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On Statistical Convergence and Lacunary Statistical Convergence of Weight g on Time Scales

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Abstract

The aim of this paper is to present new notions, namely, statistical convergence and lacunary statistical convergence and strong lacunary summability of weight on time scales. Furthermore, we investigate the relationships of these concepts and give some results.

Keywords: *Lacunary statistical convergence; Statistical convergence; Strong summability; Time scales; Weight of g.*
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1. Introduction

The idea of statistical convergence was formally introduced by Fast [8] and Steinhaus [24], independently. This concept is a generalization of the classical convergence and it depends on the density of subsets of the natural numbers $\mathbb{N}$. Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n n^{-1} |K_n|$ if the limit exists, where $|K_n|$ indicates the cardinality of K_n. A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero, i.e., for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

and written as $st - \lim x = L$.
Over the years, several generalizations and applications of statistical convergence have been discussed by various researchers [5, 7, 9, 10, 11, 14, 15, 18, 23]. Recently, Balcerzak et al. [3] show that one can further extend the notion of natural density (as well as natural density of order α) by considering natural density of weight g^* where $g^* : \mathbb{N} \to [0, \infty)$ is a function with $\lim_{n\to\infty} g^*(n) = \infty$ and $\frac{n}{g^*(n)}$ does not go to 0 as $n \to \infty$. Since then, some work has been carried out with related to this notion [16, 17].
On the other hand, the idea of statistical convergence was first studied on time scales [19] and [25], independently. Later, by inspiring from these works, various researchers have done many studies on the summability theory using the time scales calculus, see [1, 2, 20, 21, 22, 26, 27, 28]. A time scale is an arbitrary closed of the real numbers $\mathbb{R}$ in the usual topology which is denoted by $\mathbb{T}$. The theory of time scales has been constructed by Hilger [13], in order to unify continuous and discrete analysis. Afterwards, this theory has received much attention and its applications have been studied in many fields of science. More about on time scale can be seen from [4, 6, 12]. We now give a brief introduction to time scale theory:
The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

for $t \in \mathbb{T}$, and also the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. A closed interval on a time scale $\mathbb{T}$ is given by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. Open intervals or half-open intervals are defined accordingly.
In this paper, we use the Lebesgue Δ-measure by μ_Δ introduced by Guseinov [12]. In this case, it is known that if $a \in \mathbb{T} \backslash \{\max \mathbb{T}\}$, then the single point set $\{a\}$ is Δ-measurable and $\mu_\Delta(\{a\}) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leq b$, then $\mu_\Delta([a,b)_{\mathbb{T}}) = b - a$ and $\mu_\Delta((a,b)_{\mathbb{T}}) = b - \sigma(a)$; if $a, b \in \mathbb{T} \backslash \{\max \mathbb{T}\}$ and $a \leq b$, then $\mu_\Delta((a,b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$ and $\mu_\Delta([a,b]_{\mathbb{T}}) = \sigma(b) - a$, see [12].
We now recall some of the concepts defined using the time scale calculus on the summability theory:
Throughout this paper, we consider that $\mathbb{T}$ is a time scale satisfying $\inf \mathbb{T} = t_0 > 0$ and $\sup \mathbb{T} = \infty$.

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Definition 1.1. *([25]) A Δ-measurable function $f:\mathbb{T}\to\mathbb{R}$ is statistically convergent to a number L on $\mathbb{T}$, if for every $\varepsilon>0$,*

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(\{s\in[t_0,t]_\mathbb{T}:|f(s)-L|\geq\varepsilon\}\right)}{\mu_\Delta\left([t_0,t]_\mathbb{T}\right)}=0,$$

which is denoted by $st_\mathbb{T}-\lim_{t\to\infty}f(t)=L$.

Let $\theta=(k_r)$ is an increasing sequence of non-negative integers with $k_0=0$ and $\sigma(k_r)-\sigma(k_{r-1})\to\infty$ as $r\to\infty$. Then θ is called a lacunary sequence with respect to $\mathbb{T}$ [26].

