ON SOME RELATIONSHIPS AMONG PELL, PELL-LUCAS AND MODIFIED PELL SEQUENCES

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ABSTRACT

In this study, Pell, Pell-Lucas and Modified Pell numbers are investigated. Using Binet formulas for these sequences, some relationships among these sequences are obtained. Also, some sum formulas are given by these properties**.**

Key words: Pell Numbers, Pell-Lucas Numbers, Modified Pell Numbers.

AMS Subject Classification: 11B37; 11B39.

PELL, PELL-LUCAS VE MODIFIED PELL DİZİLERİ ARASINDA BAZI İLİŞKİLER

ÖZET

Bu çalışmada Pell, Pell-Lucas ve Modified Pell sayıları çalışıldı. Bu sayı dizileri için tanımlanan Binet formülleri kullanılarak, bu dizilerin birbirleriyle olan bazı ilişkileri ortaya kondu. Bulunan bu özellikler yardımıyla da bazı toplam formülleri verildi.

1. INTRODUCTION

The Fibonacci and Lucas numbers and their generalizations have very important properties and applications to almost every fields of science and art. The applications of these numbers can be seen in [7].

Some sequences, such as Pell sequences, have a similar structure with the Fibonacci sequence [1, 2, 3, 4, 6]. Pell sequence, $\{P_n\}$, can be defined as

$$P_{n+1} = 2P_n + P_{n-1} \quad ; \quad n \geq 1$$

with initial condition $P_0 = 0$, $P_1 = 1$. Moreover, the Pell sequences can be explained by matrices. In [1], Ercolana gave a matrix method for generating the Pell sequence as follows;

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n$$

Using this matrix, the following equation can be written:

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n.$$

It is seen that there is many relationships between the matrices and Pell numbers. These relations can be seen in [2, 3, 5].

Pell-Lucas sequence can be defined as

$$Q_{n+1} = 2Q_n + Q_{n-1} \; ; \; n \geq 1$$

where $Q_0 = 2$; $Q_1 = 2$. Also, Modified Pell sequence $\{q_n\}$ can be defined by the following recursive relation:

$$q_{n+1} = 2q_n + q_{n-1}, \; n \geq 1$$

where $q_0 = 1$ and $q_1 = 1$. In [4], Melham gave Binet formulas for the Pell and Pell-Lucas numbers ;

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \;\; , \;\; Q_n = \alpha^n + \beta^n$$

In [4], Horadam gave Binet formula for Modified Pell sequence,

$$q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta},$$

where α , β are the roots of the equation

$$x^2 - 2x - 1 = 0.$$

In this paper, we investigated Pell, Pell-Lucas and Modified Pell numbers. Also, we derive some miscallenous relations by using their Binet formulae.

2. MAIN RESULTS

Now, we will give the following lemma without proof. However, the proof can be easily obtained using the following equation.

$$P_n^2 = \frac{1}{8}\left(Q_{2n} + 2(-1)^{n+1}\right).$$

Lemma 1. If P_n, Q_n , q_n are *nth* Pell, Pell-Lucas and Modified Pell numbers, then for all positive integers n, we have

$$P_{2n}^2 = \frac{1}{8}\left(Q_{4n} - 2\right) = \frac{1}{4}\left(q_{4n} - 1\right),$$

and

$$P_{2n+1}^2 = \frac{1}{8}\left(Q_{4n+2} + 2\right) = \frac{1}{4}\left(q_{4n+2} + 1\right).$$

Proposition 2. If P_n, Q_n , q_n are *nth* Pell, Pell-Lucas and Modified Pell numbers, then for all positive integers n,m,k we have

$$P_{n+m} P_{n+k} = \frac{1}{8}\left(Q_{2n+m+k} + (-1)^{n+k+1} Q_{m-k}\right).$$

Proof: Considering the Binet formulas for Pell, Pell-Lucas and Modified Pell numbers, we can write

$$P_{n+m} P_{n+k} = \frac{1}{8}\left(Q_{2n+m+k} + (-1)^{n+k+1} Q_{m-k}\right)$$

$$= \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} \cdot \frac{\alpha^{n+k} - \beta^{n+k}}{\alpha - \beta}$$

$$= \frac{\alpha^{2n+m+k} + \beta^{2n+m+k}}{8} - \frac{(\alpha\beta)^n (\alpha\beta)^k (\alpha^{m-k} + \beta^{m-k})}{8}$$

So, the proof is completed.

