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Research Article

Bi-amalgamations subject to the clean and nil-clean properties

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Abstract

This paper investigates necessary and sufficient conditions for a bi-amalgamation to inherit the clean as well as the nil-clean properties. The new results recovers different settings of other constructions such as duplications and amalgamations. All results are used to build new and illustrative examples arising as bi-amalgamations.

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1. Introduction

All rings considered in this paper are commutative with unity. Let $f : A \to B$ and $g : A \to C$ be two commutative ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.$$

This construction was introduced and studied by S. Kabbaj, K. Louartiti, and M. Tamekkante in [15]. It is a generalization of the amalgamated duplication (introduced and studied by DAnna and Fontana [10, 11]) and the amalgamated algebras along an ideal (introduced and studied by M. DAnna, C. Finocchiaro and M. Fontana [7–9]). For more information about the bi-amalgamation, we refer the reader to [3, 15, 16]. Let A and B be rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. The amalgamation of A with B along J with respect to f is defined to be the subring of $A \times B$:

$$A \bowtie^{f} J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

By [15, Example 2.1], $A \bowtie^{f} J = A \bowtie^{\iota,f} (I, J)$, where $I = f^{-1}(J)$ and $\iota = id_A$. For more information about the amalgamation, we refer the reader to [4, 6–11].

Recall that a ring R is called (uniquely) clean if each element in R can be written (uniquely) as the sum of a unit and an idempotent. The concept of clean rings was introduced by Nicholson [17]. Examples of clean rings (uniquely clean rings) include all

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commutative von Neumann regular rings (Boolean rings), any homomorphic image of a clean ring is again clean. A ring R is called nil-clean if each of its elements can be written as the sum of an idempotent and a nilpotent. The notion of nil-clean ring was introduced by A. J. Diesl in [12], where he proved that the class of nil clean rings is strictly contained in the class of clean rings. Any Boolean ring is nil-clean and any homomorphic image of a nil-clean ring is again nil-clean. The following diagram of implications displays the relations among the mentioned concepts, where all the implications are irreversible in general.

$$\text{Boolean rings} \Rightarrow \text{nil-clean rings} \Rightarrow \text{clean rings}$$

In 2014, Mohamed Chhiti, Najib Mahdou, and Mohammed Tamekkante gave a characterization for the amalgamation of A with B along J with respect to f ($A \bowtie^f J$) to be clean. Namely; they proved that if $\frac{f(A)+J}{J}$ is uniquely clean, then $A \bowtie^f J$ is clean if and only if A and $f(A)+J$ are clean rings. [5]. Then, in 2018, Chahrazade Bakkari and Mohamed Es-Said gave a characterization for $A \bowtie^f J$ to be nil-clean. Namely, $A \bowtie^f J$ is nil-clean if and only if A and $f(A)+J$ are nil-clean rings. [2].

This paper establishes necessary and sufficient conditions for a bi-amalgamation $A \bowtie^{f,g} (J, J')$ to inherit the clean property as well as the nil-clean property. All obtained results recover and compare to previous works carried on the amalgamations in [2, 5].

Throughout, $Nil(A)$ denotes the nilradical of A; $Id(A)$ denotes the set of all idempotents of A; $U(A)$ denotes the set of all units of A.

2. Transfer results in the bi-amalgamation

We begin with then following preliminary lemmas.

Lemma 2.1 ([15, Proposition 4.1 parts (2) and (3)]). *Let $A \bowtie^{f,g} (J, J')$ be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) and let $I_\circ = f^{-1}(J) = g^{-1}(J')$. Then*

(1) $\frac{A \bowtie^{f,g}(J,J')}{0\times J'} \cong f(A)+J$ *and* $\frac{A \bowtie^{f,g}(J,J')}{J\times 0} \cong g(A)+J'$.

(2) $\frac{A}{I_\circ} \cong \frac{A \bowtie^{f,g}(J,J')}{J\times J'} \cong \frac{f(A)+J}{J} \cong \frac{g(A)+J'}{J'}$.

