in 1965. [6], Schurer generalized the well-known Szasz operator and the Baskakov operator for every $p \in \mathbb{N}_0, f \in E_2, x \in [0,\infty), n \geq 1$, the $n-th$ Schurer-Szász-Mirakjan operator $M_{n,p}: E_2 \to C[0,\infty)$ and the $n-th$ Baskakov-Schurer operator $A_{n,p}: E_2 \to C[0,\infty)$ as

$$M_{n,p}(f)(x) \coloneqq exp[-(n+p)x] \sum_{k=0}^{\infty} \frac{[(n+p)x]^k}{k!} f\left(\frac{k}{n}\right)$$

and

$$A_{n,p}(f)(x) \coloneqq (1+x)^{-n-p} \sum_{k=0}^{\infty} \binom{n+p+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right)$$

repectively. Where $E_2 \coloneqq \left\{f \in C[0,\infty): \frac{f(x)}{1+x^2} \, is\ convergent\ as\ x \to \infty\right\}$

Many studies have been made on various generalizations of the above operators and their convergence properties, and such studies are still ongoing, [1], [2], [10], [11], [13], [16], [17], [20], [22] .

In 1967, J. L. Durmeyer defined one-dimensional Bernstein-Durrmeyer operators as

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$$D_n(f;x) \coloneqq (n+1)\sum_{k=0}^{\infty}\binom{n}{k}x^k(1-x)^{n-k}\int_0^1\binom{n}{k}u^k(1-u)^{n-k}f(u)du, \qquad x\in[0,1], n=1,2,\ldots$$

and studied the approximation properties of these operators. Where f is any real-valued function on [0,1], which is integrable with respect to the kernel [8].

In 1995 [21],.Gupta and Srivastava studied the convergence properties of the derivatives of the Szász-Mirakyan-Baskakov type operators defined as follows;

$$M_n(f;x) \coloneqq (n-1)\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\frac{(n+k-1)!}{k!(n-1)!}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k}}f(t)dt,\ x\in[0,\infty). \tag{1.1}$$

Later, many researchers defined Durrmeyer-type extensions and many modifications of other well-known operators and studied their convergence properties, [3], [9], [12], [14], [15].

Also, in 2006, for $f\in C_\gamma[0,\infty)\equiv\{f\in C[0,\infty):|f(t)|\le Mt^\gamma,\gamma>0\}$, Gupta et al. defined the Baskakov-Durmeyerr operators as an integral modification of the Baskakov operators as follows

$$B_n(f;x) \coloneqq \sum_{k=1}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\frac{1}{B(n+1,k)}\int_0^{\infty}\frac{t^{k-1}}{(1+t)^{n+k+1}}f(t)dt+\frac{f(0)}{(1+x)^n}, \tag{1.2}$$

and studied the convergence properties of the derivatives of these operators, [8].

In 2012, Verma et al. defined a Stancu type extension of operator of (1.1) type, gave the approximation properties of these operators, the error estimation and obtained some recurrence relations [4]. In 2012, Gupta et al. examined the point convergence properties of the Stancu extension of the operators (1.2) and gave Voronovskaya type theorems,[19]. In 2014, Agraval et al. gave a different generalization of the (1.2) operators and examined the properties of uniform convergence and point convergence, [12].

In 2015, Pandey et al. a Szàsz-Baskakov Stancu type generalization of operators is defined as follows:

$$M_n(f;x) \coloneqq (n-1)\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\frac{(n+k-1)!}{k!(n-1)!}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k}}f\left(\frac{nt+\alpha}{n+\beta}\right)dt,\ x\in[0,\infty). \tag{1.3}$$

and examined various approximation properties,[5]. They also benefited from the properties of hypergeometric functions while doing these investigations.

By defining the Schurer generalization of the (1.1) operator, we wanted to achieve similar results.

For $f\in C[0,\infty)$, we define the Schurer type generalization of the Szász-Mirakyan-Baskakov type operator given in (1.1) as follows

$$A_{n,a,\beta_n}(f;x) := (n+a-1)\sum_{k=0}^{\infty}p_{n,\beta_n,k}(x)\int_0^{\infty}q_{n,a,k}(t)f(t)dt \ \ ;\ a\in N_0, x\in[0,\infty), \tag{1.4}$$

where β_n is a sequence of positive numbers such that

$$\lim_{n\to\infty}\beta_n=0 \tag{1.5}$$

and

$$p_{n,\beta_n,k}(x)=e^{-(n+\beta_n)x}\frac{[(n+\beta_n)x]^k}{k!},\ \ q_{n,a,k}(t)=\frac{(n+a+k-1)!}{k!(n+a-1)!}\frac{t^k}{(1+t)^{n+a+k}} \tag{1.6}$$

It is clear that for $x\in[0,\infty)$ the operators A_{n,a,β_n} are linear and positive.

Now, let's give some important information that we will benefit from when examining the convergence properties of the operator we have defined.

Well known Beta and Gamma functions provides the equality

$$B(x,y)=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\frac{(x-1)!(y-1)!}{(x+y-1)!}. \tag{1.7}$$

We can represent the operator defined by (1.4) in a different way by using the hyper-geometric functions defined as

$${}_1F_1(\alpha;\beta;t)=\sum_{k=0}^\infty \frac{(\alpha)_k}{(\beta)_k}\frac{x^k}{k!} \tag{1.8}$$

Where $(\alpha)_k$ is called Pochhammer symbol defined as

$$(\alpha)_k=\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1) \tag{1.9}$$

and $(1)_k=k!$. Using hyper-geometric functions, we can give an equivalent definition of the operators $A_{n,a,\beta_n}(f;x)$ as follows;

$$A_{n,a,\beta_n}(f;x)=(n+a-1)\int_0^\infty \frac{e^{-(n+\beta_n)x}}{(1+t)^{n+a}}f(t)\ {}_1F_1\left(n+a;1;\frac{(n+\beta_n)xt}{1+t}\right)dt.$$

Our aim in this study is to examine the convergence properties of the operator (1.4), which we define as a generalization of the (1.1) operator, and its derivatives, taking into account the study of Pandey et al., and evaluate the results. Let us give some recurrence relations and lemmas that we need for this.

