SOME SUMS FORMULAE FOR PRODUCTS OF TERMS OF PELL, PELL-LUCAS AND MODIFIED PELL SEQUENCES

Serpil HALICI

Sakarya Üni., Sciences and Arts Faculty, Dept. of Math., Esentepe Campus, Sakarya.

shalici@ssakarya.edu.tr

Özet

Pell, Pell-Lucas ve Modified Pell dizilerinin terimleri için bazı toplam formüllerini elde ettik. Ayrıca, bu toplamların bu dizilerin terimlerine göre yazılabileceğini de gösterdik.

Abstract

We derive some sums formulae for certain products of terms of the Pell , Pell-Lucas and modified Pell sequences. Also, we show that these sums can be rewritten in terms of these sequences.

Keywords : Pell Sequences, Binet Formulae, Recurrence Relations.

AMS Subject Classification: 11B37, 11B39.

1. INTRODUCTION

The Fibonacci and Lucas sequences can be considered as interesting classes of numbers. Applications of the Fibonacci and Lucas numbers provide a wide area to researchers. Also, Pell numbers and Pell identities have been the subject of many studies, see for instance [1, 2, 3]. For ≥ 2 , the Pell $\{P_n\}$, Pell-Lucas $\{Q_n\}$ and modified Pell sequences $\{q_n\}$ are given by the following recurrence relations:

$$P_n = 2P_{n-1} + P_{n-2}\ ,\ \ P_0 = 0\ ,\ \ P_1 = 1,$$

$$Q_n = 2Q_{n-1} + Q_{n-2}\ ,\ \ Q_0 = 2\ ,\ \ Q_1 = 2,$$

$$q_n = 2q_{n-1} + q_{n-2}\ ,\ \ q_0 = 1\ ,\ \ q = 1.$$

The Binet formulae for these sequences are

$$P_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}\ ,\ \ Q_n = \alpha^n + \beta^n\ \ ,\ \ q_n = \frac{\alpha^n+\beta^n}{\alpha+\beta},$$

where α and β are the roots of the characteristic equation for these sequences. For $n = 1, 2, 3, \ldots$

$$\{P_n\} = \{1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots\},$$

$$\{Q_n\} = \{2, 6, 14, \ \ 34, 82, 198, 478, 1154, \ldots\},$$

$$\{q_n\} = \{1, 3, 7, 17, 41, 99, 239,577, \ldots\}$$

can be written. Horadam in [1, 2] gave some identities concerning with these numbers. Some of them are

$$P_{2n+1} = P_n^2 + P_{n+1}^2,\ \ Q_n = 2q_n$$

where P_n and Q_n are the nth Pell and Pell-Lucas numbers, respectively. Also, in [3] authors gave some equations involving the Pell numbers as

$$2P_{n-1}P_n = P_{n+1}^2 - P_{n-1}^2 - 2P_{n+1}P_n,$$

$$P_1^2 + P_2^2 + \cdots + P_n^2 = \frac{P_{n+1}P_n}{2}.$$

The purpose of this paper is to derive some relationships among these numbers and obtain closed formulas for their sums. By Binet formulas for these sequences, we easily get the following equations;

$$P_nQ_m = P_{n+m} + (-1)^m P_{n-m}\ ,\ m,n \in \mathbb{Z},$$

$$P_{2n} = P_nQ_n,\ \ P_{2n+1} = P_nQ_{n+1} + (-1)^n,$$

$$P_n = \frac{Q_{n+1}+Q_{n-1}}{8}\ ,\ P_n^2 = \frac{Q_{2n}+2(-1)^{n+1}}{8},$$

$$Q_n^2 = 2(q_{2n} + (-1)^n) = (Q_{2n} + 2(-1)^n)\ \ q_n^2 = \frac{1}{2}(q_{2n} + (-1)^n),$$

$$P_nP_{n+1} = \frac{1}{4}(q_{2n+1} + (-1)^{n+1}),$$

$$P_{2n+1} = \frac{1}{2}(P_nQ_{n+1} + Q_nP_{n+1}),$$

$$Q_{2n+1} = \frac{1}{2}(8P_nP_{n+1} + Q_nQ_{n+1}),$$

$$P_nP_{n+k} = \frac{1}{8}(Q_{2n+k} + (-1)^{n+1})Q_k,$$

$$P_nP_{n+1} = \frac{1}{8}(Q_{2n+1} + 2(-1)^{n+1}),$$

$$P_nP_{n+k} = \frac{1}{4}(q_{2n+k} + (-1)^{n+1})q_k,$$

$$Q_nQ_{n+1} - Q_{2n+1} = 2(-1)^n,$$

$$P_nQ_{n+1} - P_{n+1}Q_n = 2(-1)^{n+1},$$

$$2q_n^2 - q_{2n} = (-1)^n,$$

$$2q_nq_{n+1} - q_{2n+1} = (-1)^n,$$

$$P_nq_{n+1} + P_{n+1}q_n = P_{2n+1},$$

$$P_nq_{n+1} - P_{n+1}q_n = (-1)^{n+1}.$$

2. SOME SUMS FORMULAE FOR PELL, PELL-LUCAS AND MODIFIED PELL SEQUENCES

Now, we will give the following sums formulas by using the equations given in the section one.

