ON SOME SEMİGROUPS

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Özet **- Bu çalışmada bazı yarıgrupların idempotent elemanları karakterize edildi. Ayrıca, D(I) birim aralıktan birim aralığa tanımlı türevlenebilir fonksiyonların yarıgrubu olmak üzere, D(I) yarıgrubunda iki elemanın bileşkesinin bir homeomorfizm olması için gerekli ve yeterli koşul verildi.**

Anahtar Kelimeler - yarıgrup, idempotent eleman, türevlene-bilir fonksiyon, birim.

Abstract **- In this study, we characterize idempotent elements of some semigroups. In addition to this, a necessary and sufficient condition for composition of two elements in D(I) to be a homoeomorphism was given where D(I) is the semigroup of all the differentiable function form the unit interval into the unit interval.**

Keywords – semigroup , idempotent element, differentiable function , unit.

I. INTRODUCTION

Let X be a topological space and let S(X) be the set of all continuous selfmaps of X. Let o represent composition and let $i \in S(X)$ be the function such that $i(x)=x$ for every x in X. Then (S(X),o) is a semigroup with identity i where fog is the composition of f and g. Let $f \in S(X)$. If f(x)=a for every x in X, then it is said that f is a constant function.

Let X=R or X=I where R is the real numbers and I is the unit interval and let D(X) denote the semigroup of all the differentiable selfmaps of X. We recall that f is differentiable at 0 if and only if $f'_+(0)$ exists. In addition, f is differentiable at 1 if and only if $f'_-(1)$ exists where $f \in D(I)$. It can be seen easily that (D(X),o) is a semigroup with identity i. In this study we characterize the idempotent elements of D(I).

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As a results of our study we determine a necessary and sufficient condition for fog to be a homeomorfizm, where f and g are in D(I).Let (S,o) be a semigroup and let $x,y \in S$. Then by xy we mean xoy.

II. MAIN THEOREMS

Let S be a semigroup and let $a \in S$. If $a^2 = a$, a is said to be an idempotent element of S. An element $z \in S$ is said to be a left zero of S if $za = z$ for every $a \in S$. It can be seen that $f \in S(X)$ is a left zero of S(X) if and only if f is a constant function.It is clear that if $f \in S(X)$ is a constant function, it is an idempotent element of S(X). We can give many idempotent elements in S(I) and S(R), which are different from the constant functions. The following theorem is proved in [1]. We will give a different proof.

Theorem 2.1. Let $J = \{ f \in D(I) \mid f$ is constant function or $f = i$, i is identity function $\}$. Then J is the set of all the idempotent elements of D(I).

Proof. Suppose that g is idempotent and $g \neq i$ and that g is not constant function. Let the image of I under g be [a,b]. Then it follows that a<b and $[a,b] \neq [0,1]$. Let $x \in [a,b]$. Then $x = g(y)$ for some y in [0,1]. Thus $g(x) = g(g(y)) = g^2(y) = g(y) = x$. Therefore we see that $g'(x) = 1$ for every x in [a,b]. On the other hand, it follows that

$$g(a) = a = \min\{g(x) \mid x \in I\}$$

and

$$g(b) = b = \max\{g(x) \mid x \in I\}.$$

Since $[a,b] \neq [0,1]$,either 0<a or b<1. If 0<a, then we see that $g'(a)=0$. In the same way , if b<1, then $g'(b)=0$. But this is a contradiction since $g'(x)=1$ for every x in [a,b].

The proof of Theorem 2.1 carries over easily to the semigroup D(R). Thus we can state the following theorem.

Theorem 2.2. An element of D(R) is idempotent if and only if it is identity or it is a constant function (left zero of D(R)).

Let S be a semigroup with identity e and let $x \in S$. Then x is said to be a unit if $xy = yx = e$ for some $y \in S$. An element $a \in S$ is said to be a regular element if $axa = a$ for some $x \in S$.
It can be seen that an element $f \in S(X)$ is a unit if and only if f is a homeomorphism from X to X.

Theorem 2.3. Let S be a semigroup with identity e. Suppose that the following property is satisfied :"If $x \in S$ is an idempotent element, then x=e or x is the left zero of S ". Then we get

1- xy is unit in S if and only x and y are units in S.
2- a is a regular element of S if and only if a is a unit or a is a left zero element of S.

Proof. If we show that $xy = e$ implies $yx = e$, then the proof follows. Assume that $xy = e$. Then, since $(yx)(yx) = y(xy)x = yex = yx$, yx is an idempotent.

Thus yx is a left zero or $yx = e$. If $x = e$, then $y = e$. Thus $yx = e$. Assume that $x \neq e$. We show that $yx = e$. On the contrary, if $yx \neq e$, then yx is a left zero of S. Therefore we have $yx = (yx)y = y(xy) = ye = y$ and thus we obtain $e = xy = x(yx) = (xy)x = ex = x$, which is a contradiction. So we have $yx = e$.

Now suppose that a is a regular element of S. Then $axa = a$ for some $x \in S$. We may suppose $a \neq e$. We see that ax and xa are idempotent elements of S. Suppose that $ax = e$. We assert that $xa = e$. For, if xa is a left zero element of S, then $xa = (xa)x = x(ax) = xe = x$, and thus $a = axa = ax = e$, which is a contradiction. Therefore, $xa = e$. That is a is a unit. Assume that ax is a left zero element of S. Then $ax = (ax)a = axa = a$. Thus a is a left zero element of S. If a is a unit or a is a left zero element of S, then a is a regular element.
Before main theorem, we give a theorem from[2].

Theorem 2.4. Let R^n be n-dimensional Euclidiean space and let $f,g \in S(R^n)$.Then fog is a unit if and only if f and g are units in $S(R^n)$.

However, the same is not true for $S(I^N)$,where

$$I^N = \{(x_1, x_2, \ldots, x_N) \mid 0 \leq x_i \leq 1, 1 \leq i \leq N\}$$

Let $g:[0,1] \to [0,1]$ defined by $g(x) = cx$ where $0<c<1$ and let $f: [0,1] \to [0,1]$ defined by $f(x) = \frac{1}{c}x$ for $0 \leq x \leq c$ and $f(x) = 1$ for $c \leq x \leq 1$. Then $(fog)(x) = x$ but gof is not the identity function. That is fog is a unit but f is not a unit.

Moreover let $f: I^N \to I^N$ defined by

$$f((x_1, x_2, \ldots\ldots, x_N)) = \left(\frac{1}{c}x_1, \ldots\ldots, \frac{1}{c}x_N\right) \text{ for}$$

$0 \leq x_k \leq c$ and $f((x_1, x_2, \ldots\ldots, x_N)) = 1$ otherwise and let $g: I^N \to I^N$ defined by
$g((x_1, x_2, \ldots\ldots x_N)) = (cx_1, \ldots\ldots, cx_N)$ where $0<c<1$. Then $(fog)(x) = x$.

That is, fog is a unit but f is not a unit.
In view of the Theorem 2.1 and Theorem 2.3 , we can give the following easily.

Theorem 2.5. Let I be the unit interval and let $f,g,h \in D(I)$. Then the following statements are satisfied.

1- fog is a unit in D(I) if and only if f and g are units in D(I).
2- h is a regular element of D(I) if and only if h is a constant function or h is a unit in D(I).

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