Definition 1.2. *[26] Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$. A Δ-measurable function $f:\mathbb{T}\to\mathbb{R}$ is said to be lacunary statistically convergent to a number L on $\mathbb{T}$ if for every $\varepsilon>0$,*

$$\lim_{r\to\infty}\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_\mathbb{T}:|f(s)-L|\geq\varepsilon\}\right)}{\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)}=0,$$

which is denoted by $st_{\theta-\mathbb{T}}-\lim_{t\to\infty}f(t)=L$. The set of all lacunary statistically convergent functions on $\mathbb{T}$ is denoted by $S_{\theta-\mathbb{T}}$.

Definition 1.3. *[26] Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$ and $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta-$measurable function. Then f is said to be strongly lacunary summable to L on $\mathbb{T}$, if there exists some $L\in\mathbb{R}$ such that*

$$\lim_{r\to\infty}\frac{1}{\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)}\int\limits_{(k_{r-1},k_r]_\mathbb{T}}|f(s)-L|\Delta s=0.$$

In this study, by combining the ideas of [3] and [25], we introduce the concepts of statistical convergence and lacunary statistical convergence and strong lacunary summability of weight on time scales. We then give some results related to these new notions.
Throughout this paper, we consider that functions $g:[0,\infty)\to[0,\infty)$ such that $\lim_{x\to\infty}g(x)=\infty$ and $\frac{x}{g(x)}$ does not go to 0 as $x\to\infty$. The class of all such functions g is denoted by G.

2. Main Results

We begin this part by defining the new concepts of this study. We then present the main results related to these notion.

Definition 2.1. *Let $g\in G$ and $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta-$measurable function. Then f is said to be statistically convergent of weight g (or $\left[S_\mathbb{T}^g\right]-$statistically convergent) to a number L on $\mathbb{T}$ if for every $\varepsilon>0$,*

$$\lim_{t\to\infty}\frac{1}{g\left(\mu_\Delta\left([t_0,t]_\mathbb{T}\right)\right)}\mu_\Delta\left(\{s\in[t_0,t]_\mathbb{T}:|f(s)-L|\geq\varepsilon\}\right)=0.$$

In this case, we write $\left[st_\mathbb{T}^g\right]-\lim f(t)=L$. The set of all statistically convergent functions of weight g on $\mathbb{T}$ is denoted by $\left[S_\mathbb{T}^g\right]$.

Remark 2.2. *(i) If we choose $g(x)=x$ for all $x\in[0,\infty)$, Definition 2.1 gives us the concept of statistical convergence on time scales introduced in [25].*
(ii) If we take $\mathbb{T}=[a,\infty)$ $(a>1)$ in Definition 2.1, then the concept of statistical convergence of weight g on time scales reduces to $[S^g]-$statistical convergence introduced in [17].

Definition 2.3. *Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$ and $g\in G$. A $\Delta-$measurable function $f:\mathbb{T}\to\mathbb{R}$ is said to be lacunary statistically convergent of weight g (or $\left[S_{\theta-\mathbb{T}}^g\right]-$statistically convergent) to a number L on $\mathbb{T}$ if for every $\varepsilon>0$,*

$$\lim_{r\to\infty}\frac{1}{g\left(\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)\right)}\mu_\Delta\left(\{s\in(k_{r-1},k_r]_\mathbb{T}:|f(s)-L|\geq\varepsilon\}\right)=0,$$

which is denoted by $\left[st_{\theta-\mathbb{T}}^g\right]-\lim f(t)=L$. The set of all lacunary statistically convergent functions of weight g on $\mathbb{T}$ is denoted by $\left[S_{\theta-\mathbb{T}}^g\right]$.