2. SOME MOMENTS AND RECURRENCE RELATIONS

In this section, we give some lemmas that we will use in this study.

Lemma 1. For all $x\in[0,\infty)$, the operators A_{n,a,β_n} defined by (1.4) satisfy the followings:

$$A_{n,a,\beta_n}(1;x)=1 \tag{2.1}$$

$$A_{n,a,\beta_n}(t;x)=\frac{(n+\beta_n)x+1}{(n+a-2)} \tag{2.2}$$

$$A_{n,a,\beta_n}(t^2;x)=\frac{[(n+\beta_n)x]^2+4[(n+\beta_n)x]+2}{(n+a-2)(n+a-3)} \tag{2.3}$$

The proof can be easily done from the definition of the operator A_{n,a,β_n}.

Lemma 2. Let $r\in N_0$. If the $r-th$ order moment is defined as

$$u_{n+\beta_n,r}(x)=\sum_{k=0}^\infty p_{n,\beta_n,k}(x)\left(\frac{k}{n+\beta_n}-x\right)^r=\sum_{k=0}^\infty e^{-(n+\beta_n)x}\frac{[(n+\beta_n)x]^k}{k!}\left(\frac{k}{n+\beta_n}-x\right)^r \tag{2.4}$$

then there exists a reccurence relation

$$(n+\beta_n)u_{n+\beta_n,r+1}(x)=x\left[u'_{n+\beta_n,r}(x)+ru_{n+\beta_n,r-1}(x)\right]. \tag{2.5}$$

Consequently

1. $u_{n+\beta_n,r}(x)$ is a polynomial in x of degree $\le r$.
2. $u_{n+\beta_n,r}(x)=O\left(\frac{1}{n^{\left[\frac{r+1}{2}\right]}}\right)$; as $n\to\infty$.

The proof can be done using the definition.

Lemma 3. If wedefine the central moments as

$$T_{n,a,\beta_n,r}(x) = A_{n,a,\beta_n}((t-x)^2;x) = (n+a-1)\sum_{k=0}^{\infty} p_{n,\beta_n,k}(x)\int_0^{\infty} q_{n,a,k}(t)(t-x)^r dt, \quad (2.6)$$

then

$$T_{n,a,\beta_n,0}(x) = 1$$

$$T_{n,a,\beta_n,1}(x) = \frac{(\beta_n - a + 2)x + 1}{(n+a-2)}$$

$$T_{n,a,\beta_n,2}(x) = \frac{[(n-a-3) + (\beta_n - a + 3)^2]x^2 + (2n + 4\beta_n - 2a + 6)x + 2}{(n+a-2)(n+a-3)}$$

and for $n > 1$ we have the following recurrence relation;

$$(n+a-r-2)T_{n,a,\beta_n,r+1}(x) = xT'_{n,a,\beta_n,r}(x) + [(2r + a - \beta_n + 2)x + r + 1]T_{n,a,\beta_n,r}(x) + (x^2+2x)rT_{n,a,\beta_n,r-1}(x). \quad (2.7)$$

From the recurrence relation, it can be easily verified that for all $x \in [0,\infty)$, we have $T_{n,a,\beta_n,r}(x) = O\left(\frac{1}{n^{\left[\frac{r+1}{2}\right]}}\right)$.

Proof. Since the operatör A_{n,a,β_n} is linear, we get $T_{n,a,\beta_n,0}(x) = 1$, $T_{n,a,\beta_n,1}(x) = \frac{(\beta_n - a + 2)x + 1}{(n+a-2)}$,

$T_{n,a,\beta_n,2}(x) = \frac{[(n-a-3)+(\beta_n-a+3)^2]x^2+(2n+4\beta_n-2a+6)x+2}{(n+a-2)(n+a-3)}$. From (2.6), we can write

$$T'_{n,a,\beta_n,r}(x) = \frac{k}{x}T_{n,a,\beta_n,r}(x) - (n+\beta_n)T_{n,a,\beta_n,r}(x) - rT_{n,a,\beta_n,r-1}(x)$$

$\Longrightarrow$

$$xT'_{n,a,\beta_n,r}(x) = [k - (n+\beta_n)]T_{n,a,\beta_n,r}(x) - xrT_{n,a,\beta_n,r-1}(x).$$

On the other hand, using $t(1+t)q'_{n,a,k}(t) = [k-(n+a)t]q_{n,a,k}(t)$ and $xp'_{n,\beta_n,k}(x) = [k-(n+\beta_n)x]p_{n,\beta_n,k}(x)$, we obtain

$$\begin{aligned} xT'_{n,a,\beta_n,r}(x) + rxT_{n,a,\beta_n,r-1}(x) &= (n+a-1)\sum_{k=0}^{\infty} p_{n,\beta_n,k}(x)\int_0^{\infty} q'_{n,a,k}(t)(t-x)^{r+2}dt \\ &+(n+a-1)(2x+1)\sum_{k=0}^{\infty} p_{n,\beta_n,k}(x)\int_0^{\infty} q'_{n,a,k}(t)(t-x)^{r+1}dt \\ &+(n+a-1)(3x^2+x)\sum_{k=0}^{\infty} p_{n,\beta_n,k}(x)\int_0^{\infty} q'_{n,a,k}(t)(t-x)^{r}dt \\ &+(n+a)T_{n,a,\beta_n,r+1}(x) - (\beta_n - a)xT_{n,a,\beta_n,r}(x). \end{aligned}$$

If partial integration is applied to the integrals here, we get

$$\begin{aligned} &xT'_{n,a,\beta_n,r}(x) + rxT_{n,a,\beta_n,r-1}(x) \\ &= (n+a-r-2)T_{n,a,\beta_n,r+1}(x) - (2xr + 2x + r + 1 - \beta_n x + ax)T_{n,a,\beta_n,r}(x) - (x^2+x)rT_{n,a,\beta_n,r-1}(x). \end{aligned}$$

Then we obtain (2.7) and

$$T_{n,a,\beta_n,r}(x) = O\left(n^{-\left[\frac{r+1}{2}\right]}\right).$$

Lemma 4. For all $r \in N_0$,

$$A_{n,a,\beta_n}(t^r;x) = \frac{r!\,(n+a-r-2)!}{(n+a-2)!}\sum_{j=0}^{r}\binom{r}{j}\frac{[(n+\beta_n)x]^j}{j!}$$

is a polynomial in x of degree exactly r.