Proposition 1. If P_n and Q_n are the nth Pell and Pell-Lucas numbers, respectively, then we have

$$\sum_{k=1}^{n} P_kQ_k = \frac{P_{2n+1}-1}{2}.$$

Proof. If we write the sum $\sum_{k=1}^{n}(P_k+Q_k)^2$ in the following form,

$$\sum_{k=1}^{n}(P_k+Q_k)^2 = \frac{(P_n+Q_n)\ (P_{n+1}+Q_{n+1})}{2} - (P_1+Q_1),$$

then, we can write

$$\sum_{k=1}^{n}(P_k+Q_k)^2 = \frac{P_n\ P_{n+1} + P_nQ_{n+1} + P_{n+1}Q_n + Q_{n+1}Q_n}{2} - 3,$$

$$\sum_{k=1}^{n}(P_k+Q_k)^2 = \frac{1}{2}\left[\frac{1}{8}Q_{2n+1} - \frac{(-1)^n}{4} + 2P_{2n+1} + Q_{2n+1} + 2(-1)^n\right] - 3,$$

$$\sum_{k=1}^{n}(P_k+Q_k)^2 = P_{2n+1} + \frac{9}{16}Q_{2n+1} + \frac{7}{8}(-1)^n - 3.$$

On the other hand, we can write

$$\sum_{k=1}^{n}(P_k+Q_k)^2 = \sum_{k=1}^{n}(P_k)^2 + 2\sum_{k=1}^{n}P_kQ_k + \sum_{k=1}^{n}(Q_k)^2,$$

$$\sum_{k=1}^{n}(P_k+Q_k)^2 = \frac{P_nP_{n+1}}{2} + 2\sum_{k=1}^{n}P_kQ_k + \frac{Q_{2n+1}+2(-1)^n-4}{2}$$

$$P_{2n+1}+\frac{9}{16}Q_{2n+1}+\frac{7}{8}(-1)^n-3 = 2\sum_{k=1}^{n} P_kQ_k + \frac{9Q_{2n+1}+14\,(-1)^n-32}{16}.$$

By the certain arrangements, we get

$$\sum_{k=1}^{n} P_kQ_k = \frac{P_{2n+1}-1}{2}.$$

Thus, the proof of the proposition is completed. QED.

Corollary 1. Let P_n and q_n are the nth Pell and Modified Pell numbers, respectively. Then, for all positive integers n

$$\sum_{k=1}^{n} P_kq_k = \frac{P_{2n+1}-1}{4},$$

$$\sum_{k=1}^{n} P_k^2 = \frac{q_{2n+1}+(-1)^{n+1}}{8},$$

$$\sum_{k=1}^{n} P_k\,P_{k+1} = \frac{q_{2n}-2+(-1)^n}{8},$$

$$\sum_{k=1}^{n} P_k^2 = \frac{Q_{2n+1}+2(-1)^{n+1}}{16}.$$

Proposition 2. If P_n , Q_n are the nth Pell and Pell-Lucas numbers, then we have

$\sum_{k=1}^{n} P_i\,P_{i+k} = \frac{1}{16}(Q_{2n+k+1}-Q_{k+1})$; if n is even.

$\sum_{k=1}^{n} P_i\,P_{i+k} = \frac{1}{16}(Q_{2n+k+1}-Q_{k-1})$; if n is odd.

Proof. Using the equation $P_nP_{n+k}=\frac{1}{8}(Q_{2n+k}+(-1)^{n+1})Q_k$, we can write the following equations;

$$P_1P_{k+1}=\frac{1}{8}(Q_{k+2}+Q_k),$$

$$P_2P_{k+2}=\frac{1}{8}(Q_{k+4}-Q_k),$$

$$P_3P_{k+3}=\frac{1}{8}(Q_{k+6}+Q_k),$$

$$\vdots \qquad\qquad \vdots$$

$$P_nP_{n+k}=\frac{1}{8}(Q_{2n+k}+(-1)^{n+1}\,Q_k).$$

Then, we obtain that

$$P_1P_{k+1}+P_2P_{k+2}+\cdots+P_nP_{n+k} = \frac{1}{8}(Q_{k+2}+Q_{k+4}+\cdots+Q_{2n+k}+\delta),$$

where $\delta=\begin{cases}Q_k & if\ n\ is\ even,\\ 0 & if\ n\ is\ odd\end{cases}$.