Theorem 2.4. *Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$ and $g_1,g_2\in G$ be such that there exists $M>0$ and $r_0\in\mathbb{N}$ such that $\frac{g_1(\mu_\Delta((k_{r-1},k_r]_\mathbb{T}))}{g_2(\mu_\Delta((k_{r-1},k_r]_\mathbb{T}))}\leq M$ for all $r\geq r_0$. Then $\left[S_{\theta-\mathbb{T}}^{g_1}\right]\subseteq\left[S_{\theta-\mathbb{T}}^{g_2}\right]$.*

Proof. Let $\varepsilon>0$ be given. Then, we may write

$$\begin{aligned}\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_\mathbb{T}:|f(s)-L|\geq\varepsilon\}\right)}{g_2\left(\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)\right)}&=\frac{g_1\left(\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)\right)}{g_2\left(\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)\right)}\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_\mathbb{T}:|f(s)-L|\geq\varepsilon\}\right)}{g_1\left(\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)\right)}\\&\leq M\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_\mathbb{T}:|f(s)-L|\geq\varepsilon\}\right)}{g_1\left(\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)\right)}\end{aligned}$$

for all $r\geq r_0$. If $f\in\left[S_{\theta-\mathbb{T}}^{g_1}\right]$, then the right hand side tends to 0 as $r\to\infty$ and consequently

$$\lim_{r\to\infty}\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_\mathbb{T}:|f(s)-L|\geq\varepsilon\}\right)}{g_2\left(\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)\right)}=0,$$

and so $f\in\left[S_{\theta-\mathbb{T}}^{g_2}\right]$. Hence, $\left[S_{\theta-\mathbb{T}}^{g_1}\right]\subseteq\left[S_{\theta-\mathbb{T}}^{g_2}\right]$. □

In the following two theorems, we aim to examine the relationship between the concepts of statistical convergence and lacunary statistical convergence of weight g on time scales.

Theorem 2.5. *Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$ and $g\in G$. Then $\left[S_{\mathbb{T}}^{g}\right]\subseteq\left[S_{\theta-\mathbb{T}}^{g}\right]$ if $\liminf\limits_{r\to\infty}\frac{g(\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}}))}{g(\mu_\Delta([t_0,k_r]_{\mathbb{T}}))}>1$.*

Proof. Suppose that $\liminf\limits_{r\to\infty}\frac{g(\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}}))}{g(\mu_\Delta([t_0,k_r]_{\mathbb{T}}))}>1$. Then, we can find a $H>1$ such that for sufficiently large r we have $\frac{g(\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}}))}{g(\mu_\Delta([t_0,k_r]_{\mathbb{T}}))}\geq H$. For any $\varepsilon>0$, we have

$$\begin{aligned}\frac{\mu_\Delta\left(\{s\in[t_0,k_r]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)}{g\left(\mu_\Delta\left([t_0,k_r]_{\mathbb{T}}\right)\right)}&\geq\frac{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}{g\left(\mu_\Delta\left([t_0,k_r]_{\mathbb{T}}\right)\right)}\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}\\&\geq H\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}\end{aligned}$$

for sufficiently $r\in\mathbb{N}$. If $\left[st_{\mathbb{T}}^{g}\right]-\lim f(t)=L$, then the left hand side of the last inequality tends to 0 as $r\to\infty$, which gives that

$$\lim_{r\to\infty}\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}=0,$$

and so $\left[st_{\theta-\mathbb{T}}^{g}\right]-\lim f(t)=L$. Therefore, $\left[S_{\mathbb{T}}^{g}\right]\subseteq\left[S_{\theta-\mathbb{T}}^{g}\right]$. □

Theorem 2.6. *Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$ and $g\in G$. Then $\left[S_{\theta-\mathbb{T}}^{g}\right]\subseteq\left[S_{\mathbb{T}}^{g}\right]$ if $\limsup\limits_{r\to\infty}\frac{\mu_\Delta([t_0,k_r]_{\mathbb{T}})}{g(\mu_\Delta([t_0,k_{r-1}]_{\mathbb{T}}))}<\infty$ where g is further assumed to be monotonically increasing such that $\lim\limits_{r\to\infty}\frac{g(r)}{r}=l\,(say)<\infty$.*