Proposition 3. If P_n, q_n are *nth* Pell and Modified Pell numbers, then for all positive integers n, we have

$$P_n q_{n+2} = \frac{1}{2} P_{2n+2} - (-1)^n .$$

Proof: Using the Binet formulas of Pell and Modified Pell, we get

$$P_n q_{n+2} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \; \frac{\alpha^{n+2} + \beta^{n+2}}{\alpha + \beta}$$

$$= \frac{\alpha^{2n+2} + \alpha^n \beta^{n+2} - \alpha^{n+2} \beta^n - \beta^{2n+2}}{(\alpha - \beta)(\alpha + \beta)}$$

By simple computation, we obtain that

$$\alpha\beta = -1, \quad \alpha^4 - 1 = 4\sqrt{2}\alpha^2,$$

$$\alpha + \beta = 2, \quad \alpha - \beta = 2\sqrt{2}, \quad \beta^4 - 1 = -4\sqrt{2}\beta^2$$

So, the proof is completed.

Proposition 4. If P_n, Q_n, q_n are *nth* Pell, Pell-Lucas and Modified Pell numbers, then for all integers n, we have

$$\sum_{i=1}^{n} P_{2i}q_{2i+2} = \frac{1}{4}P_{2n}P_{2n+4} - n,$$

and

$$\sum_{i=1}^{n} P_{2i}Q_{2i+2} = \frac{1}{2}P_{2n}P_{2n+4} - 2n.$$

Proof: Firstly, let us define a new sequence as follows;

$$a_n = \left(\frac{1}{4}P_{2n}P_{2n+4} - n\right) - \left(\frac{1}{4}P_{2n-2}P_{2n+2} - (n-1)\right)$$

From the definition of Binet formula for Pell numbers, we can write

$$a_n = \left(\frac{1}{4}P_{2n}P_{2n+4} - n\right) - \left(\frac{1}{4}P_{2n-2}P_{2n+2} - (n-1)\right)$$

$$= \frac{1}{4}\left(P_{2n}P_{2n+4} - P_{2n-2}P_{2n+2} - 4\right)$$

$$= \frac{1}{4}\left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \cdot \frac{\alpha^{2n+4} - \beta^{2n+4}}{\alpha - \beta}\right.$$

$$\left. - \frac{\alpha^{2n-2} - \beta^{2n-2}}{\alpha - \beta} \cdot \frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha - \beta} - 4\right)$$

$$= \frac{1}{4}\left(\frac{\alpha^{4n}\left(\alpha^4 - 1\right) + \beta^{4n}\left(\beta^4 - 1\right)}{(\alpha - \beta)^2} - 4\right)$$

$$= \frac{1}{4}\left(\frac{\alpha^{4n}4\sqrt{2}\alpha^2 - \beta^{4n}4\sqrt{2}\beta^2}{(\alpha - \beta)^2} - 4\right)$$

$$= \frac{1}{4}\left(\frac{\alpha^{4n+2} - \beta^{4n+2}}{\sqrt{2}} - 4\right)$$

$$= \frac{1}{2}P_{4n+2} - 1$$

$$= P_{2n}q_{2n+2}$$

Now, using the idea of "creative telescoping" [5], we conclude

$$\sum_{i=1}^{n} P_{2i}q_{2i+2} = \sum_{i=1}^{n} a_i$$

$$= \sum_{i=1}^{n}\left[\left(\frac{1}{4}P_{2i}P_{2i+4} - i\right)\right] - \sum_{i=1}^{n}\left(\frac{1}{4}P_{2i-2}P_{2i+2} - (i-1)\right)$$

$$= \left[\left(\frac{1}{4}P_{2n}P_{2n+4} - n\right) - \left(\frac{1}{4}P_{2n-2}P_{2n+2} - (n-1)\right)\right]$$

$$+ \cdots + \left[\left(\frac{1}{4}P_2 P_6 - 1\right) - \left(\frac{1}{4}P_0 P_2 - 0\right)\right]$$

$$= \frac{1}{4}P_{2n}P_{2n+4} - n,$$

which is desired.

Proposition 5. If P_n is n th Pell number, then we have the following equation;

$$\left(P_{n+1}-P_n\right)^2 = 2P_n^2 + \left(-1\right)^n.$$

Proof: Considering the Binet formulas for Pell numbers, we get

$$\left(P_{n+1}-P_n\right)^2 = \left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} - \frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2$$

$$= \left(\frac{\alpha^n\left(\alpha-1\right)-\beta^n\left(\beta-1\right)}{\alpha-\beta}\right)^2$$

On the other hand, we know that

$$\alpha-1=\sqrt{2}, \quad \beta-1=-\sqrt{2}. \quad \text{, we get}$$

$$\left(P_{n+1}-P_n\right)^2 = \left(\frac{\alpha^n+\beta^n}{2}\right)^2.$$

Also, we obtain that

$$2P_n^2 + \left(-1\right)^n = 2\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2 + \left(-1\right)^n$$

$$= \frac{\alpha^{2n}+\beta^{2n}-2\left(\alpha\beta\right)^n}{4} + \left(-1\right)^n$$

$$= \frac{\alpha^{2n}+\beta^{2n}-2\left(-1\right)^n+4\left(-1\right)^n}{4}$$

$$= \frac{\alpha^{2n}+\beta^{2n}+2\left(-1\right)^n}{4}$$

$$= \left(\frac{\alpha^n+\beta^n}{2}\right)^2$$

So, the proof is completed.