Lemma 2.2. *Using the notation of Lemma 2.1,*

If $a \in U(A)$, $f(a)+j \in U(B)$, and $g(a)+j' \in U(C)$, then $(f(a)+j, g(a)+j') \in U(A \bowtie^{f,g} (J, J'))$.

Proof. Since $f(a)+j \in U(B)$, and $g(a)+j' \in U(C)$, there are $b \in B$ and $c \in C$ such that $(f(a)+j)b = 1$ and $(g(a)+j')c = 1$. So

$$(f(a)+j, g(a)+j')(f(a^{-1}) - bf(a^{-1})j, g(a^{-1}) - cg(a^{-1})j') = (1, 1).$$

□

From now until the end of this article, in all the results we will be using the notation of Lemma 2.1. Assume that A is clean ring and $\frac{A}{I_\circ}$ is uniquely clean ring (using the notation of Lemma 2.1). The following theorem establishes the necessary and sufficient conditions under which the bi-amalgamation $A \bowtie^{f,g} (J, J')$ will be clean.

Theorem 2.3. *Consider $A \bowtie^{f,g} (J, J')$, where A is clean.*

(1) *If $A \bowtie^{f,g} (J, J')$ is clean, then $f(A)+J$ and $g(A)+J'$ are clean rings.*

(2) *Suppose that $\frac{A}{I_\circ}$ is uniquely clean, then $A \bowtie^{f,g} (J, J')$ is clean if and only if $f(A)+J$ and $g(A)+J'$ are clean rings*

Proof. (1) Since $f(A)+J$ and $g(A)+J'$ are homomorphic image of $A \bowtie^{f,g} (J, J')$ (by Lemma 2.1) and $A \bowtie^{f,g} (J, J')$ is clean, then they are clean.

(2) Let $a \in A$ and $(j, j') \in J \times J'$. Then $a = u + e$, where $u \in U(A)$ and $e \in I(A)$ (since A is clean). Since $f(A) + J$ and $g(A) + J'$ are clean, then $f(a) + j = f(x_1) + j_1 + f(x_2) + j_2$ and $g(a) + j' = g(y_1) + j_1' + g(y_2) + j_2'$, where $f(x_1) + j_1$ and $f(x_2) + j_2$ ($g(y_1) + j_1'$ and $g(y_2) + j_2'$) are respectively unit and idempotent elements of $f(A) + J$ ($g(A) + J'$). Clearly, $\overline{f(x_1)} = \overline{f(x_1) + j_1}$ and $\overline{f(u)}$ ($\overline{g(y_1)} = \overline{g(y_1) + j_1'}$ and $\overline{g(u)}$) are units in $\frac{f(A)+J}{J}$ ($\frac{g(A)+J'}{J'}$). Also, $\overline{f(x_2)} = \overline{f(x_2) + j_2}$ and $\overline{f(e)}$ ($\overline{g(y_2)} = \overline{g(y_2) + j_2'}$ and $\overline{g(e)}$) are idempotents in $\frac{f(A)+J}{J}$ ($\frac{g(A)+J'}{J'}$). So we have

$$\overline{f(a)} = \overline{f(u)} + \overline{f(e)} = \overline{f(x_1)} + \overline{f(x_2)}, \text{ and}$$

$$\overline{g(a)} = \overline{g(u)} + \overline{g(e)} = \overline{g(y_1)} + \overline{g(y_2)}.$$

Since $\frac{A}{I_\circ}$ is uniquely clean, then by Lemma 2.1, $\frac{f(A)+J}{J}$ and $\frac{g(A)+J'}{J'}$ are uniquely clean, so

$$\overline{f(u)} = \overline{f(x_1)} \text{ and } \overline{f(e)} = \overline{f(x_2)}, \text{ and}$$

$$\overline{g(u)} = \overline{g(y_1)} \text{ and } \overline{g(e)} = \overline{g(y_2)}.$$

