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Some Geometric Properties of Bicomplex Sequence Spaces $l_p(\mathbb{BC})$

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Abstract

In this article, we examine some geometric properties such as convexity, strictly convexity, uniformly convexity of bicomplex sequence spaces $l_p(\mathbb{BC})$ with Euclidean norm by proving some significant inequalities. We also furnish some nontrivial examples that support our findings for geometric properties not provided in some of these bicomplex sequence spaces.

Keywords: *Bicomplex sequence spaces, convexity, strictly convexity, uniformly convexity.*
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1. Introduction and Preliminaries

Segre [1], in his complex geometry studies, defined the concept of bicomplex numbers in 1892. In 1991, Price [2] published a book on bicomplex numbers, multicomplex spaces and their function theory. In recent years, many studies have been done on bicomplex analysis and it has become a subject of research in physics and mathematics by attracting considerable interest of researchers thanks to its huge applications in different fields of mathematical sciences. The most important of these studies are [3, 4, 5].
Sager and Sağır [6], by defining bicomplex sequence spaces with Euclidean norm in the set of bicomplex numbers, studied completeness of them. For some works on the geometric properties of other sequence spaces we refer the reader to [7, 8, 9, 10].
Our aim in this study is to add new properties to bicomplex sequence spaces. We present the study in two parts. In the first one, we start with a number of known results needed in this paper. In the second one, we give some of geometric properties of bicomplex sequence spaces $l_p(\mathbb{BC})$ with Euclidean norm by proving some necessary inequalities.
Let i and j be independent imaginary units such that $i^2 = j^2 = -1$, $ij = ji$ and $\mathbb{C}(i)$ be the set of complex numbers with the imaginary unit i. The set of bicomplex numbers $\mathbb{BC}$ is defined by

$$\mathbb{BC} = \{s = s_1 + js_2 : s_1, s_2 \in \mathbb{C}(i)\}.$$

The set $\mathbb{BC}$ forms a Banach space and a ring with respect to the addition, scalar multiplication, multiplication and Euclidean norm for all $s = s_1 + js_2, t = t_1 + jt_2 \in \mathbb{BC}$ and for all $\lambda \in \mathbb{R}$ defined as

$$\begin{aligned} s+t &= (s_1+js_2)+(t_1+jt_2) = (s_1+t_1)+j(s_2+t_2), \\ \lambda.s &= \lambda.(s_1+js_2) = \lambda s_1 + j\lambda s_2, \\ s\times t &= st = (s_1+js_2)(t_1+jt_2) = (s_1t_1 - s_2t_2) + j(s_1t_2+s_2t_1), \\ \|.\|_{\mathbb{BC}} &: \mathbb{BC} \to \mathbb{R}, s \to \|s\|_{\mathbb{BC}} = \sqrt{|s_1|^2+|s_2|^2}. \end{aligned}$$

A sequence in $\mathbb{BC}$ (a bicomplex sequence) is a function defined by $z : \mathbb{N} \to \mathbb{BC}$, $n \to s_n$. This sequence converges to a point $s^* \in \mathbb{BC}$ if and only if to each $\varepsilon > 0$ there corresponds an $n_0(\varepsilon) \in \mathbb{N}$ such that $\|s_n - s^*\|_{\mathbb{BC}} < \varepsilon$ for all $n \geq n_0(\varepsilon)$.
Let $(\zeta_k)_{k\in\mathbb{N}}$ be a bicomplex sequence. Then, the infinite sum

$$\sum_{k=1}^{\infty} \zeta_k = \sum_{k=1}^{\infty} (\zeta_{1k} + j\zeta_{2k}) = \zeta_1 + \zeta_2 + \ldots + \zeta_n + \ldots$$

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is called an infinite series in $\mathbb{BC}$. Define the sequence $s:\mathbb{N}\to\mathbb{BC}, n\to s_n$ by setting $s_n=\sum_{k=1}^{n}\zeta_k$ for all $n\in\mathbb{N}$. The infinite series converges if and only if $\lim_{n\to\infty}s_n$ exists; if the limit does not exist, the series diverges. If $\lim_{n\to\infty}s_n=\zeta^*$ then, ζ^* is called the sum of series, and we write $\sum_{k=1}^{\infty}\zeta_k=\zeta^*$ [2].