Proof. Suppose first that $\limsup\limits_{r\to\infty}\frac{\mu_\Delta([t_0,k_r]_{\mathbb{T}})}{g(\mu_\Delta([t_0,k_{r-1}]_{\mathbb{T}}))}<\infty$. Then, there is a $K>0$ such that $\frac{\mu_\Delta([t_0,k_r]_{\mathbb{T}})}{g(\mu_\Delta([t_0,k_{r-1}]_{\mathbb{T}}))}\leq K$ for all $r\in\mathbb{N}$. Now, let $\left[st_{\theta-\mathbb{T}}^{g}\right]-\lim f(t)=L$ and $\lim\limits_{r\to\infty}\frac{g(r)}{r}=l$. Then, for any $\varepsilon>0$,
$\lim\limits_{r\to\infty}\frac{U_r}{g(\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}}))}=0$,
where $U_r:=U_r(\varepsilon)=\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)$. Hence, there exists a $r_0=r_0(\varepsilon)\in\mathbb{N}$ such that
$\frac{g(r)}{r}<l+\varepsilon$ and $\frac{U_r}{g(\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}}))}<\varepsilon$
for all $r>r_0$. For any given $t\in\mathbb{T}$, we may find an interval $(k_{r-1},k_r]_{\mathbb{T}}$ such that $t\in(k_{r-1},k_r]_{\mathbb{T}}$. Now letting $N=\max\{U_1,U_2,\ldots,U_{r_0}\}$. Then, for sufficiently large r, we have

$$\begin{aligned}&\frac{\mu_\Delta\left(\{s\in[t_0,t]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)}{g\left(\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)\right)}\\&\leq\frac{\mu_\Delta\left(\{s\in[t_0,k_r]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)}{g\left(\mu_\Delta\left([t_0,k_{r-1}]_{\mathbb{T}}\right)\right)}\\&\leq\frac{U_1+U_2+\ldots+U_{r_0}+U_{r_0+1}+\ldots+U_r}{g\left(\mu_\Delta\left([t_0,k_{r-1}]_{\mathbb{T}}\right)\right)}\\&\leq\frac{r_0N}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}+\frac{1}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}\left\{U_{r_0+1}+\ldots+U_r\right\}\\&=\frac{r_0N}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}+\frac{1}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}\left\{\frac{U_{r_0+1}}{g\left(\sigma\left(k_{r_0+1}\right)-\sigma\left(k_{r_0}\right)\right)}\frac{g\left(\sigma\left(k_{r_0+1}\right)-\sigma\left(k_{r_0}\right)\right)}{\sigma\left(k_{r_0+1}\right)-\sigma\left(k_{r_0}\right)}\left(\sigma\left(k_{r_0+1}\right)-\sigma\left(k_{r_0}\right)\right)\right.\\&\qquad\left.+\ldots+\frac{U_r}{g\left(\sigma\left(k_r\right)-\sigma\left(k_{r-1}\right)\right)}\frac{g\left(\sigma\left(k_r\right)-\sigma\left(k_{r-1}\right)\right)}{\sigma\left(k_r\right)-\sigma\left(k_{r-1}\right)}\left(\sigma\left(k_r\right)-\sigma\left(k_{r-1}\right)\right)\right\}\\&\leq\frac{r_0N}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}+\frac{1}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}\left[\varepsilon\left(l+\varepsilon\right)\left(\sigma\left(k_{r_0+1}\right)-\sigma\left(k_{r_0}\right)\right)+\ldots+\varepsilon\left(l+\varepsilon\right)\left(\sigma\left(k_r\right)-\sigma\left(k_{r-1}\right)\right)\right]\\&=\frac{r_0N}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}+\frac{1}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}\varepsilon\left(l+\varepsilon\right)\left[\sigma\left(k_r\right)-\sigma\left(k_{r_0}\right)\right]\\&\leq\frac{r_0N}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}+\varepsilon\left(l+\varepsilon\right)\frac{\sigma\left(k_r\right)-t_0}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}\\&\leq\frac{r_0N}{g\left(\sigma\left(k_{r-1}\right)-t_0\right)}+\varepsilon\left(l+\varepsilon\right)K.\end{aligned}$$