Then there exist $(j_3, j_3') \in J \times J'$ and $(j_4, j_4') \in J \times J'$ such that

$$f(x_1) = f(u) + j_3 \text{ and } g(y_1) = g(u) + j_3', \text{ and}$$

$$f(x_2) = f(e) + j_4 \text{ and } g(y_2) = g(e) + j_4'.$$

Now,

$$\begin{aligned}(f(a) + j, g(a) + j') &= (f(x_1) + j_1 + f(x_2) + j_2, g(y_1) + j_1' + g(y_2) + j_2') \\ &= (f(u) + j_3 + j_1 + f(e) + j_4 + j_2, g(u) + j_3' + j_1' + g(e) + j_4' + j_2') \\ &= (f(u) + j_3 + j_1, g(u) + j_3' + j_1') + (f(e) + j_4 + j_2, g(e) + j_4' + j_2')\end{aligned}$$

Clearly, $(f(e) + j_4 + j_2, g(e) + j_4' + j_2') = (f(x_2) + j_2, g(y_2) + j_2')$ is an idempotent in $A \bowtie^{f,g} (J, J')$. On the other hand, since u, $f(u) + j_3 + j_1$, and $g(u) + j_3' + j_1'$ are units in A, $f(A) + J$, and $g(A) + J'$ respectively, then by Lemma 2.2, $(f(u) + j_3 + j_1, g(u) + j_3' + j_1')$ is a unit in $A \bowtie^{f,g} (J, J')$. Therefore, $A \bowtie^{f,g} (J, J')$ is clean.

□

Recall that if I is an ideal of a ring R such that $I \subseteq Nil(R)$, then R is clean (nil-clean) if and only if $\frac{R}{I}$ is clean (nil-clean) [14]. Using the notation of Lemma 2.1, the following result establishes another necessary and sufficient conditions under which the bi-amalgamation will be clean.

Theorem 2.4. (1) *Suppose that $I_\circ \subseteq Nil(A)$. If $A \bowtie^{f,g} (J, J')$ is clean, then A is clean.*

(2) *Suppose that $J \subseteq Nil(B)$ and $J' \subseteq Nil(C)$. If A is clean, then $A \bowtie^{f,g} (J, J')$ is clean.*

Proof. (1) If $A \bowtie^{f,g} (J, J')$ is clean, then $\frac{A}{I_\circ} \cong \frac{A \bowtie^{f,g}(J,J')}{J \times J'}$ is clean but $I_\circ \subseteq Nil(A)$, so A is clean.

(2) If $J \subseteq Nil(B)$ and $J' \subseteq Nil(C)$, then $J \times J' \subseteq Nil(A \bowtie^{f,g} (J, J'))$. Since A is clean, then so is $\frac{A}{I_\circ}$ but $\frac{A}{I_\circ} \cong \frac{A \bowtie^{f,g}(J,J')}{J \times J'}$, hence $\frac{A \bowtie^{f,g}(J,J')}{J \times J'}$ is clean but since $J \times J' \subseteq Nil(A \bowtie^{f,g} (J, J'))$, we have $A \bowtie^{f,g} (J, J')$ is clean.

□

The following lemma is used frequently in this article.

Lemma 2.5. *Let A be a ring, S be a subring of A, and I be an ideal of A. If $s \in Id(S)$ and $I \subseteq Id(A)$, then $s + i \in Id(A)$ for all $i \in I$.*

Proof. Since $I \subseteq Id(A)$, then $2i = 0$ and $i^2 = i$ for all $i \in I$. So if $s \in Id(S)$ and $i \in I$, then $s^2 = s$, $2i = 0$, and $i^2 = i$. Thus

$$(s + i)^2 = s^2 + 2is + i^2 = s + i.$$

□

The following proposition establishes (idempotent) conditions under which the bi-amalgamation will be clean.