Proof. By using (1.4), (1.9) and the equations $\Gamma(k+r+1) = \Gamma(r+1)(r+1)_k = r!\,(r+1)_k$, $k! = (1)_k$, we have

$$A_{n,a,\beta_n}(t^r;x) = (n+a-1)\sum_{k=0}^{\infty} e^{-(n+\beta_n)x}\frac{[(n+\beta_n)x]^k}{k!}\frac{(n+a+k-1)!}{k!\,(n+a-1)!}\int_0^{\infty}\frac{t^{k+r}}{(1+t)^{n+a+k}}dt$$

$$= (n+a-1)\sum_{k=0}^{\infty} e^{-(n+\beta_n)x}\frac{[(n+\beta_n)x]^k}{k!}\frac{(n+a+k-1)!}{k!\,(n+a-1)!}\frac{r!\,(r+1)_k(n+a-r-2)!}{(n+a+k-1)!}$$

$$= e^{-(n+\beta_n)x}\frac{r!\,(n+a-r-2)!}{(n+a-2)!}\sum_{k=0}^{\infty}\frac{[(n+\beta_n)x]^k}{k!}\frac{(r+1)_k}{k!}$$

$$= e^{-(n+\beta_n)x}\frac{r!\,(n+a-r-2)!}{(n+a-2)!}\ {}_1F_1(r+a;1;(n+\beta_n)x)$$

Using the Kummer transformation

$${}_1F_1(\alpha;\beta;x) = e^x\,{}_1F_1(\beta-\alpha;\beta;-x)$$

and

$${}_1F_1(1;1;x) = e^x$$

We have

$$e^{-(n+\beta_n)x}\,{}_1F_1(r+a;1;(n+\beta_n)x) = {}_1F_1(-r;1;-(n+\beta_n)x).$$

Then, we can write

$$A_{n,a,\beta_n}(t^r;x) = \frac{r!\,(n+a-r-2)!}{(n+a-2)!}\,{}_1F_1(-r;1;-(n+\beta_n)x).$$

Since the confluent hypergeometric function is related with generalized Laguerre polynomial with the relation

$$L_n^m(x) = \binom{m+n}{n}\,{}_1F_1(-n;m+1;x) = \frac{(m+n)!}{m!\,n!}\,{}_1F_1(-n;m+1;x),$$

and

$$L_r^0(x) = L_r(x) = {}_1F_1(-r;1;x) \Longrightarrow L_r(-(n+\beta_n)x) = {}_1F_1(-r;1;-(n+\beta_n)x)$$

we obtain

$$A_{n,a,\beta_n}(t^r;x) = \frac{r!(n+a-r-2)!}{(n+a-2)!}L_r(-(n+\beta_n)x),$$

where

$L_r(-(n+\beta_n)x) = \sum_{j=0}^{r}(-1)^j \binom{r}{r-j}\frac{[-(n+\beta_n)x]^j}{j!} = \sum_{j=0}^{r}\binom{r}{j}\frac{[(n+\beta_n)x]^j}{j!}.$

Then we have

$$A_{n,a,\beta_n}(t^r;x) = \frac{r!(n+a-r-2)!}{(n+a-2)!}\sum_{j=0}^{r}\binom{r}{j}\frac{[(n+\beta_n)x]^j}{j!} = \frac{(n+\beta_n)^r(n+a-r-2)!}{(n+a-2)!}x^r + \frac{r^2(n+\beta_n)^{r-1}(n+a-r-2)!}{(n+a-2)!}x^{r-1} + \frac{r(r-1)(n+\beta_n)^{r-2}(n+a-r-2)!}{(n+a-2)!}x^{r-2} + O(n^{-2}).$$

So the proof is completed.

Lemma 5. [21] There exist a polynomial $\phi_{i,j,r}(x)$ independent of n and k such that

$$x^r\frac{d^r}{dx^r}\left[e^{-(n+\beta_n)x}[(n+\beta_n)x]^k\right] = \sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+\beta_n)^i[k-(n+\beta_n)x]^j\phi_{i,j,r}(x)e^{-(n+\beta_n)x}[(n+\beta_n)x]^k.$$

The proof can be easily done using the induction method.

Lemma 6.Let f be r-times differentiable on $[0,\infty)$ such that $f^{(r-1)} = O(t^\alpha)$, for some $\alpha > 0$ as $t \to \infty$. Then for $r = 1,2,...$ and $n > \alpha + r$, we have

$$A_{n,a,\beta_n}^{(r)}(f;x) = \frac{(n+\beta_n)^r(n+a-r-1)!}{(n+a-2)!}\sum_{k=0}^{\infty}p_{n,\beta_n,k}(x)\int_0^\infty(-1)^r q_{n-r,a,k+r}(t)f^{(r)}(t)dt.$$

Proof. Using Leibnitz theorem, we can write

$$A_{n,a,\beta_n}^{(r)}(f;x) =$$

$$(n+a-1)\sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}\frac{(-1)^{r-i}(n+\beta_n)^r[(n+\beta_n)x]^{k-i}}{(k-i)!}e^{-(n+\beta_n)x}\frac{(n+a+k-1)!}{k!\,(n+a-1)!}\int_0^\infty\frac{t^k}{(1+t)^{n+a+k}}f(t)dt$$

$$= (n+a-1)\sum_{k=0}^{\infty}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{r-i}(n+\beta_n)^r[(n+\beta_n)x]^k}{k!}e^{-(n+\beta_n)x}\frac{(n+a+k+i-1)!}{(k+i)!\,(n+a-1)!}\int_0^\infty\frac{t^k}{(1+t)^{n+a+k+i}}f(t)dt$$

$$= (n+a-1)\sum_{k=0}^{\infty}p_{n,\beta_n,k}(x)\int_0^\infty(-1)^r\sum_{i=0}^{r}\binom{r}{i}(-1)^i(n+\beta_n)^r(-1)^r q_{n,a,k+i}(t)f(t)dt.$$