Taking limit as $r\to\infty$ on both sides of the last inequality and also using the hypothesis, since ε is arbitrary, we obtain that

$$\lim_{t\to\infty}\frac{1}{g\left(\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)\right)}\mu_\Delta\left(\{s\in[t_0,t]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)=0,$$

which means $\left[st_{\mathbb{T}}^{g}\right]-\lim f(t)=L$. Hence, the proof is completed.

□

Definition 2.7. *Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$ and $g\in G$. A $\Delta-$measurable function $f:\mathbb{T}\to\mathbb{R}$ is said to be strongly lacunary summable of weight g to L on $\mathbb{T}$, if there exists some $L\in\mathbb{R}$ such that*

$$\lim_{r\to\infty}\frac{1}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}\int_{(k_{r-1},k_r]_{\mathbb{T}}}|f(s)-L|\,\Delta s=0.$$

In this case, we can write $\left[W^g_{\theta-\mathbb{T}}\right]-\lim f(t)=L$. The set of all strong lacunary summable functions of weight g on $\mathbb{T}$ is denoted by $\left[W^g_{\theta-\mathbb{T}}\right]$.

Theorem 2.8. *Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$ and $g\in G$. Then we have the following:*

(i) If f is strongly lacunary summable of weight g to L, then $\left[st^g_{\theta-\mathbb{T}}\right]-\lim f(t)=L$.

(ii) $\left[W^g_{\theta-\mathbb{T}}\right]$ is a proper subset of $\left[S^g_{\theta-\mathbb{T}}\right]$.

Proof.

(i) Let f be strongly lacunary summable of weight g to L. For any $\varepsilon>0$, we have

$$\begin{aligned}\frac{1}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}\int_{(k_{r-1},k_r]_{\mathbb{T}}}|f(s)-L|\,\Delta s &\geq \frac{1}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}\int_{\substack{(k_{r-1},k_r]_{\mathbb{T}}\\ |f(s)-L|\geq\varepsilon}}|f(s)-L|\,\Delta s\\ &\geq \varepsilon\,\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}.\end{aligned}$$

Letting $r\to\infty$ in here, we get that $\left[st^g_{\theta-\mathbb{T}}\right]-\lim f(t)=L$.

(ii) Let f be defined in each intervals $(k_{r-1},\sigma(k_r)]_{\mathbb{T}}$ as follows

$$f(t)=\begin{cases}1, & \text{if } t\in(k_{r-1},\sigma(k_{r-1})+1)_{\mathbb{T}},\\ 2, & \text{if } t\in[\sigma(k_{r-1})+1,\sigma(k_{r-1})+2)_{\mathbb{T}},\\ \vdots & \\ \left[\sqrt{g(u_r)}\right], & \text{if } t\in\left[\sigma(k_{r-1})+\left[\sqrt{g(u_r)}\right]-1,\sigma(k_{r-1})+\left[\sqrt{g(u_r)}\right]\right)_{\mathbb{T}},\\ 0, & \text{otherwise,}\end{cases}$$

where $u_r=\sigma(k_r)-\sigma(k_{r-1})$ and $[x]$ denotes the largest integer not exceeding x.
Then, we have

$$\begin{aligned}\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|f(s)-0|\geq\varepsilon\}\right)}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)} &= \frac{\mu_\Delta\left(\left(k_{r-1},\sigma(k_{r-1})+\left[\sqrt{g(u_r)}\right]\right)_{\mathbb{T}}\right)}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}\\ &\leq \frac{\left[\sqrt{g(u_r)}\right]}{g(u_r)}\to 0 \quad (\text{as } r\to\infty),\end{aligned}$$