Lemma 1.1. *[6][Bicomplex Hölder's Inequality] Let p and q be real numbers with $1<p<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $s_k,t_k\in\mathbb{BC}$ for $k\in\{1,2,...,n\}$. Then*

$$\sum_{k=1}^{n}\|s_kt_k\|_{\mathbb{BC}}\leq\sqrt{2}\left(\sum_{k=1}^{n}\|s_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}\|t_k\|_{\mathbb{BC}}^{q}\right)^{\frac{1}{q}}.$$

Lemma 1.2. *[6][Bicomplex Minkowski's Inequality] Let p be a real number with $1<p<\infty$ and $s_k,t_k\in\mathbb{BC}$ for $k\in\{1,2,...,n\}$. Then*

$$\left(\sum_{k=1}^{n}\|s_k+t_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\leq\left[\left(\sum_{k=1}^{n}\|s_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}\|t_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\right].$$

Definition 1.3. *[6]*

$$\begin{aligned} l_\infty(\mathbb{BC}) &:= \left\{s=(s_k)\in w(\mathbb{BC}):\sup_{k\in\mathbb{N}}\|s_k\|_{\mathbb{BC}}<\infty\right\},\\ l_p(\mathbb{BC}) &:= \left\{s=(s_k)\in w(\mathbb{BC}):\sum_{k=1}^{\infty}\|s_k\|_{\mathbb{BC}}^{p}<\infty\right\} \text{ for } 0<p<\infty, \end{aligned}$$

where $w(\mathbb{BC})$ denotes all bicomplex sequences.

Theorem 1.4. *[6] $l_\infty(\mathbb{BC})$ is a Banach space with the norm $\|.\|_{l_\infty(\mathbb{BC})}$ defined by*

$$\|s\|_{l_\infty(\mathbb{BC})}=\sup_{k\in\mathbb{N}}\|s_k\|_{\mathbb{BC}}$$

for all $s=(s_k)\in l_\infty(\mathbb{BC})$.

Theorem 1.5. *[6] The space $l_p(\mathbb{BC})$ is a Banach space for $1\leq p<\infty$ with the norm $\|.\|_{l_p(\mathbb{BC})}$ defined by*

$$\|s\|_{l_p(\mathbb{BC})}=\left(\sum_{k=1}^{\infty}\|s_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}$$

for all $s=(s_k)\in l_p(\mathbb{BC})$, and the space $l_p(\mathbb{BC})$ is a $p-$Banach space for $0<p<1$ with the $p-$norm $|||.|||_{l_p(\mathbb{BC})}$ defined by

$$\|s\|_{l_p(\mathbb{BC})}=\sum_{k=1}^{\infty}\|s_k\|_{\mathbb{BC}}^{p}$$

for all $s=(s_k)\in l_p(\mathbb{BC})$. Here, we refer to [11] and [12] for the definitions of $p-$norm and $p-$Banach space.

Definition 1.6. *[7] Let C be a subset of a linear space X. Then C is said to be convex if $(1-\lambda)x+\lambda y\in C$ for all $x,y\in C$ and all scalar $\lambda\in[0,1]$.*

Definition 1.7. *[7] A Banach space X is said to be strictly convex if $x,y\in S_X$ with $x\neq y$ implies that $\|(1-\lambda)x+\lambda y\|_X<1$ for all $\lambda\in(0,1)$.*

Definition 1.8. *[7] A Banach space X is said to be uniformly convex if for any ε with $0<\varepsilon\leq 2$, the inequalities $\|x\|\leq 1,\|y\|\leq 1$ and $\|x-y\|\geq\varepsilon$ imply that there exists a $\delta=\delta(\varepsilon)>0$ such that $\left\|\frac{x+y}{2}\right\|\leq 1-\delta$.*

Lemma 1.9. *[13] Let $p\in(0,1)$. Then, for $a\geq 0$ and $b\geq 0$ we have $(a+b)^p\leq a^p+b^p$.*

Theorem 1.10. *[7] Let X be a Banach space. Then, the following statements are equivalent:*
(a) X is strictly convex.
(b) For every $1<p<\infty$, $\|\lambda x+(1-\lambda)y\|^p<\lambda\|x\|^p+(1-\lambda)\|y\|^p$ for all $x,y\in X$, $x\neq y$ and $\lambda\in(0,1)$.