By Leibnitz theorem, we have

$$q_{n-r,a,k+r}^{(r)}(t) = \frac{(n+a-1)!}{(n+a-r-1)!}\sum_{i=0}^{r}\binom{r}{i}(-1)^i q_{n,a,k+i}(t)$$

If this expression is substituted in $A_{n,a,\beta_n}^{(r)}(f;x)$, we get

$$A_{n,a,\beta_n}^{(r)}(f;x) = \frac{(n+\beta_n)^r(n+a-r-1)!}{(n+a-2)!}\sum_{k=0}^{\infty}p_{n,\beta_n,k}(x)\int_0^\infty(-1)^r q_{n-r,a,k+r}^{(r)}(t)f(t)dt.$$

If partial integration is applied to this expression, we obtain the desired result

$$A_{n,a,\beta_n}^{(r)}(f;x) = \frac{(n+\beta_n)^r(n+a-r-1)!}{(n+a-2)!}\sum_{k=0}^{\infty}p_{n,\beta_n,k}(x)\int_0^\infty(-1)^r q_{n-r,a,k+r}(t)f^{(r)}(t)dt. \qquad (2.8)$$

Remark 1. A simple consequence of (2.8) is

$$A_{n,a,\beta_n}(t^r;x) = \frac{r!(n+\beta_n)^r(n+a-r-2)!}{(n+a-2)!} \tag{2.9}$$

Lemma 7. If we define

$$\lambda_r(n) = \frac{\prod_{j=0}^{r}(n+\beta_n)^j(n+a-j-2)}{(n+a-2)!} = \frac{(n+\beta_n)^r(n+a-r-2)!}{(n+a-2)!} \tag{2.10}$$

then we have the following recurrence relations

$$[\lambda_{r+1}(n) - \lambda_r(n)]x = \frac{r+1}{n+\beta_n}\lambda_{r+1}(n) = \lambda_r(n)\left[\frac{(\beta_n-a+r+2)x+(r+1)}{n+a-r-2}\right] \tag{2.11}$$

$$[\lambda_r(n) - 2\lambda_{r+1}(n) - \lambda_{r+2}(n)] = \lambda_r(n)\left[\frac{(r-a)^2+(r+\beta_n)^2-r^2+5(r-a)+6(1+\beta_n)-2a\beta_n+n}{(n+a-r-2)(n+a-r-3)}\right] \tag{2.12}$$

$$\frac{r+2}{n+\beta_n}\lambda_{r+1}(n) - \frac{r+1}{n+\beta_n}\lambda_{r+2}(n) = \lambda_r(n)\left[\frac{r(a-\beta_n)-r^2-5r+2a-6+\beta_n+n}{(n+a-r-2)(n+a-r-3)}\right]. \tag{2.13}$$

The proof can be easily done with simple operations.
Let us now give some definitions that we will use.

Definition 1. [19] The $m-th$ order modulus of continuity $\omega_m(f;\delta:[a,b])$ for a function f which is continuous on $[a,b]$ is defined by

$$\omega_m(f;\delta:[a,b]) = sup\{|\Delta_h^m f(x)|; |h| < \delta; x, x+h \in [a,b]\}. \tag{2.14}$$

For $m = 1, \omega_m(f;\delta)$ is usual modulus of continuity.

Definition 2. [5] Let $C_\gamma[0,\infty) = \{f \in [0,\infty); |f(t)| \le Mt^\gamma,\ \gamma > 0\}$. The norm ($\|.\|$) on $C_\gamma[0,\infty)$ is defined by;

$$\|f\|_\gamma = \sup_{0\le t<\infty} |f(t)|t^{-\gamma}. \tag{2.15}$$

Definition 3. [19] Let $0 < a < a_1 < b_1 < b < \infty$. For sufficiently small $\eta > 0$, the $2-nd$ order Steklov mean $f_{\eta,2}$ corresponding to $f \in C_\gamma[a,b]$ and $t \in I_1$ is defined as;

$$f_{\eta,2}(t) = \eta^{-2}\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}[f(t) - \Delta_h^2 f(t)]dt_1 dt_2, \tag{2.16}$$

Where $h = \frac{t_1+t_2}{2}$ and Δ_h^2 is the second order forward difference operator with step lenght h. For $f \in C[a,b]$, $f_{\eta,2}$ satisfy the following properties:

1. $f_{\eta,2}$, has continuous derivatives up to order 2 over $[a_1,b_1]$.
2. $\|f_{\eta,2}\|_{C[a_1,b_1]} \le C\omega_r(f;\delta:[a,b]),\ r = 1,2$
3. $\|f - f_{\eta,2}\|_{C[a_1,b_1]} \le C\omega_2(f;\delta:[a,b])$
4. $\|f_{\eta,2}\|_{C[a_1,b_1]} \le C\eta^{-2}\|f\|_{C[a,b]}$
5. $\|f_{\eta,2}\|_{C[a_1,b_1]} \le C\|f\|_\gamma$

where C are certain constants which are different in each occurence and are independent of f and η.