which means $\left[st^g_{\theta-\mathbb{T}}\right]-\lim f(t)=0$.
On the other hand, we get

$$\begin{aligned}&\frac{1}{g\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}\int_{(k_{r-1},k_r]_{\mathbb{T}}}|f(s)-0|\,\Delta s\\ &=\frac{\mu_\Delta\left((k_{r-1},\sigma(k_{r-1})+1)_{\mathbb{T}}\right)}{g(\sigma(k_r)-\sigma(k_{r-1}))}+\frac{1}{g(\sigma(k_r)-\sigma(k_{r-1}))}\sum_{m=2}^{\left[\sqrt{g(u_r)}\right]}m\mu_\Delta\left([\sigma(k_{r-1})+m-1,\sigma(k_{r-1})+m)_{\mathbb{T}}\right)\\ &=\frac{1}{g(u_r)}\sum_{m=1}^{\left[\sqrt{g(u_r)}\right]}m\\ &=\frac{\left[\sqrt{g(u_r)}\right]\left(\left[\sqrt{g(u_r)}\right]+1\right)}{2g(u_r)}\to\frac{1}{2}\quad(\text{as } r\to\infty).\end{aligned}$$

This means that f is not strongly lacunary summable of weight g to 0 and completes the proof. □

Theorem 2.9. *Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$ and $g\in G$ with $\lim_{r\to\infty}\frac{r}{g(r)}<\infty$. If $\left[st^g_{\theta-\mathbb{T}}\right]-\lim f(t)=L$ and f is a bounded function, then f is strongly lacunary summable of weight g to L.*

Proof. Let $\left[st^{g}_{\theta-\mathbb{T}}\right]-\lim f(t)=L$ and f be a bounded function. Then, there is a positive number M such that $|f(s)-L|\leq M$ for all $s\in\mathbb{T}$. For any given $\varepsilon>0$, we may write

$$\frac{1}{g\left(\mu_{\Delta}\left((k_{r-1},k_{r}]_{\mathbb{T}}\right)\right)}\int\limits_{(k_{r-1},k_{r}]_{\mathbb{T}}}|f(s)-L|\Delta s=\frac{1}{g\left(\mu_{\Delta}\left((k_{r-1},k_{r}]_{\mathbb{T}}\right)\right)}\int\limits_{\substack{(k_{r-1},k_{r}]_{\mathbb{T}}\\ |f(s)-L|\geq\varepsilon}}|f(s)-L|\Delta s+\frac{1}{g\left(\mu_{\Delta}\left((k_{r-1},k_{r}]_{\mathbb{T}}\right)\right)}\int\limits_{\substack{(k_{r-1},k_{r}]_{\mathbb{T}}\\ |f(s)-L|<\varepsilon}}|f(s)-L|\Delta s$$

$$\leq\frac{1}{g\left(\mu_{\Delta}\left((k_{r-1},k_{r}]_{\mathbb{T}}\right)\right)}\mu_{\Delta}\left(\{s\in(k_{r-1},k_{r}]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}\right)M+\frac{1}{g\left(\mu_{\Delta}\left((k_{r-1},k_{r}]_{\mathbb{T}}\right)\right)}\mu_{\Delta}\left((k_{r-1},k_{r}]_{\mathbb{T}}\right)\varepsilon.$$

Taking limit as $r\to\infty$ on both sides of the last inequality and using the hypothesis, since ε is arbitrary, we obtain that

$$\lim_{r\to\infty}\frac{1}{g\left(\mu_{\Delta}\left((k_{r-1},k_{r}]_{\mathbb{T}}\right)\right)}\int\limits_{(k_{r-1},k_{r}]_{\mathbb{T}}}|f(s)-L|\Delta s=0.$$

This completes the proof. □

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