Theorem 1.11. *[7] Every uniformly convex Banach space is strictly convex.*

2. Some Geometric Properties of the Banach space $l_p(\mathbb{BC})$

Lemma 2.1. *Let $s,t \in \mathbb{BC}$. Then, we have*

$$\|s+t\|_{\mathbb{BC}}^2 + \|s-t\|_{\mathbb{BC}}^2 = 2\left(\|s\|_{\mathbb{BC}}^2 + \|t\|_{\mathbb{BC}}^2\right).$$

Proof. The proof is direct application of definition of $\|.\|_{\mathbb{BC}}$. □

Lemma 2.2. *Let p be a real number with $0 < p \leq 1$ and $s,t \in \mathbb{BC}$. Then, we have*

$$\|s+t\|_{\mathbb{BC}}^p \leq \|s\|_{\mathbb{BC}}^p + \|t\|_{\mathbb{BC}}^p.$$

Proof. Let p be a real number with $0 < p \leq 1$ and $s,t \in \mathbb{BC}$. Then, we have by Lemma 1.9

$$\|s+t\|_{\mathbb{BC}}^p \leq (\|s\|_{\mathbb{BC}} + \|t\|_{\mathbb{BC}})^p \leq \|s\|_{\mathbb{BC}}^p + \|t\|_{\mathbb{BC}}^p.$$

□

Theorem 2.3. *The sets $\mathbb{BC}$ and $w(\mathbb{BC})$ are convex.*

Proof. The proof is clear from definition of convexity. □

Lemma 2.4. *The set $\mathbb{BC}$ is uniformly convex and strictly convex.*

Proof. Let $s,t \in \mathbb{BC}, \varepsilon \in (0,2]$, $\|s\|_{\mathbb{BC}} \leq 1, \|t\|_{\mathbb{BC}} \leq 1$ and $\varepsilon \leq \|s-t\|_{\mathbb{BC}}$. Then, by using Lemma 2.1 we have

$$\begin{aligned} \|s+t\|_{\mathbb{BC}}^2 &= 2\left(\|s\|_{\mathbb{BC}}^2 + \|t\|_{\mathbb{BC}}^2\right) - \|s-t\|_{\mathbb{BC}}^2 \\ &\leq 4 - \varepsilon^2 \end{aligned}$$

and so,

$$\left\|\frac{s+t}{2}\right\|_{\mathbb{BC}} = \left[\frac{1}{2^2}\|s+t\|_{\mathbb{BC}}^2\right]^{\frac{1}{2}} \leq \left[\frac{1}{2^2}\left(4-\varepsilon^2\right)\right]^{\frac{1}{2}} \leq \left[1-\left(\frac{\varepsilon}{2}\right)^2\right]^{\frac{1}{2}}.$$

If we take $\delta(\varepsilon) = 1 - \left[1-\left(\frac{\varepsilon}{2}\right)^2\right]^{\frac{1}{2}}$, then we say that $\mathbb{BC}$ is uniformly convex. By Theorem 1.11, $\mathbb{BC}$ is also strictly convex. □

Lemma 2.5. *Let p be a real number with $1 < p < \infty$, $s,t \in \mathbb{BC}$, $s \neq t$ and $\lambda \in (0,1)$. Then, we have*

$$\|\lambda s + (1-\lambda)t\|_{\mathbb{BC}}^p < \lambda\|s\|_{\mathbb{BC}}^p + (1-\lambda)\|t\|_{\mathbb{BC}}^p.$$

Proof. The proof is consequence of Lemma 2.4 and Theorem 1.10. □

Lemma 2.6. *Let p be a real number with $2 \leq p < \infty$ and $s,t \in \mathbb{BC}$. Then, we have*

$$\|s+t\|_{\mathbb{BC}}^p + \|s-t\|_{\mathbb{BC}}^p \leq 2^{p-1}\left(\|s\|_{\mathbb{BC}}^p + \|t\|_{\mathbb{BC}}^p\right).$$