3. DIRECT RESULTS

Theorem 1. (Pointwise convergence) Let $a \in N_0$, $x \in [0,\infty)$ and β_n be a sequence satisfying (1.5). If $r \in N_0, f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty}\left|A_{n,a,\beta_n}^{(r)}(f;x)\right| = f^{(r)}(x).$$

Proof. By Taylor expansion of f, we have

$$f(t) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. If the operator $A_{n,a,\beta_n}(f;x)$ is applied to this expression, we get

$$A_{n,a,\beta_n}(f;x) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}A_{n,a,\beta_n}((t-x)^i;x) + A_{n,a,\beta_n}(\varepsilon(t,x)(t-x)^r;x).$$

By taking the derivative of this expression r times, we obtain

$$\frac{d^r}{dx^r}\left[A_{n,a,\beta_n}(f;x)\right] = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}A_{n,a,\beta_n}^{(r)}((t-x)^i;x) + A_{n,a,\beta_n}^{(r)}(\varepsilon(t,x)(t-x)^r;x) = I_1 + I_2$$

Using Lemma 4, Lemma 5 and (2.9),

$$\begin{aligned} I_1 &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}A_{n,a,\beta_n}^{(r)}((t-x)^i;x) \\ &= \sum_{i=0}^{r-1}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{i!\,(n+\beta_n)^i(n+a-i-2)!}{(n+a-2)!} + \frac{f^{(r)}(x)}{r!}\sum_{j=0}^{r-1}\binom{r}{j}(-x)^{r-j}A_{n,a,\beta_n}^{(r)}(t^j;x) + \frac{f^{(r)}(x)}{r!}A_{n,a,\beta_n}^{(r)}(t^r;x) \\ &= I_1 + I_2 + I_3 \end{aligned}$$

is obtained. Since $I_3 = O((1/n)), I_4 = O((1/n)), I_5 = \frac{f^{(r)}(x)}{r!}A_{n,a,\beta_n}^{(r)}(t^r;x)$, we have

$$I_1 = \frac{f^{(r)}(x)}{r!}A_{n,a,\beta_n}^{(r)}(t^r;x) = \frac{f^{(r)}(x)}{r!}\frac{r!\,(n+\beta_n)^r(n+a-r-2)!}{(n+a-2)!} = \frac{(n+\beta_n)^r(n+a-r-2)!}{(n+a-2)!}f^{(r)}(x).$$

thus $f^{(r)}(x)$ for $n \to \infty$. Now let's set an upper bound for I_2 by using Lemma 6.

$$\begin{aligned} I_2 &= A_{n,a,\beta_n}^{(r)}(\varepsilon(t,x)(t-x)^r;x) \\ &= (n+a-1)\sum_{\substack{2i+j\le r\\ i,j\ge 0}}\frac{(n+\beta_n)^i}{x^r}\left|\phi_{i,j,r}(x)\right|\sum_{k=0}^{\infty}[k-(n+\beta_n)x]^j\, e^{-(n+\beta_n)x}\frac{[(n+\beta_n)x]^k}{k!} \\ &\quad\times\frac{(n+a+k-1)!}{k!(n+a-1)!}\int_0^\infty \varepsilon(t,x)\frac{t^k(t-x)^r}{(1+t)^{n+a+k}}dt \end{aligned}$$

So we can write I_2 as:

$$\begin{aligned} |I_2| &\le (n+a-1)\sum_{\substack{2i+j\le r\\ i,j\ge 0}}\frac{(n+\beta_n)^i}{x^r}\left|\phi_{i,j,r}(x)\right|\sum_{k=0}^{\infty}\left|[k-(n+\beta_n)x]^j\right| e^{-(n+\beta_n)x}\frac{[(n+\beta_n)x]^k}{k!} \\ &\quad\times\frac{(n+a+k-1)!}{k!(n+a-1)!}\int_0^\infty|\varepsilon(t,x)|\frac{t^k|t-x|^r}{(1+t)^{n+a+k}}dt \end{aligned}$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\varepsilon(t,x)|$ whenever $|t-x| < \delta$, morever if $\lambda \geq max\{\gamma, r\}$ is any integer, then we find a constant $K > 0$ such that $|\varepsilon(t,x)||t-x|^r \leq K|t-x|^\gamma$. Thus we obtain

$$|I_2| \leq (n+a-1)C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+\beta_n)^i \sum_{k=0}^{\infty} \left|[k-(n+\beta_n)x]^j\right| e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!}$$
$$\times \left\{\frac{(n+a+k-1)!}{k!(n+a-1)!}\int_{|t-x|<\delta} |\varepsilon(t,x)| \frac{t^k|t-x|^r}{(1+t)^{n+a+k}} dt + \frac{(n+a+k-1)!}{k!(n+a-1)!}\int_{|t-x|\geq\delta} K \frac{t^k|t-x|^r}{(1+t)^{n+a+k}} dt\right\} = I_6 + I_7$$

where

$$C_1 = \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{\left|\phi_{i,j,r}(x)\right|}{x^r} > 0$$

and K is a constant independent of C_1. If we use the Schwarz inequality first for the integration and then for the summation to calculate I_6, we have

$$I_6 \leq \varepsilon C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+\beta_n)^i \left[\sum_{k=0}^{\infty} [k-(n+\beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!}\right]^{\frac{1}{2}}$$
$$\times \left[(n+a-1)\sum_{k=0}^{\infty} [k-(n+\beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!} [(n+\beta_n)x]^k \frac{(n+a+k-1)!}{k!(n+a-1)!}\int_0^\infty \frac{t^k(t-x)^r}{(1+t)^{n+a+k}} dt\right]^{\frac{1}{2}},$$

as

$$\int_0^\infty q_{n,a,k}(t)dt = \frac{1}{(n+a-1)}$$

By using Lemma 2, we get

$$\sum_{k=0}^{\infty} [k-(n+\beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!} = (n+\beta_n)^{2j} \sum_{k=0}^{\infty} \left[\frac{k}{(n+\beta_n)}\right]^{2j} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!}$$
$$= (n+\beta_n)^{2j}[O((n+\beta_n)^{-j})] = O((n+\beta_n)^j).$$

Since

$$\left[(n+a-1)\sum_{k=0}^{\infty} [k-(n+\beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!} [(n+\beta_n)x]^k \frac{(n+a+k-1)!}{k!(n+a-1)!}\int_0^\infty \frac{t^k(t-x)^r}{(1+t)^{n+a+k}} dt\right]^{\frac{1}{2}} = O((n+\beta_n)^{-r})$$

can be written as a result of Lemma 2, we have

$$I_6 \leq \varepsilon C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+\beta_n)^i O\left((n+\beta_n)^{\frac{j}{2}}\right) O\left((n+\beta_n)^{-\frac{r}{2}}\right) = \varepsilon O(1).$$

