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XMod$_{\mathbf{Lie}}$ Fibred Over Lie Algebras

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Abstract — In this work, we showed that the category of crossed modules over Lie algebras is fibred over the category of Lie algebras by illustrating that the forgetful functor is a fibration.

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1. Introduction

The crossed module is defined by Whitehead on groups in [1]. In his work, algebraic structures that CW-complex has homotopy 2-type were the crossed modules. In the following years, Lichtenbaum, Schlessinger [2] and Gerstenhaber [3] have made different definitions on the crossed modules of Lie algebras. The definition and some of the categorical properties of crossed modules over algebras can be found in [4] Shammu's work. The crossed modules of commutative algebras could be seen in the work of Porter [5].

Lie algebras were first studied by Marius Sophus Lie in 1870s and independently by Wilhelm Killing in the 1880s to create infinitely small transformations. Lie algebras is defined by Hermann Weyl in 1930s. Also, crossed modules of Lie algebras studied by Kassel and Loday [6]. They investigated this notion with computational algebraic structures that are equivalent to simplicial Lie algebras with Moore complex of length 1. For more details [7–9]. Akça and Arvasi examined simplicial Lie algebras and crossed Lie algebras equivalency and applied to Lie crossed squares [10].

To show the fibration feature, the characteristics of the functor defined between the categories should be examined. The first known definition of fibration was described by Heinz Hopf [11] in his article as Hopf Fibration. In this study, the category of crossed modules on Lie algebras is referred to **XMod**$_{\mathbf{Lie}}$ that this structure obtained before.

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2. Preliminary

Definition 2.1. Let A be a commutative ring with identity if the bilinear function

$$[,]: M \times M \longrightarrow M$$

called multiplication satisfies

L1. $[x,x]=0$

L2. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$

for $x,y,z \in M$ then M is called a Lie algebra over A with $[,]$. Property $L1$ with bilinearity implies the following two conditions.

L3. $[x,y]=-[y,x]$,

L4. $[x,[y,z]]=[[x,y],z]+[y,[x,z]]$

Recall that if $\alpha: N \longrightarrow M$ is a Lie algebra morphism then for all $m_1, m_2 \in M$

$$\alpha([m_1,m_2])=[\alpha(m_1),\alpha(m_2)]$$

we will denote the category of Lie algebras with **Lie** [12].

Definition 2.2. Let A and N be two Lie k-algebras. If the N-algebra morphism

$$\mu: N \to A$$

with Lie action of A on N given by

$$\begin{gathered} A \times N \to N \\ (a,n) \mapsto a \cdot n \end{gathered}$$

satisfies

$XMod_{Lie}1.$ $\mu(a \cdot n)=[a,\mu(n)]$

$XMod_{Lie}2.$ $n' \cdot \mu(n)=[n',n]$

then the triple (N,A,μ) is called crossed modules of Lie algebras [6].

Definition 2.3. Let (A,N,μ), (A',N',μ') be two crossed modules of Lie algebras. A morphism

$$(f,\phi):(A,N,\mu) \longrightarrow (A',N',\mu')$$

of crossed modules of Lie algebras consists of pair of Lie algebra morphism $f: N \longrightarrow N$ and $\phi: A \longrightarrow A'$ such that

$$f(n \cdot a)=f(n) \cdot \phi(a)$$

for all $a \in A$ and $n \in N$ and the following diagram

$$\begin{array}{ccc} N & \xrightarrow{f} & N' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ A & \xrightarrow[\phi]{} & A' \end{array}$$

commutes. That is $\mu' f = \phi\mu$. We will denote this category with $\mathbf{XMod_{Lie}}$.

Example 2.4. Let R be a Lie algebra and I be an ideal of R.

$$\begin{aligned} \partial : \quad I &\to R \\ i &\mapsto i \end{aligned}$$

and the action of R on I is given as Lie product

$$\begin{aligned} R \times I &\to I \\ (r, i) &\mapsto [r, i] \end{aligned}$$

CM1.

$$\begin{aligned} \partial(r \cdot i) &= \partial[r, i] \\ &= [r, i] \\ &= [r, \partial i] \end{aligned}$$

CM2.

$$\begin{aligned} (\partial r) \cdot r' &= [\partial r, r'] \\ &= [r, r'] \end{aligned}$$

So, (I, R, ∂) is a crossed module of Lie algebras.