Proof. If we take $t = \frac{\|s+t\|_{\mathbb{BC}}}{\|s-t\|_{\mathbb{BC}}}$ in the proof of Lemma 3.67 in [8], we get

$$\left(\|s+t\|_{\mathbb{BC}}^p + \|s-t\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} \leq \left(\|s+t\|_{\mathbb{BC}}^2 + \|s-t\|_{\mathbb{BC}}^2\right)^{\frac{1}{2}}$$

for all $s,t \in \mathbb{BC}$ and $2 \leq p < \infty$. By Lemma 2.1,

$$\left(\|s+t\|_{\mathbb{BC}}^2 + \|s-t\|_{\mathbb{BC}}^2\right)^{\frac{1}{2}} = \left(2\left(\|s\|_{\mathbb{BC}}^2 + \|t\|_{\mathbb{BC}}^2\right)\right)^{\frac{1}{2}} = \sqrt{2}\left(\|s\|_{\mathbb{BC}}^2 + \|t\|_{\mathbb{BC}}^2\right)^{\frac{1}{2}}.$$

Then, by real Hölder's inequality for $\frac{2}{p} + \frac{p-2}{p} = 1$, we have

$$\begin{aligned} \|s\|_{\mathbb{BC}}^2 + \|t\|_{\mathbb{BC}}^2 &\leq \left(\|s\|_{\mathbb{BC}}^p + \|t\|_{\mathbb{BC}}^p\right)^{\frac{2}{p}}(1+1)^{\frac{p-2}{p}} \\ &= 2^{\frac{p-2}{p}}\left(\|s\|_{\mathbb{BC}}^p + \|t\|_{\mathbb{BC}}^p\right)^{\frac{2}{p}} \end{aligned}$$

and so,

$$\begin{aligned} \sqrt{2}\left(\|s\|_{\mathbb{BC}}^2 + \|t\|_{\mathbb{BC}}^2\right)^{\frac{1}{2}} &\leq 2^{\frac{1}{2}+\frac{p-2}{2p}}\left(\|s\|_{\mathbb{BC}}^p + \|t\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} \\ &= 2^{\frac{p-1}{p}}\left(\|s\|_{\mathbb{BC}}^p + \|t\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}. \end{aligned}$$

This implies that $\left(\|s+t\|_{\mathbb{BC}}^p + \|s-t\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}\left(\|s\|_{\mathbb{BC}}^p + \|t\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}$. Therefore, $\|s+t\|_{\mathbb{BC}}^p + \|s-t\|_{\mathbb{BC}}^p \leq 2^{p-1}\left(\|s\|_{\mathbb{BC}}^p + \|t\|_{\mathbb{BC}}^p\right)$ for all $s,t \in \mathbb{BC}$ and $2 \leq p < \infty$. The proof is completed. □

Theorem 2.7. *The sets $l_p(\mathbb{BC})$ for $0 < p < \infty$ and $l_\infty(\mathbb{BC})$ are convex.*

Proof. Let $s,t \in l_p(\mathbb{BC})$ and $\lambda \in \mathbb{R}$ satisfying $0 \leq \lambda \leq 1$. Then, the series $\sum_{n=1}^{\infty} \|s_n\|_{\mathbb{BC}}^p$ and $\sum_{n=1}^{\infty} \|t_n\|_{\mathbb{BC}}^p$ converges.
If $1 < p < \infty$, we have by Lemma 1.2

$$\begin{aligned}\sum_{n=1}^{\infty} \|\lambda s_n + (1-\lambda) t_n\|_{\mathbb{BC}}^p &\leq \left[\left(\sum_{n=1}^{\infty} \|\lambda s_n\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} \|(1-\lambda) t_n\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}\right]^p \\ &= \left[\lambda\left(\sum_{n=1}^{\infty} \|s_n\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} + (1-\lambda)\left(\sum_{n=1}^{\infty} \|t_n\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}\right]^p\end{aligned}$$

