A NEW METHOD TO SHOW ISOMORPHISMS OF FINITE GROUPS

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ABSTRACT

The concept of a group is of fundamental importance in the study of algebra. Groups which are, from the point of view of algebraic structure, essentially the same are said to be isomorphic. The ideal aim of finite group theory is to find all finite groups: that is, to show how to construct finite groups of every possible type, and to establish effective procedures which will determine whether two given finite groups are of the same type. We added a new one on all present techniques: Group Matrices. This technique is easier and shorter than the all present techniques to obtain all finite groups of the same finite order. Since this technique includes matrices, the theory can be translated to computational programming in the future.

Key Words - Finite group, Isomorphism.

SONLU GRUPLARIN İZOMORFLUĞUNU GÖSTEREN YENİ BİR METOD

ÖZET

Bir grup yapısı cebir çalışmaları için temel önem taşır. Cebirsel yapılar olarak bakıldığında karakteristik olarak aynı olan gruplar izomorfik olarak adlandırılır. Tüm sonlu grupları bulmak, sonlu grup teorinin önemli bir amacıdır. Bu ise, tüm durumlar için sonlu grup yapısını inşa etmek, yani verilen iki sonlu grubun aynı tipte olup olmadığını veren prosedürleri ortaya koymaktır. Bu makalede, kısmen bu amaca hizmet eden, şu an bilinen tekniklere bir yenisi ilave edilmiştir: Grup Matrisleri. Bu teknik aynı mertebeli tüm sonlu grupları tespit etmede hali hazırda bilinen tekniklere göre daha kolay ve kısadır. Bu teknik matrisleri içerdiğinden, burada teoride verilenler gelecekte bilgisayar programlamaya aktarılabilir.

Anahtar Kelimeler - Sonlu grup, İzomorfizm

1. INTRODUCTION

In this paper, the capital letters G, H, K,... will always denote groups. We shall usually denote the identity element of a group by e. Then if G is a group, the subset $\{e\}$ consisting only of the identity element of G forms a subgroup which we call the trivial subgroup of G. Let $g \in G$. If there are distinct positive integers r, s such that $g^r = g^s$, we say that g is of finite order in G: then there is a positive integer n such that $g^n = e$ and we call the least such n the order of G, denoted o(g). If there is an element g of G such that every element of G is expressible as a power g^m of g (m is an integer), we say that G is a cyclic group and that g generates G: then we write $G=\langle g \rangle$. Let X be any set. If X is finite, we denote by $|X|$ the number of elements in X.

For any group G, we call $|G|$ the order of G. Further details are to be found in [1] and [2].

The results we give are proven by some well known results in finite group theory which can be found in [3], [4], [5], [6] and [7].

The new technique which is introduced in Section II gives that how many different (up to isomorphisms) groups of the same finite order and that any two groups of the same finite order are isomorphic or not. This way is easier and shorter than the all present techniques to obtain all finite groups of the same finite order. Since this technique includes matrices, the theory can be translated to computational programming in the future.

2. GROUP MATRICES

2.1. Definition. Let G be a group of order n and e is the identity element of G. We will denote the elements which order k in G by g_k and g_{k_j} will denote the j-th element which order k in G. If the matrice $\theta = \{a_{ij}\}_{nxn}$ consists of the elements

$$a_{11} = e, a_{12} = a_{13} = \ldots = a_{1n} = a_{21} = a_{31} = \ldots = a_{n1} = 0$$

and

$$g_{k_j} = a_{k(j+1)} \text{ where } 1 \le j \le n-1,\ 2 \le k \le n$$

then we will call it a group matrice of G and we denote this by θ_G.

2.2. Example. We consider the Klein 4-group $V = \{e, a, b, c\}$, that is $a^2 = b^2 = c^2 = e$. Then

$$\theta_V = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

of course we can change the place of a,b and c.