Proposition 2.6. *Suppose that $J \subseteq Id(B)$ and $J' \subseteq Id(C)$. If A is clean, then $A \bowtie^{f,g} (J, J')$ is clean.*

Proof. Let $a \in A$ and $(j, j') \in J \times J'$. Then $a = u+e$ where $u \in U(A)$ and $e \in Id(A)$ (since A is clean). Now, since u is a unit in A, we have $(f(u), g(u))$ is a unit in $A \bowtie^{f,g} (J, J')$. Also, since e is an idempotent of A, then $f(e), g(e)$ are idempotents in B and C resspectively, and since $J \subseteq Id(B)$ and $J' \subseteq Id(C)$, then by Lemma 2.5, $(f(e)+j, g(e)+j')$ is an idempotent in $A \bowtie^{f,g} (J, J')$. So, we have $(f(a) + j, g(a) + j') = (f(u), g(u)) + (f(e) + j, g(e) + j')$ is a sum of a unit and an idempotent in $A \bowtie^{f,g} (J, J')$. Therefore, $A \bowtie^{f,g} (J, J')$ is clean. □

Next, we appeal to the previous results to construct new examples of clean rings. Let A be a ring and M an A-module. The trivial ring extension of A by M is the commutative ring $A \ltimes M = A \times M$ with component-wise addition and multiplication given by $(a, m)(a', m') = (aa', am' + a'm)$. [1].

Example 2.7. Let $A = \mathbb{Z}_3$, $B = C = A \ltimes A$, and $J = J' = 0 \ltimes \mathbb{Z}_3$. Consider the canonical embedding $i_A : A \hookrightarrow A \ltimes A$ $(a \mapsto (a, 0))$. Since A is clean and $0 \ltimes \mathbb{Z}_3 = Nil(\mathbb{Z}_3 \ltimes \mathbb{Z}_3)$, then by Theorem 2.4 (2), $A \bowtie^{\iota_A,\iota_A} (J, J')$ is clean.

The following theorem investigates a necessary and sufficient conditions for the bi-amalgamation to be Boolean.

Theorem 2.8. *$A \bowtie^{f,g} (J, J')$ is Boolean if and only if $f(A), g(A)$ are Boolean, $J \subseteq Id(B)$, and $J' \subseteq Id(C)$.*

Proof. Assume $A \bowtie^{f,g} (J, J')$ is Boolean, then $(f(a), g(a))^2 = (f(a), g(a))$ for all $a \in A$, so $(f(a))^2 = f(a)$ and $(g(a))^2 = g(a)$ for all $a \in A$, it follows that $f(A)$ and $g(A)$ are Boolean. Next, let $(j, j') \in (J, J')$. Then $(j, 0)^2 = (j, 0)$ and $(0, j')^2 = (0, j')$, so $j^2 = j$ and $j'^2 = j'$. Hence, $J \subseteq Id(B)$ and $J' \subseteq Id(C)$. Conversely, let $a \in A$ and $(j, j') \in J \times J'$. Since $f(a), g(a)$ are idempotents in $f(A)$ and $g(A)$, respectively, $J \subseteq Id(B)$ and $J' \subseteq Id(C)$, then by Lemma 2.5, $f(a) + j$ and $g(a) + j'$ are idempotents in B and C, respectively. Thus $(f(a) + j, g(a) + j')$ is idempotent in $A \bowtie^{f,g} (J, J')$. Therefore, $A \bowtie^{f,g} (J, J')$ is Boolean. □

The following is a special case of Theorem 2.8 (when f or g is injective).