which implies that $\lambda s + (1-\lambda) t \in l_p(\mathbb{BC})$.
If $0 < p \leq 1$, we have by Lemma 2.2

$$\begin{aligned}\sum_{n=1}^{\infty} \|\lambda s_n + (1-\lambda) t_n\|_{\mathbb{BC}}^p &\leq \sum_{n=1}^{\infty} \left(\|\lambda s_n\|_{\mathbb{BC}}^p + \|(1-\lambda) t_n\|_{\mathbb{BC}}^p\right) \\ &= \lambda^p \sum_{n=1}^{\infty} \|s_n\|_{\mathbb{BC}}^p + (1-\lambda)^p \sum_{n=1}^{\infty} \|t_n\|_{\mathbb{BC}}^p\end{aligned}$$

which implies that $\lambda s + (1-\lambda) t \in l_p(\mathbb{BC})$.
Let $s,t \in l_\infty(\mathbb{BC})$ and $\lambda \in \mathbb{R}$ satisfying $0 \leq \lambda \leq 1$. Then, $\sup\{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\}$ and $\sup\{\|t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\}$ are finite. Then, we have

$$\begin{aligned}\sup\{\|\lambda s_n + (1-\lambda) t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} &\leq \sup\{\lambda \|s_n\|_{\mathbb{BC}} + (1-\lambda)\|t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} \\ &= \lambda \sup\{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} + (1-\lambda) \sup\{\|t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\}\end{aligned}$$

which implies that $\lambda s + (1-\lambda) t \in l_\infty(\mathbb{BC})$. Consequently, $l_p(\mathbb{BC})$ for $0 < p < \infty$ and $l_\infty(\mathbb{BC})$ are convex. □

Theorem 2.8. *The sequence spaces $l_p(\mathbb{BC})$ for $1 < p < \infty$ are strictly convex.*

Proof. Let $s,t \in S_{l_p(\mathbb{BC})}$, $s \neq t$ and $\lambda \in (0,1)$. Then, we get by Lemma 2.5

$$\begin{aligned}\|\lambda s + (1-\lambda) t\|_{l_p(\mathbb{BC})}^p &= \sum_{n=1}^{\infty} \|\lambda s_n + (1-\lambda) t_n\|_{\mathbb{BC}}^p \\ &< \sum_{n=1}^{\infty} \left[\lambda \|s_n\|_{\mathbb{BC}}^p + (1-\lambda)\|t_n\|_{\mathbb{BC}}^p\right] \\ &= \lambda \sum_{n=1}^{\infty} \|s_n\|_{\mathbb{BC}}^p + (1-\lambda) \sum_{n=1}^{\infty} \|t_n\|_{\mathbb{BC}}^p \\ &= \lambda \|s\|_{l_p(\mathbb{BC})}^p + (1-\lambda)\|t\|_{l_p(\mathbb{BC})}^p = 1\end{aligned}$$

which implies that $l_p(\mathbb{BC})$ for $1 < p < \infty$ is strictly convex. □

Example 2.9. *The sequence space $l_\infty(\mathbb{BC})$ is not strictly convex.*

Let

$$\begin{aligned}(s_n) &= (1, j, 0, 0, ...), \\ (t_n) &= (-1, j, 0, 0, ...).\end{aligned}$$

Then, we have $\|s\|_{l_\infty(\mathbb{BC})} = \|t\|_{l_\infty(\mathbb{BC})} = 1$ and

$$\begin{aligned}&\|\lambda s + (1-\lambda) t\|_{l_\infty(\mathbb{BC})} = \sup\{\|\lambda s_n + (1-\lambda) t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} \\ &= \sup\{\|(2\lambda - 1, j, 0, 0, ...)\|_{\mathbb{BC}} : n \in \mathbb{N}\} \\ &= \sup\{|2\lambda - 1|, 1\} = 1\end{aligned}$$

for all $\lambda \in (0,1)$. That is to say that $l_\infty(\mathbb{BC})$ is not strictly convex.