Notice that there are two different cases according to the choose of k. That is, k is an odd integer number such that $k=2p-1,\ p\in \mathrm{Z}$, then

$$\sum_{i=1}^{n} P_i\,P_{i+k} = \frac{1}{8}\big(Q_{2p+1}+Q_{2p+3}+\cdots+Q_{2n+2p-1}+\delta\big),$$

$$\sum_{i=1}^{n} P_i\,P_{i+k} = \frac{1}{8}\left(\sum_{i=1}^{n+p-1} Q_{2i+1}-\sum_{i=1}^{p-1} Q_{2i+1}+\delta\right)$$

$$\sum_{i=1}^{n} P_i\,P_{i+k} = \frac{1}{8}\left(\frac{Q_{2n+2p}-6}{2}-\frac{Q_{2p}-6}{2}+\delta\right)$$

$$\sum_{i=1}^{n} P_i\,P_{i+k} = \frac{1}{16}(Q_{2n+k+1}-Q_{k+1}+2\delta)$$

can be obtained. And then, we consider k is an even integer number such that $k = 2p,\ p \in \mathrm{Z}$. Thus,

$$\sum_{i=1}^{n} P_i P_{i+k} = \frac{1}{8}\left(Q_{2p+2} + Q_{2p+4} + \cdots + Q_{2n+2p} + \delta\right),$$

$$\sum_{i=1}^{n} P_i P_{i+k} = \frac{1}{8}\left(\sum_{i=1}^{n+p} Q_{2i} - \sum_{i=1}^{p} Q_{2i} + \delta\right),$$

$$\sum_{i=1}^{n} P_i P_{i+k} = \frac{1}{8}\left(\frac{Q_{2n+2p+1}-2}{2} - \frac{Q_{2p+1}-2}{2} + \delta\right),$$

$$\sum_{i=1}^{n} P_i P_{i+k} = \frac{1}{16}(Q_{2n+k+1} - Q_{k+1} + 2\delta).$$

Thus, the proof is completed. QED.

Moreover, we can get some sums for Modified Pell numbers;

If n is a even number, then we can write

$$\sum_{i=1}^{n} P_i P_{i+k} = \frac{1}{8}(q_{2n+k+1} - q_{k+1});$$

If n is a odd number, then we can write

$$\sum_{i=1}^{n} P_i P_{i+k} = \frac{1}{8}(q_{2n+k+1} - q_{k-1}).$$

Proposition 3. If $\mathrm{Q_n}$ is the n^{th} Pell-Lucas number, then we have

$$\mathrm{Q_{n+1}Q_{n-1} - Q_{2n}} = 6(-1)^{n+1}.$$

Proof. For $1, 2, 3, \ldots, n-1$ we write the following equation;

$$2\sum_{k=1}^{n-1} P_k P_{k+1} = P_{n+1}^2 + P_n^2 - P_1^2 - P_2^2 - 2\sum_{k=1}^{n-1} P_k P_{k+1} - 2P_n P_{n+1} + 2P_1 P_2.$$

Taking $P_1 = 1,\ P_2 = 2$ in the last equation and using the identity $4P_n = \mathrm{Q_n + Q_{n-1}}$ we can get

$$\sum_{k=1}^{n-1} P_k P_{k+1} = \frac{\left(\frac{\mathrm{Q_{n+1}+Q_n}}{4}\right)^2 + \left(\frac{\mathrm{Q_{n-1}+Q_n}}{4}\right)^2}{4} + \frac{-\left(\frac{\mathrm{Q_{n+1}+Q_n}}{4}\right)\left(\frac{\mathrm{Q_{n-1}+Q_n}}{4}\right) - 1}{2}.$$

Here, if we use $\mathrm{Q_n}^2 = \mathrm{Q_{2n}} + 2(-1)^n$, then we have

$$\sum_{k=1}^{n-1} P_k P_{k+1} = \frac{\mathrm{Q_{2n+2} + Q_{2n-2} - 2Q_{n+1}Q_{n-1}} + 4(-1)^{n+1} - 16}{64}.$$

On the other hand, we know that $\sum_{k=1}^{n-1} P_k P_{k+1} = \frac{\mathrm{Q_{2n}} - 4 + 2(-1)^n}{64}$.

If we equal the right sides of the last two equations, then we have

$$\mathrm{Q_{n+1}Q_{n-1} - Q_{2n}} = 6(-1)^{n+1}.$$

Thus, the proof is completed. So, the next corollary can be given without proof.

Corollary 2. If $\mathrm{q_n}$ is the nth modified Pell number, then we have

$$\mathrm{q_{2n+2} + q_{2n-2}} - 4(\mathrm{q_{2n} + q_{n+1}q_{n-1}}) = 6(-1)^n.$$

REFERENCES

[1] Horadam, A. F., Applications of Modified Pell Numbers to Representations, Ulam Quarterly, **3(1),** (1994), 34-53.

[2] Horadam, A. F., Pell Identities, The Fibonacci Quart., **9(3),** (1971), 245-252.

[3] Melham, R., Sums Involving Fibonacci and Pell Numbers, Portugaliae Math., **56(3),** (1999), 309-317.

[4] Vajda, S., Fibonacci and Lucas numbers and the Golden section: Theory and Applications, Ellies Horwood Ltd., (1989).

[5] T. Koshy, Fibonacci and Lucas Numbers with Applications, AWiley-Intersection Publication, 2001.