SOME NEW SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

Tuncay BÖYÜK , Metin BAŞARIR

Özet-**Bu çalışmada, regüler bir matris ve bir Orlicz fonksiyon dizisi yardımıyla üç yeni dizi uzayı tanımlayıp bazı özellikleri incelendi.**

Anahtar Kelimeler- **Dizi uzayları,Orlicz fonksiyonu.**

Abstract-**In this paper, we introduce and examine some properties of three sequence spaces defined by using a regular matrix and a sequence of Orlicz functions.**

Key words-**Sequence spaces, Orlicz function.**

I.INTRODUCTION

Let l_∞ and c denote the Banach spaces of real bounded and convergent sequences $x=(x_n)$ normed by $\|x\|=\sup_n|x_n|$ respectively.

An Orlicz function is a function M: $[0,\infty)\to[0,\infty)$, which is continuous, non-decreasing and convex with M(0)=0, M(x)>0 for x>0 and $M(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function M is replaced by $M(x+y)\leq M(x)+M(y)$ then this function is called modulus function defined and discussed by Ruckle[1] and Maddox[2].

Lindenstrauss and Tzafriri[3] used the idea of Orlicz function to construct sequence space

$$l_M=\left\{x\in\omega:\sum_{k=1}^{\infty}\left(M\left(\frac{|x_k|}{\rho}\right)\right)<\infty\ ,\ for\ some\ \rho>0\right\}$$

The space l_M with norm

$$\|x\|=\inf\left\{\rho>0:\sum_{k=1}^{\infty}\left(M\left(\frac{|x_k|}{\rho}\right)\right)\leq 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space, where ω be the family of real or complex sequences.

An Orlicz function M can always be represented (see Krasnoselskii and Rutitsky[4],p.5) in the integral form

$$M(x)=\int_0^x q(t)dt$$ where q, known as the kernel of M, is right-differentiable for $t\geq 0$, q(0)=0, q(t)>0 for t>0, q is nondecreasing , and $q(t)\to\infty$ as $t\to\infty$.The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x)=x^p$; $1\leq p<\infty$.

Recently, Parashar and Choudhary[5] introduced and examined some properties of following four sequence spaces defined by Orlicz function M:

Let $p=(p_k)$ be any sequence of positive real numbers.

$l_M(p)=$

$$\left\{x\in\omega:\sum_{k=1}^{\infty}\left(M\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}<\infty\ ,\ for\ some\ \rho>0\right\}$$

$$W(M,p)=\left\{\begin{array}{l}x\in\omega:\ \frac{1}{n}\sum_{k=1}^{n}\left(M\left(\frac{|x_k-\ell|}{\rho}\right)\right)^{p_k}\to 0\ \text{as}\ n\to\infty\ ,\\ for\ some\ \rho>0\ \text{and}\ \ell>0\end{array}\right\}$$

$W_0(M,p)=$

$$\left\{\begin{array}{l}x\in\omega:\ \frac{1}{n}\sum_{k=1}^{n}\left(M\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}\to 0\ \text{as}\ n\to\infty\ ,\\ for\ some\ \rho>0\end{array}\right\}$$

$$W_\infty(M,p)=\left\{\begin{array}{l}x\in\omega:\ \sup_n\sum_{k=1}^{n}\left(M\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}<\infty\\ for\ some\ \rho>0\end{array}\right\}$$

When $p_k=1$, for all k, then $l_M(p)$ becomes l_M. If M(x)=x then the family of sequences defined above become $l(p)$, $[c,1,p]$, $[c,1,p]_0$ and $[c,1,p]_\infty$ respectively. We denote W(M,p), $W_0(M,p)$ and $W_\infty(M,p)$ as W(M), $W_0(M)$ and $W_\infty(M)$ when $p_k=1$, for each k.

Let $M=(M_k)$ be a sequence of Orlicz functions and suppose that $A=(a_{nk})$ be a regular matrix. We define

$W_0(A,M,p)=$

$$\left\{\begin{array}{l}x\in\omega:\ \sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}\to 0\ \text{as}\ n\to\infty\\ \text{for some}\ \rho>0\end{array}\right\}$$

$W(A,M,p)=$

T. Böyük; Kuzuluk M.Soykan Elementary School, Akyazı-Sakarya
M. Başarır; Department of Mathematics, Sakarya University

$$\left\{ x\in\omega:\ \sum_k a_{nk}\left(M_k\left(\frac{|x_k-\ell|}{\rho}\right)\right)^{p_k}\to 0 \text{ as } n\to\infty \text{ for some } \rho>0 \text{ and } \ell>0 \right\}$$

$W_\infty(A,M,p)=$

$$\left\{ x\in\omega:\ \sup_n\sum_k a_{nk}\left(M_k\left(\frac{|x_k-\ell|}{\rho}\right)\right)^{p_k}<\infty \text{ for some } \rho>0 \right\}$$

When $M_k(x)=x$ for all k, then the family of sequences defined above becomes $[A,p]_0$, $[A,p]$ and $[A,p]_\infty$, respectively.