Example 2.10. *The sequence space $l_1(\mathbb{BC})$ is not strictly convex.*

Let

$$\begin{aligned}(s_n) &= (i, 0, 0, ...), \\ (t_n) &= (0, -i, 0, 0, ...).\end{aligned}$$

Then, we have $\|s\|_{l_1(\mathbb{BC})} = \|t\|_{l_1(\mathbb{BC})} = 1$ and

$$\begin{aligned}\|\lambda s + (1-\lambda) t\|_{l_1(\mathbb{BC})} &= \sum_{n=1}^{\infty} \|\lambda s_n + (1-\lambda) t_n\|_{\mathbb{BC}} \\ &= \|\lambda i\|_{\mathbb{BC}} + \|(1-\lambda)(-i)\|_{\mathbb{BC}} \\ &= \lambda + (1-\lambda) = 1\end{aligned}$$

for all $\lambda \in (0,1)$. That is to say that $l_1(\mathbb{BC})$ is not strictly convex.

Theorem 2.11. *The sequence spaces $l_p(\mathbb{BC})$ for $2 \leq p < \infty$ are uniformly convex.*

Proof. Let $s,t \in l_p(\mathbb{BC}), \varepsilon \in (0,2]$, $\|s\|_{l_p(\mathbb{BC})} \leq 1, \|t\|_{l_p(\mathbb{BC})} \leq 1$ and $\varepsilon \leq \|s-t\|_{l_p(\mathbb{BC})}$. Then, we have by Lemma 2.6

$$\begin{aligned}
\|s+t\|^p_{l_p(\mathbb{BC})} + \|s-t\|^p_{l_p(\mathbb{BC})} &= \sum_{n=1}^{\infty} \|s_n+t_n\|^p_{\mathbb{BC}} + \sum_{n=1}^{\infty} \|s_n-t_n\|^p_{\mathbb{BC}} \\
&= \sum_{n=1}^{\infty} \left(\|s_n+t_n\|^p_{\mathbb{BC}} + \|s_n-t_n\|^p_{\mathbb{BC}}\right) \\
&\leq \sum_{n=1}^{\infty} 2^{p-1}\left(\|s_n\|^p_{\mathbb{BC}} + \|t_n\|^p_{\mathbb{BC}}\right) \\
&= 2^{p-1}\left[\sum_{n=1}^{\infty} \|s_n\|^p_{\mathbb{BC}} + \sum_{n=1}^{\infty} \|t_n\|^p_{\mathbb{BC}}\right] \\
&= 2^{p-1}\left[\|s\|^p_{l_p(\mathbb{BC})} + \|t\|^p_{l_p(\mathbb{BC})}\right] \leq 2^p.
\end{aligned}$$

Thus, we can write

$$\|s+t\|^p_{l_p(\mathbb{BC})} \leq 2^p - \|s-t\|^p_{l_p(\mathbb{BC})} \leq 2^p - \varepsilon^p,$$

and so,

$$\left\|\frac{s+t}{2}\right\|_{l_p(\mathbb{BC})} = \left[\frac{1}{2^p}\|s+t\|^p_{l_p(\mathbb{BC})}\right]^{\frac{1}{p}} \leq \left[1-\left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}}.$$

If we take $\delta(\varepsilon) = 1 - \left[1-\left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}}$, we say that $l_p(\mathbb{BC})$ for $2 \leq p < \infty$ is uniformly convex. □

Example 2.12. *The sequence space $l_\infty(\mathbb{BC})$ is not uniformly convex.*

Let

$$\begin{aligned}
(s_n) &= (i, j, i, 0, 0, \ldots), \\
(t_n) &= (i, j, -i, , 0, 0, \ldots).
\end{aligned}$$

Then, we have $\|s\|_{l_\infty(\mathbb{BC})} = \|t\|_{l_\infty(\mathbb{BC})} = 1$,

$$\begin{aligned}
\|s-t\|_{l_\infty(\mathbb{BC})} &= \sup\{\|s_n-t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} \\
&= \sup\{\|(0,0,2i,0,\ldots)\|_{\mathbb{BC}} : n \in \mathbb{N}\} \\
&= \sup\{0,2\} = 2
\end{aligned}$$

and $\varepsilon \leq \|s-t\|_{l_\infty(\mathbb{BC})} = 2$. On the other hand, since