2.3. Example. We take the group $Z_3 = \{\bar{0}, \bar{1}, \bar{2}\}$, then

$$\theta_{Z_3} = \begin{bmatrix} \bar{0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \bar{1} & \bar{2} \end{bmatrix}.$$

2.4. Definition. Let G and H be groups of the same finite order n and these group matrices are θ_G and θ_H ,respectively. If for every $1 \le i \le n$ the i-th rows of θ_G and θ_H have non-zero elements of same number, then we will call these matrices are equivalent and denote $\theta_G \approx \theta_H$. If these are not equivalent, we will denote this by $\theta_G \not\approx \theta_H$.

2.5. Example. We take the group V as in example II.2 and take another group $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. Then

$$\theta_{Z_4} = \begin{bmatrix} \bar{0} & 0 & 0 & 0 \\ 0 & \bar{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \bar{1} & \bar{3} & 0 \end{bmatrix}$$

hence by Definition II.4 $\theta_V \not\approx \theta_{Z_4}$.

We want to obtain isomorphisms of finite groups by group matrices. Since this technique includes matrices, all distinct groups of the same finite order can be obtained by computational in the future. Before giving some results, we want to explain what we mean with an example: it is well known that there are two distinct groups of order 4 [7]. If we take a group of order 4 then its group matrice will be equivalent to θ_V or θ_{Z_4}. For example we consider the group $K = \{1, -1, i, -i\}$ with usual multiplication where $i = \sqrt{-1}$ .Then its group matrice is

$$\theta_K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & -i & 0 \end{bmatrix}.$$

Hence it is seen that $\theta_K \approx \theta_{Z_4}$. But we already know that $K \cong Z_4$ (K is isomorphic to Z_4).

II.6 Lemma. The relation "$\approx$" on the set of all group matrices is an equivalance relation.

The proof of Lemma II.6 is easy, hence omitted.

It is seen that every $n \times n$ matrices is not represent a group. So we consider only group matrices, i.e. these matrices represent groups. We have the following;

II.7 Theorem. Let G and H be groups of the same finite order n and these group matrices are θ_G and θ_H ,respectively. Then $G \cong H$ if and only if $\theta_G \approx \theta_H$.

Proof If $G \cong H$ then both of these groups have the elements of same order which numbers are equal. Therefore by Definition II.4 $\theta_G \approx \theta_H$. Conversely if

$_G \approx \theta_H$, then we can define a one-to-one correspondence between the elements of G and H by using order of elements. Since one to one homomorphisms between two groups of the same finite order is also onto, it is seen that $G \cong H$.

We will denote the set of all non-equivalent $n \times n$ group matrices by θ_n, from Lemma II.6 it can easily seen that θ_n the set of all equivalance classes of nxn group matrices. Thus we can do a classification of group matrices. $|\theta_n|$ denotes the number of elements of θ_n. Undefined sets can be found in [7]. In the following table, given in the next page, some groups are Q_8: Quaternion group, A_n: Alternating group of n letters, D_n: Dihedral group of degree n and T: A group generated by elements a,b such that $o(a)=6, b^2=a^3, ba=a^{-1}b$. We have the following table;

Table1. The order of group matrices of some groups

	Groups	**Group Matrices**
Order (n)	Distinct Groups	Order ($\|\theta_n\|$)
1	$\langle e \rangle$	1
2	Z_2	1
3	Z_3	1
4	V, Z_4	2
5	Z_5	1
6	Z_6, D_3	2
7	Z_7	1
8	$Z_2 \oplus Z_2 \oplus Z_2, Z_2 \oplus Z_4, Z_8, Q_8, D_4$	5
9	$Z_3 \oplus Z_3, Z_9$	2
10	Z_{10}, D_5	2
11	Z_{11}	1
12	$Z_2 \oplus Z_6, Z_{12}, A_4, D_6, T$	5
13	Z_{13}	1
14	Z_{14}, D_7	2
15	Z_{15}	1

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