Next, using Lemma 2 and Schwarz inequality for the integration and summation, in view of of the above results, we have

$$I_7 \leq (n+a-1)C_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+\beta_n)^i \sum_{k=0}^{\infty} \left|[k-(n+\beta_n)x]^j\right| e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!}$$
$$\times \left[\frac{(n+a+k-1)!}{k!\,(n+a-1)!}\int_{|t-x|\geq\delta} \frac{t^k}{(1+t)^{n+a+k}} dt\right]^{\frac{1}{2}} \left[\frac{(n+a+k-1)!}{k!\,(n+a-1)!}\int_{|t-x|\geq\delta} \frac{t^k(t-x)^{2r}}{(1+t)^{n+a+k}} dt\right]^{\frac{1}{2}}$$
$$\leq C_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+\beta_n)^i \left[\sum_{k=0}^{\infty} [k-(n+\beta_n)x]^{2j} e^{-(n+\beta_n)x} \frac{[(n+\beta_n)x]^k}{k!}\right]^{\frac{1}{2}}$$

$$\times\left[\sum_{k=0}^{\infty} e^{-(n+\beta_n)x}\frac{[(n+\beta_n)x]^k}{k!}\frac{(n+a+k-1)!}{k!\,(n+a-1)!}\int_{|t-x|\geq\delta}\frac{t^k(t-x)^{2r}}{(1+t)^{n+a+k}}dt\right]^{\frac{1}{2}}$$
$$=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+\beta_n)^i O\left((n+\beta_n)^{\frac{j}{2}}\right)O\left((n+\beta_n)^{-\frac{r}{2}}\right)=O(1).$$

Thus the proof is completed by using the results obtained for I_1 and I_2.

Theorem 2. (Asymptotic expansion) Let $f\in C_\gamma[0,\infty)$ be bounded for every finite sub-interval of $[0,\infty)$ and has the derivative of order $(r+2)$ at a fixed $x\in (0,\infty)$. For some $\gamma>0$, let $f(t)=O(t^\gamma)$ as $t\to\infty$. Then we have,

$$\lim_{n\to\infty} n\left[A_{n,a,\beta_n}^{(r)}(f;x)-f^{(r)}(x)\right]=[(2+r-a)x+r+1]\,f^{(r+1)}(x)+\left(x+\frac{x^2}{2}\right)f^{(r+2)}(x).$$

Proof. By Taylor expansion of f, we have

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^{r+2}$$

where $\varepsilon(t,x)\to 0$ as $t\to x$. If the operator $A_{n,a,\beta_n}^{(r)}(f;.)$ is applied to this expression, we have

$$A_{n,a,\beta_n}^{(r)}(f;x)=\left\{\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}A_{n,a,\beta_n}^{(r)}((t-x)^i;x)\right\}+A_{n,a,\beta_n}^{(r)}[\varepsilon(t,x)(t-x)^{r+2};x]=I_1+I_2$$

Using Lemma 7, we get

$$I_1=\left\{\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}A_{n,a,\beta_n}^{(r)}((t-x)^i;x)\right\}=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\left\{A_{n,a,\beta_n}^{(r)}\left(\sum_{j=1}^{i}\binom{i}{j}(-x)^{i-j}t^i;x\right)\right\}$$
$$=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=1}^{i}\binom{i}{j}(-x)^{i-j}A_{n,a,\beta_n}^{(r)}(t^i;x)$$
$$=\sum_{i=0}^{r-1}\frac{f^{(i)}(x)}{i!}\sum_{j=1}^{i}\binom{i}{j}(-x)^{i-j}A_{n,a,\beta_n}^{(r)}(t^i;x)+\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{j=1}^{i}\binom{i}{j}(-x)^{i-j}A_{n,a,\beta_n}^{(r)}(t^r;x)$$
$$+\frac{f^{(r+1)}(x)}{(r+1)!}\sum_{j=1}^{r+1}\binom{r+1}{j}(-x)^{r+1-j}A_{n,a,\beta_n}^{(r)}(t^r;x)+\frac{f^{(r+2)}(x)}{(r+2)!}\sum_{j=1}^{r+2}\binom{r+2}{j}(-x)^{r+2-j}A_{n,a,\beta_n}^{(r)}(t^r;x)$$
$$=\sum_{i=0}^{r-1}\frac{f^{(i)}(x)}{i!}O\left(\frac{1}{n}\right)+\frac{f^{(r)}(x)}{r!}A_{n,a,\beta_n}^{(r)}(t^r;x)+\frac{f^{(r+1)}(x)}{(r+1)!}\left[(r+1)(-x)A_{n,a,\beta_n}^{(r)}(t^r;x)+A_{n,a,\beta_n}^{(r)}(t^{r+1};x)\right]$$
$$+\frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+1)(r+2)}{2}x^2A_{n,a,\beta_n}^{(r)}(t^r;x)+(r+2)(-x)A_{n,a,\beta_n}^{(r)}(t^{r+1};x)+A_{n,a,\beta_n}^{(r)}(t^{r+2};x)\right]$$
$$=\frac{f^{(r)}(x)}{r!}\frac{(n+\beta_n)^r(n+a-r-2)!\,r!}{(n+a-2)!}+O(n^{-1})+\frac{f^{(r+1)}(x)}{(r+1)!}\left[\frac{(r+1)(-x)(n+\beta_n)^r(n+a-r-2)!\,r!}{(n+a-2)!}\right.$$
$$\left.+\frac{(n+\beta_n)^{r+1}(n+a-r-3)!\,(r+1)!}{(n+a-2)!}x+(r+1)^2\frac{(n+\beta_n)^{r+1}(n+a-r-3)!\,r!}{(n+a-2)!}x+O(n^{-1})\right]$$
$$+\frac{f^{(r+2)}(x)}{(r+2)!}\left\{\frac{(r+1)(r+2)}{2}x^2\frac{(n+\beta_n)^r(n+a-r-2)!\,r!}{(n+a-2)!}+(r+2)(-x)\left[\frac{(n+\beta_n)^{r+1}(n+a-r-3)!\,(r+1)!}{(n+a-2)!}x\right.\right.$$
$$\left.+(r+1)^2\frac{(n+\beta_n)^r(n+a-r-3)!\,r!}{(n+a-2)!}\right]+\frac{(n+\beta_n)^{r+2}(n+a-r-4)!\,(r+2)!}{(n+a-2)!}x$$