$$\begin{aligned}
\left\|\frac{s+t}{2}\right\|_{l_\infty(\mathbb{BC})} &= \sup\left\{\left\|\frac{s_n+t_n}{2}\right\|_{\mathbb{BC}} : n \in \mathbb{N}\right\} \\
&= \sup\{(i, j, 0, 0, \ldots)\} = 1,
\end{aligned}$$

there doesn't exist $\delta(\varepsilon) > 0$ such that $\left\|\frac{s+t}{2}\right\|_{\mathbb{D}, l^{\mathbb{k}}_\infty(\mathbb{BC})} \leq 1-\delta$. That is to say that $l_\infty(\mathbb{BC})$ is not uniformly convex.

Example 2.13. *The sequence space $l_1(\mathbb{BC})$ is not uniformly convex.*

Let

$$\begin{aligned}
(s_n) &= (i, 0, 0, \ldots), \\
(t_n) &= (0, -j, 0, 0, \ldots).
\end{aligned}$$

Then, $\|s\|_{l_1(\mathbb{BC})} = \|t\|_{l_1(\mathbb{BC})} = 1$ and

$$\|s-t\|_{l_1(\mathbb{BC})} = \sum_{n=1}^{\infty} \|s_n-t_n\|_{\mathbb{BC}} = \|i\|_{\mathbb{BC}} + \|j\|_{\mathbb{BC}} = 2$$

and $\varepsilon \leq \|s-t\|_{l_1(\mathbb{BC})} = 2$. On the other hand, since

$$\left\|\frac{s+t}{2}\right\|_{l_1(\mathbb{BC})} = \sum_{n=1}^{\infty}\left\|\frac{s_n+t_n}{2}\right\|_{\mathbb{BC}} = \left\|\frac{i}{2}\right\|_{\mathbb{BC}} + \left\|\frac{-j}{2}\right\|_{\mathbb{BC}} = 1,$$

there doesn't exist $\delta(\varepsilon) > 0$ such that $\left\|\frac{s+t}{2}\right\|_{l_1(\mathbb{BC})} \leq 1-\delta$. That is to say that $l_1(\mathbb{BC})$ is not uniformly convex.

References

[1] C. Segre, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, Math. Ann., Vol:40, No.3 (1892), 413-467.
[2] G. B. Price, *An introduction to multicomplex spaces and functions*, M. Dekker, 1991.
[3] M. E. Luna-Elizarraras, M. Shapiro, D. C. Struppa and A. Vajiac, Bicomplex numbers and their elementary functions, Cubo (Temuco), Vol:14, No.2 (2012), 61-80.
[4] D. Alpay, M. E. Luna-Elizarrarás, M. Shapiro and D. C. Struppa, *Basics of functional analysis with bicomplex scalars, and bicomplex Schur analysis*, Springer Science & Business Media, 2014.
[5] M. E. Luna-Elizarrarás, M. Shapiro, D. C. Struppa and A. Vajiac, *Bicomplex holomorphic functions: The algebra, geometry and analysis of bicomplex numbers*, Birkhäuser, 2015.
[6] N. Sager and B. Sağır, On completeness of some bicomplex sequence spaces, Palest. J. Math., Vol:9, No.2 (2020), 891-902.
[7] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications (Vol. 6)*, New York:Springer, 2006.
[8] R. E. Castillo and H. Rafeiro, *An introductory course in Lebesgue spaces*, Switzerland: Springer, 2016.
[9] B. Sağır and İ Alaşalvar, On geometric properties of weighted Lebesgue sequence spaces, Ikonion Journal of Mathematics, Vol:1, No.1 (2019), 18-25.
[10] N. Güngör, Some geometric properties of the non-Newtonian sequence spaces $l_p(N)$, Math. Slovaca, Vol:70, No.3 (2020), 689-696.
[11] G. Köthe, *Topological vector spaces I*, Springer-Verlag Berlin, Heidelberg, 1983.
[12] H. Jarchow, *Locally convex space*, BG Teubner, Stuttgart, 1981.
[13] J. Yeh, *Real Analysis: Theory of measure and integration second edition*, World Scientific Publishing Company, 2006.