3. $\mathbf{XMod_{Lie}}$ Fibred Over Lie Algebras

In this section, we will show that the forgetful functor

$$\theta : \mathbf{XMod_{Lie}} \to \mathbf{Lie}$$

which takes $\mu : M \to N \in Ob(\mathbf{XMod_{Lie}})$ in its base Lie Algebra N is a fibration. That is given a map of Lie algebras, we can obtain the crossed modules of Lie algebras can be constructed via pullback. Furthermore, the functor θ has a left adjoint.

Proposition 3.1. The functor

$$\theta : \mathbf{XMod_{Lie}} \to \mathbf{Lie}$$

is fibred.

Proof.

To prove that θ is fibred we will give the construction with pullback crossed module Lie algebras.

Let $L \to N$ be an object in $\mathbf{XMod_{Lie}}$ and $M \to N$ is a homomorphism of Lie algebras

$$\begin{array}{ccc} & & L \\ & & \downarrow \mu \\ M & \xrightarrow[\sigma]{} & N \end{array}$$

Define $P = \{(l,m) : \mu(l) = \sigma(m)\} \subset L \times M$ and mappings

$$\begin{array}{ll} i_1 : P \to L & \qquad i_2 : P \to M \\ (l,m) \mapsto l & \qquad (l,m) \mapsto m \end{array}$$

That is we obtain the following diagram.

$$\begin{array}{ccc} P & \xrightarrow{i_1} & L \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle \mu} \\ M & \xrightarrow[\sigma]{} & N \end{array}$$

It is clear that i_1 and i_2 are Lie algebra morphims. For $(l,m) = p \in P$ we have

$$\begin{aligned} (\sigma \circ i_2)(p) &= (\sigma \circ i_2)(l,m) \\ &= \sigma(i_2(l,m)) \\ &= \sigma(m) \\ &= \mu(l) \\ &= \mu(i_1(l,m)) \\ &= (\mu \circ i_1)(l,m) \\ &= (\mu \circ i_1)(p) \end{aligned}$$

the diagram is commutative.

Claim $\eta : P \to N \in Ob(\mathbf{XMod_{Lie}})$.

For $p = (l,m), p' = (l',m') \in P$ defining the Lie bracket as

$$[(l,m),(l',m')] = ([l,l'],[m,m'])$$

the action is

$$\begin{array}{rcl} N \times P & \to & P \\ (n,p) & \mapsto & n \cdot p = n \cdot (l,m) = (n \cdot l, n \cdot m) \end{array}$$

and η is

$$\begin{array}{rcl} P & \to & N \\ p & \mapsto & \eta(p) = \eta(l,m) = \mu(l) = \sigma(m) \end{array}$$

The conditions are:

XMLie1. For $n \in A$ and $p \in P$

$$\begin{aligned}\eta(n\cdot p) &= \eta(n\cdot(l,m))\\ &= \eta(n\cdot l, n\cdot m)\\ &= \sigma(n\cdot m)\\ &= [n,\sigma(m)]\\ &= [n,\eta(l,m)]\\ &= [n,\eta(p)]\end{aligned}$$

XMLie2. For $p, p' \in P$

$$\begin{aligned}\eta(p)\cdot p' &= \eta((l,m)\cdot(l',m'))\\ &= \sigma(m)\cdot(l',m')\\ &= (\sigma(m)\cdot l', \sigma(m)\cdot m')\\ &= (\mu(l)\cdot l', \sigma(m)\cdot m')\\ &= ([l,l'],[m,m'])\\ &= [(l,m),(l',m')]\end{aligned}$$

Claim 1: Let $X \in Ob(\mathbf{Lie})$ and $\alpha_1 : X \to L, \alpha_2 : X \to M \in Mor(\mathbf{Lie})$. Then there exists

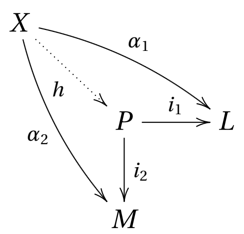
$$\begin{aligned}h : X &\to P\\ x &\mapsto (\alpha_1(x), \alpha_2(x))\end{aligned}$$

Since $\alpha_1, \alpha_2 \in Mor(\mathbf{Lie})$ we get $h \in Mor(\mathbf{Lie})$. For $x \in X$ we have

$$(i_1 \circ h)_{(x)} = i_1(h(x)) = i_1(\alpha_1(x), \alpha_2(x)) = \alpha_1(x)$$

$$(i_2 \circ h)_{(x)} = i_2(h(x)) = i_2(\alpha_1(x), \alpha_2(x)) = \alpha_2(x)$$

that is the diagram



is commutative.