Corollary 2.9. *Suppose that f or g is injective. Then $A \bowtie^{f,g} (J, J')$ is Boolean if and only if A is Boolean, $J \subseteq Id(B)$, and $J' \subseteq Id(C)$.*

Proof. Suppose $A \bowtie^{f,g} (J, J')$ is Boolean, then by Theorem 2.8, $f(A), g(A)$ are Boolean, $J \subseteq Id(B)$, and $J' \subseteq Id(C)$. If f or g is injective, then $A \cong f(A)$ or $A \cong g(A)$ respectively, and in both cases A is Boolean. Conversely, suppose A is Boolean, then $f(A)$ and $g(A)$ are Boolean, and since $J \subseteq Id(B)$, and $J' \subseteq Id(C)$, then by Theorem 2.8, $A \bowtie^{f,g} (J, J')$ is Boolean. □

Corollary 2.9 can be used to recover an old result by Mohamed Chhiti, Najib Mahdou and Mohammed Tamekkante in [5]

Corollary 2.10 ([5, Proposition 2.19 (2)])**.** *Let $f : A \to B$ be a ring homomorphism and J be an ideal B. Then $A \bowtie^f J$ is Boolean if and only if A is Boolean and $J \subseteq Id(B)$.*

The following lemma is useful in testing the transfer of the nil-clean in the bi-amalgamation.

Lemma 2.11. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Then*

(1) *If A is nil-clean and $J \subseteq Nil(B)$, then $f(A) + J$ is nil-clean.*
(2) *If A is nil-clean and $J \subseteq Id(B)$, $f(A) + J$ is nil-clean.*

Proof. (1) Let $a \in A$ and $j \in J$. Since A is nil-clean, there exists an idempotent e and a nilpotent n such that $a = e + n$. So, $f(a) + j = f(e) + f(n) + j$. Clearly, $f(e) \in Id(f(A) + J)$ and $f(n) \in Nil(f(A) + J)$, but since $j \in J \subseteq Nil(B)$, we have $f(n) + j \in Nil(f(A) + J)$. Hence, $f(a) + j = f(e) + (f(n) + j)$ is sum of an idempotent and a nilpotent in $f(A) + J$. Therefore, $f(A) + J$ is nil-clean.

(2) Let $a \in A$ and $j \in J$. Since A is nil-clean, there exists an idempotent e and a nilpotent n such that $a = e+n$. So, $f(a)+j = f(e)+f(n)+j = (f(e)+j)+f(n)$. Clearly, $f(n) \in Nil(f(A)+J)$, and since $f(e)$ is idempotent in $f(A)$ and $J \subseteq Id(B)$, then by Lemma 2.5, $f(e) + j$ is an idempotent in $f(A) + J$. Hence, $f(a) + j = (f(e)+j)+f(n)$ is sum of an idempotent and a nilpotent in $f(A)+J$. Therefore, $f(A) + J$ is nil-clean.

□

Recall that a commutative ring R is nil-clean if and only if for all $r \in R$, $r - r^2$ is nilpotent [13, Theorem 3]. The following theorem provides the necessary and sufficient conditions for the bi-amalgamation to inherit the nil-clean property.

Theorem 2.12. . *Let $A \bowtie^{f,g} (J, J')$ be the bi-amalgamation of A. Then the following conditions are equivalent:*

(1) *$A \bowtie^{f,g} (J, J')$ is nil-clean.*
(2) *$f(A) + J$ and $g(A) + J'$ are nil-clean.*

Proof. (1) $\Rightarrow$ (2). Assume $A \bowtie^{f,g} (J, J')$ is nil-clean. Then $f(A) + J$ and $g(A) + J'$ are nil-clean rings since by Lemma 2.1, they are homomorphic images of $A \bowtie^{f,g} (J, J')$ which is nil-clean.

(2) $\Rightarrow$ (1). Let $a \in A$ and $(j, j') \in J \times J'$. By (2), $(f(a) + j) - (f(a) + j)^2$ and $(g(a) + j') - (g(a) + j')^2$ are nilpotents in$f(A) + J$ and $g(A) + J'$, respectively. Hence, $(f(a) + j, g(a) + j') - (f(a) + j, g(a) + j')^2$ is nilpotent in $A \bowtie^{f,g} (J, J')$. Therefore, $A \bowtie^{f,g} (J, J')$ is nil-clean. □

The following result was proved by Chahrazade Bakkari and Mohamed Es-Said in [2] which can be recovered by using Theorem 2.12.