$$+\frac{(r+1)^2(n+\beta_n)^{r+1}(n+a-r-4)!(r+1)!}{(n+a-2)!}x+(r+1)(r+2)\frac{(n+\beta_n)^r(n+a-r-4)!r!}{(n+a-2)!}+O(n^{-1})\Big\}.$$

and using (2.10), (2.11), (2.12) and (2.13) we obtain

$$\begin{aligned}
I_1 &= \frac{f^{(r)}(x)}{r!}\left[\lambda_r(n)r!+O\left(\frac{1}{n}\right)\right]+\frac{f^{(r+1)}(x)}{(r+1)!}\left[(r+1)(-x)\lambda_r(n)r!+\lambda_{r+1}(n)(r+1)!\,x+\frac{r+1}{n+\beta_n}\lambda_{r+1}(n)+O\left(\frac{1}{n}\right)\right]\\
&+\frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+2)!}{2}x^2\lambda_r(n)-(r+2)!\,x^2\lambda_{r+1}(n)+\frac{(r+2)!}{2}x^2\lambda_{r+2}(n)+\left(\frac{r+2}{n+\beta_n}\lambda_{r+1}(n)-\frac{r+1}{n+\beta_n}\lambda_{r+2}(n)\right)x\right.\\
&\left.+\frac{1}{(n+\beta_n)^2}\lambda_{r+2}(n)+O\left(\frac{1}{n}\right)\right]\\
&= f^{(r)}(x)\lambda_r(n)+f^{(r+1)}(x)\left[\left(\lambda_{r+1}(n)-\lambda_r(n)\right)x-\frac{r+1}{n+\beta_n}\lambda_{r+1}(n)\right]+f^{(r+2)}(x)\left\{[\lambda_r(n)-2\lambda_{r+1}(n)+\lambda_{r+2}(n)]\frac{x^2}{2}\right.\\
&\left.+\left(\frac{r+2}{n+\beta_n}\lambda_{r+1}(n)-\frac{r+1}{n+\beta_n}\lambda_{r+2}(n)\right)x+\frac{1}{(n+\beta_n)^2}\lambda_{r+2}(n)+O\left(\frac{1}{n}\right)\right\}\\
&= f^{(r)}(x)\lambda_r(n)+f^{(r+1)}(x)\left[\lambda_r(n)\frac{(\beta_n-a+r+2)x+(r+1)}{n+a-r-2}\right]\\
&+f^{(r+2)}(x)\left\{\lambda_r(n)\left[\frac{(r-a)^2+(r+\beta_n)^2-r^2+5(r-a)+6(1+\beta_n)-2a\beta_n+n}{(n+a-r-2)(n+a-r-3)}\right]\frac{x^2}{2}+\lambda_r(n)\left[\frac{r(a-\beta_n)-r^2-5r+2a-6+\beta_n+n}{(n+a-r-2)(n+a-r-3)}\right]x\right.\\
&\left.+\frac{\lambda_r(n)}{(n+a-r-2)(n+a-r-3)}\right\}+O\left(\frac{1}{n}\right)
\end{aligned}$$

Taking limit for $n\to\infty$, we obtain

$$\lim_{n\to\infty} n\left[A_{n,a,\beta_n}^{(r)}(f;x)-f^{(r)}(x)\right]=[(2+r-a)x+r+1]\,f^{(r+1)}(x)+\left(x+\frac{x^2}{2}\right)f^{(r+2)}(x).$$

Theorem 3. (Eror estimation) Let $f\in C_\gamma[0,\infty)$ for some $\gamma>0$ and $0<a<a_1<b_1<b<\infty$. Then for sufficiently large n, we have

$$\left\|A_{n,a,\beta_n}^{(r)}(f;.)-f^{(r)}\right\|_{C[a_1,b_1]}\le C_1\omega_2\left(f^{(r)},n^{-\frac{1}{2}}[a,b]\right)+C_2n^{-1}\|f\|_\gamma,$$

where $C_1=C_1(r)$ and $C_2=C_2(r,f)$.

Proof. The equation

$$f(t)=f^{(r)}(x)+f(t)-f_{\eta,2}(t)+f_{\eta,2}(t)-f_{\eta,2}^{(r)}(x)+f_{\eta,2}^{(r)}(x)-f^{(r)}(x)$$
$$f(t)-f^{(r)}(x)=f(t)-f_{\eta,2}(t)+f_{\eta,2}(t)-f_{\eta,2}^{(r)}(x)+f_{\eta,2}^{(r)}(x)-f^{(r)}(x)$$

can be written by considering (2.16). If we apply the operator $A_{n,a,\beta_n}^{(r)}$ to each side of this equation and using the linearity of the operator, we can write

$$\begin{aligned}
&A_{n,a,\beta_n}^{(r)}(f;.)-f^{(r)}(x)A_{n,a,\beta_n}^{(r)}(1;.)\\
&\qquad=A_{n,a,\beta_n}^{(r)}\left(f-f_{\eta,2};.\right)+A_{n,a,\beta_n}^{(r)}\left(f_{\eta,2};.\right)-f_{\eta,2}^{(r)}(x)A_{n,a,\beta_n}^{(r)}(1;.)+f_{\eta,2}^{(r)}(x)-f^{(r)}(x)A_{n,a,\beta_n}^{(r)}(1;.)
\end{aligned}$$

If we first take the absolute value of the equation and then add both sides of the equation on $C[a_1,b_1]$, we obtain

$$\begin{aligned}
\left\|A_{n,a,\beta_n}^{(r)}(f;.)-f^{(r)}\right\|_{C[a_1,b_1]} &\le \left\|A_{n,a,\beta_n}^{(r)}\left(f-f_{\eta,2};.\right)\right\|_{C[a_1,b_1]}+\left\|A_{n,a,\beta_n}^{(r)}\left(f_{\eta,2};.\right)-f_{\eta,2}^{(r)}\right\|_{C[a_1,b_1]}+\left\|f^{(r)}-f_{\eta,2}^{(r)}\right\|_{C[a_1,b_1]}\\
&:= H_1+H_2+H_3
\end{aligned}$$

By property (3) of Steklov mean, we get

$H_3 \leq K\omega_2(f,[a,b])$.