Claim 2: h is unique.
Let $h' : X \to P \in Mor(\mathbf{Lie})$ defined as $h(x) = (l,m) = p$ for $x \in X$ such that $i_1 \circ h' = \alpha_1$ and $i_2 \circ h' = \alpha_2$.
For $x \in X$ we have

$$(i_1 \circ h')_{(x)} = i_1(h'(x)) = i_1(l,m) = l = \alpha_1(x)$$

$$(i_2 \circ h')_{(x)} = i_2(h'(x)) = i_2(l,m) = m = \alpha_2(x)$$

that is

$$h'(x) = (l,m) = (\alpha_1(x), \alpha_2(x)) = h(x)$$

As a result in the diagram

$$\begin{array}{ccc} P & \xrightarrow{i_1} & L \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle \mu} \\ M & \xrightarrow[\sigma]{} & N \end{array}$$

the morphism (i_1,σ) becomes cartesian morphism over $\phi(i_1,\sigma) = \sigma'$ which shows ϕ is a fibration of categories.

Theorem 3.2. The functor $F : \mathbf{XMod_{Lie}} \longrightarrow \mathbf{Lie}$ which is given by $F(L,N,\mu) = N$(base Lie algebra) has a right adjoint.

Proof.

Let $G : \mathbf{Lie} \longrightarrow \mathbf{XMod_{Lie}}$ be defined as $G(M) = (M,M,id)$. Then for any crossed module of Lie algebra say (L,N,μ) and a Lie Algebra. M define the morphism

$$\phi : \mathbf{Lie}(F(L,N,\mu),M) \longrightarrow \mathbf{XMod_{Lie}}((L,N,\mu),G(M))$$

as follows, if $\alpha : N \longrightarrow M$ is a Lie algebra morphism then

$$\phi(\alpha) = (\alpha\circ\mu,\alpha) : (L,N,\mu) \longrightarrow (M,M,id)$$

is a morphism in $\mathbf{XMod_{Lie}}$.

$$\begin{array}{ccc} L & \xrightarrow{\alpha\circ\mu} & M \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle id} \\ N & \xrightarrow[\alpha]{} & M \end{array}$$

i)

$$\begin{aligned} \alpha\circ\mu(n\cdot l) &= \alpha(n,l) \\ &= [\alpha(n),\alpha(l)] \\ &= \alpha(n)\cdot\alpha\circ\mu(l) \end{aligned}$$

ii) The commutativity of this diagram is obvious.

Conversely, for any morphism in $\mathbf{XMod_{Lie}}$ say

$$(\alpha,\beta) : (L,N,\mu) \longrightarrow (M,M,id)$$

Let us define $\psi(\alpha,\beta) = \beta : N \longrightarrow M$ which is a Lie algebra morphism. Then we get $\psi\circ\phi = 1_{\mathbf{Lie}}$ and $\phi\circ\psi = 1_{\mathbf{XMod_{Lie}}}$.

Therefore, ϕ is a bijection.

Now let us show that naturality in (L,N,μ) and M.

Moreover, for any morphism $(\alpha,\beta) : (L,N,\mu) \longrightarrow (P,S,\partial)$ and a Lie algebra morphism $h : A \longrightarrow B$ the following diagrams are commutative.

$$\begin{array}{ccc}
\mathbf{Lie}(F(L,N,\mu),A) & \longrightarrow & \mathbf{XMod_{Lie}}((L,N,\mu),G(A)) \\
\downarrow & & \downarrow \\
\mathbf{Lie}(F(L,N,\mu),B) & \longrightarrow & \mathbf{XMod_{Lie}}((L,N,\mu),G(B))
\end{array}$$

$$\begin{array}{ccc}
\mathbf{Lie}(F(L,N,\mu),A) & \longrightarrow & \mathbf{XMod_{Lie}}((L,N,\mu),G(A)) \\
\downarrow & & \downarrow \\
\mathbf{Lie}(F(P,S,\partial),A) & \longrightarrow & \mathbf{XMod_{Lie}}((P,S,\partial),G(A))
\end{array}$$

Thus ϕ is an isomorphism.

4. Conclusion

In this paper we show that the category $\mathbf{XMod_{Lie}}$ is fibred over Lie algebras. Further work can be done by investigating induced structure on $\mathbf{XMod_{Lie}}$ for a cofibration. Then, it is of interest to investigate the functor

$$\psi^{*} : \mathbf{XMod_{Lie}}/N \to \mathbf{XMod_{Lie}}/M$$

has a right adjoint

$$\psi_{*} : \mathbf{XMod_{Lie}}/M \to \mathbf{XMod_{Lie}}/N$$

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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