Corollary 2.13 ([2, Theorem 2.1])**.** *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Then the following conditions are equivalent:*

(1) *$A \bowtie^f J$ is nil-clean.*
(2) *A and $f(A) + J$ are nil-clean.*

The following is the special case of Theorem 2.12 when A is nil-clean.

Corollary 2.14. *Let $A \bowtie^{f,g} (J, J')$ be the bi-amalgamation of A.*

(1) *Suppose that $J \subseteq Nil(B)$ and $J' \subseteq Nil(C)$. If A is nil-clean, then $A \bowtie^{f,g} (J, J')$ is nil-clean.*
(2) *Suppose that $J \subseteq Id(B)$ and $J' \subseteq Id(C)$. If A is nil-clean, then $A \bowtie^{f,g} (J, J')$ is nil-clean.*

Proof. (1) This follows from Lemma 2.11 (1) and Theorem 2.12.

(2) This follows from Lemma 2.11 (2) and Theorem 2.12.

□

The following example uses Corollary 2.14 to build new examples of nil-clean rings.

Example 2.15. Let $A = \mathbb{Z}_6 \ltimes \mathbb{Z}_6$. Since $\mathbb{Z}_6$ is a nil-clean ring, then by [12, Corollary 3.23 (1)], A is nil-clean. Now, consider the ring homomorphism $p_1 : A \to \mathbb{Z}_6$ ($(a, b) \mapsto a$). Since $\langle 3 \rangle \subset Id(\mathbb{Z}_6)$, then by Corollary 2.14 (2), $A \bowtie^{p_1,p_1} (\langle 3 \rangle, \langle 3 \rangle)$ is nil-clean.

Proposition 2.16. *Suppose that $I_\circ \subseteq Nil(A)$. If $A \bowtie^{f,g} (J, J')$ is nil-clean, then A is nil-clean.*

Proof. Assume that $A \bowtie^{f,g} (J, J')$ is nil-clean, then $\frac{A}{I_\circ} \cong \frac{A \bowtie^{f,g}(J,J')}{J \times J'}$ is nil-clean but $I_\circ \subseteq Nil(A)$, so A is nil-clean. □

The case of f and g are surjective maps can give a necessary and sufficient conditions for $A \bowtie^{f,g} (J, J')$ to be (nil-)clean.

Proposition 2.17. *Consider $A \bowtie^{f,g} (J, J')$ with f and g are surjective maps. If $I_\circ \subseteq Nil(A)$, then $A \bowtie^{f,g} (J, J')$ is clean (nil-clean) if and only if A is clean (nil-clean).*

Proof. Since $I_\circ \subseteq Nil(A)$, then by Theorem 2.4 (1) (Proposition 2.16), $A \bowtie^{f,g} (J, J')$ is clean (nil-clean) implies A is clean (nil-clean). Conversely, since f and g are surjective and $I_\circ \subseteq Nil(A)$, then $J \subseteq Nil(B)$ and $J' \subseteq Nil(C)$, so by Theorem 2.4 (2) (Corollary 2.14 (1)), A is clean (nil-clean) implies $A \bowtie^{f,g} (J, J')$ is clean (nil-clean). □

Lastly, we use Theorem 2.12 to provide another necessary and sufficient conditions for the bi-amalgamation to inherit the nil-clean property.

Theorem 2.18. *Let $A \bowtie^{f,g} (J, J')$ be the bi-amalgamation of A. Then the following conditions are equivalent:*

(1) *$A \bowtie^{f,g} (J, J')$ is nil-clean.*
(2) *Any proper homomorphic image of $A \bowtie^{f,g} (J, J')$ is nil-clean.*

Proof. Since any proper homomorphic image of a nil-clean ring is nil-clean, then the implication (1) ⇒ (2) holds. For the other implication, since $f(A) + J$ and $g(A) + J'$ are proper homomorphic image of $A \bowtie^{f,g} (J, J')$, they are nil-clean, hence by Theorem 2.12, $A \bowtie^{f,g} (J, J')$ is nil-clean. □

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