Next, using Theorem 2, we obtain

$$H_2 \leq K_1 n^{-1} \sum_{j=r}^{r+2} \left\| f_{\eta,2}^{(j)} \right\|_{C[a,b]} \leq K_1 n^{-1} \left[\left\| f_{\eta,2} \right\|_{C[a,b]} + \left\| f_{\eta,2}^{(r+2)} \right\|_{C[a,b]} \right].$$

Now by properties (2) and (4) of Steklov mean, we have

$$H_2 \leq K_2 n^{-1} \left[\eta^{-2} \|f\|_{C[a,b]} + \eta^{-2} \omega_r\left(f^{(r)};\eta;[a,b]\right) \right] \leq K_3 n^{-1} \left[\|f\|_\gamma + \eta^{-2} \omega_r\left(f^{(r)};\eta;[a,b]\right) \right].$$

Finally, we estimate H_1 choosing a,b satisfiying the condition $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. Let $\chi(t)$ denotes the characteristic function in the interval $[a^*,b^*]$, then

$$H_1 \leq \left\| A_{n,a,\beta_n}^{(r)} \left[\chi(t)\left(f - f_{\eta,2};.\right) \right] \right\|_{C[a_1,b_1]} + \left\| A_{n,a,\beta_n}^{(r)} \left[\left(1-\chi(t)\right)\left(f - f_{\eta,2};.\right) \right] \right\|_{C[a_1,b_1]} := H_4 + H_5$$

From (2.8), we have

$$A_{n,a,\beta_n}^{(r)} \left[\chi(t)\left(f(t) - f_{\eta,2}(t);x\right) \right] = \frac{(n+\beta_n)^r (n+a-r-1)!}{(n+a-2)!} \sum_{k=0}^{\infty} p_{n,\beta_n,k}(x) \int_0^\infty q_{n-r,a,k+r}(t) \left(\chi(t)\left(f^{(r)}(t) - f_{\eta,2}^{(r)}(t);x\right) \right) dt.$$

Hence

$$H_4 \leq K_4 \left\| f^{(r)} - f_{\eta,2}^{(r)} \right\|_{C[a^*,b^*]}.$$

Now for $x \in [a_1,b_1]$ and $t \in [0,\infty)\backslash[a^*,b^*]$, we choose a $\delta_1 > 0$ satisfying $|t-x| > \delta_1$. By Lemma 5 and Schwarz inequality, we have

$$\left| \frac{d^r}{dx^r} A_{n,a,\beta_n}\left[\left(1-\chi(t)\right)\left(f(t) - f_{\eta,2}(t);x\right) \right] \right|$$
$$\leq (n+a-1) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} \frac{(n+\beta_n)^i}{x^r} \phi_{i,j,r}(x) \sum_{k=0}^{\infty} [k-(n+\beta_n)x]^j e^{-(n+\beta_n)x} [(n+\beta_n)x]^k$$
$$\times \int_0^\infty q_{n,a,k}(t) \left(\left(1-\chi(t)\right) \left| f(t) - f_{\eta,2}(t) \right| \right) dt.$$
$$\leq K_5 \|f\|_\gamma (n+a-1) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+\beta_n)^i \sum_{k=0}^{\infty} [k-(n+\beta_n)x]^j e^{-(n+\beta_n)x} [(n+\beta_n)x]^k \int_{|t-x|\geq\delta_1} q_{n,a,k}(t) dt$$
$$\leq \frac{K_5}{\delta^{2l}} \|f\|_\gamma (n+a-1) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+\beta_n)^i \sum_{k=0}^{\infty} [k-(n+\beta_n)x]^j e^{-(n+\beta_n)x} [(n+\beta_n)x]^k$$
$$\times \left[\int_0^\infty q_{n,a,k}(t) dt \right]^{\frac{1}{2}} \left[\int_0^\infty q_{n,a,k}(t)(t-x)^{4l} dt \right]^{\frac{1}{2}}$$
$$\leq \frac{K_5}{\delta^{2l}} \|f\|_\gamma (n+a-1) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+\beta_n)^i \left[\sum_{k=0}^{\infty} [k-(n+\beta_n)x]^j e^{-(n+\beta_n)x} [(n+\beta_n)x]^k \right]^{\frac{1}{2}}$$
$$\times \left[(n+a-1) \sum_{k=0}^{\infty} e^{-(n+\beta_n)x} [(n+\beta_n)x]^k \int_0^\infty q_{n,a,k}(t)(t-x)^{4l} dt \right]^{\frac{1}{2}}.$$

Hence by Lemma 1 and Lemma 2 , we get

$$H_5 \leq K_6\|f\|_\gamma \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+\beta_n)^{i+\frac{j}{2}-1} \leq K_6\|f\|_\gamma (n+\beta_n)^{-v} \leq K_6\|f\|_\gamma (n+\beta_n)^{-1},$$

where $v = \left(l - \frac{r}{2}\right) > 1, \eta > 0.$

Therefore, by property (3) of Steklov mean, we obtain

$$H_1 \leq K_7\left\|f^{(r)} - f_{\eta,2}^{(r)}\right\|_{C[a^*,b^*]} + K_6\|f\|_\gamma (n+\beta_n)^{-1} \leq K_8\omega_2(f;\eta;[a,b])K_6\|f\|_\gamma (n+\beta_n)^{-1}.$$

The proof is done by choosing $\eta = (n+\beta_n)^{-\frac{1}{2}}$.

As a result, when the results obtained for the operators defined by (1.4) are compared with the results given for the (1.1) operator, it is seen that the convergence properties are preserved and the error rate decreases by changing compared to β_n.

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