Proposition 6. If P_n, Q_n, q_n are *nth* Pell, Pell-Lucas and Modified Pell numbers, then for all integers n, we have

$$P_{2n}q_{2n+2} - P_{2n-2}q_{2n} = Q_{4n},$$

and

$$P_{2n}Q_{2n+2} - P_{2n-2}Q_{2n} = 2Q_{4n}.$$

Proof: If we use the Binet formlas of the Pell and Modified Pell numbers and $\alpha\beta = -1$, then we have

$$P_{2n}q_{2n+2} - P_{2n-2}q_{2n}$$

$$= \frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta} \frac{\alpha^{2n+2}+\beta^{2n+2}}{\alpha+\beta} - \frac{\alpha^{2n-2}-\beta^{2n-2}}{\alpha-\beta} \frac{\alpha^{2n}+\beta^{2n}}{\alpha+\beta}$$

$$= \frac{\alpha^{4n+2}-\beta^{4n+2}-\alpha^{4n-2}+\beta^{4n-2}}{\alpha^2-\beta^2}$$

By simple computation, we get

$$\alpha^4 - 1 = 4\sqrt{2}\alpha^2, \quad \beta^4 - 1 = -4\sqrt{2}\beta^2.$$

And

$$\alpha^2 - \beta^2 = 4\sqrt{2},$$

So, the proof is completed. Similarly the other equation can be obtained by the equation $Q_n = 2q_n$.

Proposition 7. If P_n, Q_n, q_n are *nth* Pell, Pell-Lucas and Modified Pell numbers, then for all integers n, we have

$$\sum_{k=1}^{n} Q_{4k} = P_{2n}q_{2n+2}.$$

Proof: Considering proposition 4, we can write

$$\sum_{k=1}^{n} Q_{4k} = \sum_{k=1}^{n} \left(P_{2k}q_{2k+2} - P_{2k-2}q_{2k}\right)$$

$$= \left(P_{2n}q_{2n+2} - P_{2n-2}q_{2n}\right) + \left(P_{2n-2}q_{2n} - P_{2n-4}q_{2n-2}\right)$$

$$+ \cdots + \left(P_2 q_4 - P_0 q_2\right).$$

Thus, the proof is completed.

Proposition 8. If P_n, q_n are *nth* Pell and Modified Pell numbers, then for all integers n, , we have

$$\sum_{k=1}^{n} P_{2k}^2 = \frac{1}{8}\left(P_{2n}q_{2n+2} - 2n\right) = \frac{1}{16}\left(P_{2n}Q_{2n+2} - 4n\right).$$

Proof: In [2], author give following relation between the Pell and Pell-Lucas numbers.

$$P_n^2 = \frac{1}{8}\left(Q_{2n} + 2(-1)^{n+1}\right).$$

If we take *2n* instead of *n* in last equation, then we have

$$P_{2n}^2 = \frac{1}{8}\left(Q_{4n} - 2\right).$$

Thus, we obtain that

$$\sum_{k=1}^{n} P_{2k}^2 = \frac{1}{8}\sum_{k=1}^{n}\left(Q_{4k} - 2\right) = \frac{1}{8}\left(P_{2n}q_{2n+2} - 2n\right).$$

which is desired. Similarly, the other equation can be obtained.

Proposition 9. Let α, β be the root of $x^2 - 2x - 1 = 0$. Then

$$\alpha^n = q_n + P_n\sqrt{2},$$

and

$$\beta^n = q_n - P_n\sqrt{2}.$$

Proof: We will prove the theorem by induction method on n. By the definitions of Pell and Modified Pell numbers, we have

$$\alpha = q_1 + P_1\sqrt{2}.$$

We suppose that the claim is true for n. Now, we will show that the claim is true for $n + 1$. Using by our assumption, we can write

$$\alpha^{n+1} = \alpha^n \alpha = \left(q_n + P_n\sqrt{2}\right)\left(1 + \sqrt{2}\right)$$
$$= q_n + 2P_n + q_n\sqrt{2} + P_n\sqrt{2}.$$

In [2], author gave a relationships such that $P_n + q_n = P_{n+1}$, $P_{n+1} + P_n = q_{n+1}$. Therefore, we obtain that

$$\alpha^{n+1} = q_{n+1} + P_{n+1}.$$

So, the proof is completed. Also, we can write

$$\beta^n = q_n - P_n\sqrt{2}.$$

Thus, we get

$$\alpha^n = q_n + P_n\sqrt{2}, \ \beta^n = q_n - P_n\sqrt{2}$$

which is desired.

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