When $M_k(x)=x$ for all k and $A=(C,1)$ Cesaro matrix, we have the sequence spaces $W_0(M,p)$, $W(M,p)$ and $W_\infty(M,p)$ that are defined by S.D. Parashar and Choudhary.

II.MAIN RESULTS

Theorem1: Let $p=(p_k)$ be bounded. Then $W_0(A,M,p)$, $W(A,M,p)$ and $W_\infty(A,M,p)$ are linear spaces over the set of complex numbers C.

Proof: We shall only prove for $W_0(A,M,p)$. The others can be treated similarly. Let $x,y\in W_0(A,M,p)$ and $\alpha,\beta\in C$. In order to prove the result we need to find some ρ_3 such that

$$\sum_k a_{nk}\left(M_k\left(\frac{|\alpha x_k+\beta y_k|}{\rho_3}\right)\right)^{p_k}\to 0,\ \text{as } n\to\infty.$$

Since $x,y\in W_0(A,M,p)$, therefore there exist some ρ_1 and ρ_2 such that

$$\sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho_1}\right)\right)^{p_k}\to 0,\ \text{as } n\to\infty$$

and

$$\sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho_2}\right)\right)^{p_k}\to 0,\ \text{as } n\to\infty.$$

Define $\rho_3=\max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non decreasing and convex,

$$\sum_k a_{nk}\left(M_k\left(\frac{|\alpha x_k+\beta y_k|}{\rho_3}\right)\right)^{p_k}\le$$

$$\sum_k a_{nk}\left(M_k\left(\frac{|\alpha x_k|}{\rho_3}+\frac{|\beta y_k|}{\rho_3}\right)\right)^{p_k}$$

$$\le\sum_k \frac{1}{2^{p_k}}a_{nk}\left(M_k\left(\frac{|x_k|}{\rho_1}\right)+M_k\left(\frac{|y_k|}{\rho_2}\right)\right)^{p_k}\le$$

$$C\sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho_1}\right)\right)^{p_k}+C\sum_k a_{nk}\left(M_k\left(\frac{|y_k|}{\rho_2}\right)\right)^{p_k}\to 0 \text{ as } n\to\infty,$$

where $C=\max(1,2^{H-1})$. This proves that $W_0(A,M,p)$ is linear.

Theorem2: Let $H=\max(1,\sup p_k)$. Then $W_0(A,M,p)$ is a linear topological space paranormed by

$$G(x)=\inf\left\{\rho^{p_n/H}:\left(\sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}\right)^{\frac{1}{H}}\le 1,\ n=1,2,3,\ldots\right\}$$

Proof: Clearly $G(x)=G(-x)$. By using *Theorem1* for $\alpha=\beta=1$, we get $G(x+y)\le G(x)+G(y)$. Since $M(0)=0$, we get $\inf\{\rho^{p_n/H}\}=0$ for $x=0$. Conversely, suppose $G(x)=0$, then

$$\inf\left\{\rho^{p_n/H}:\left(\sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}\right)^{\frac{1}{H}}\le 1\right\}=0.$$

This implies that for a given $\varepsilon>0$, there exists some $\rho_\varepsilon(0<\rho_\varepsilon<\varepsilon)$ such that

$$\left(\sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho_\varepsilon}\right)\right)^{p_k}\right)^{\frac{1}{H}}\le 1.$$

Thus

$$\left(\sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\varepsilon}\right)\right)^{p_k}\right)^{\frac{1}{H}}\le$$

$$\left(\sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho_\varepsilon}\right)\right)^{p_k}\right)^{\frac{1}{H}}\le 1.$$

Suppose $x_{n_m}\neq 0$ for some m. Let $\varepsilon\to 0$

Then $\left(\frac{|x_{n_m}|}{\rho}\right)\to\infty$ it follows that

$$\left(\sum_m a_{nk}\left(M_k\left(\frac{|x_{n_m}|}{\rho}\right)\right)^{p_m}\right)^{\frac{1}{H}} \to\infty \text{ which is a}$$

contradiction. Therefore $x_{n_m}=0$ for each m.

Finally, we prove scalar multiplication is continuous. Let λ be any number. By definition,

$$G(\lambda x)=\inf\left\{\rho^{p_n/H}:\left(\sum_k a_{nk}\left(M_k\left(\frac{|\lambda x_k|}{\rho}\right)\right)^{p_k}\right)^{\frac{1}{H}}\le 1,\ n=1,2,3,\ldots\right\}$$

Then

$G(\lambda x)=\inf$

$$\left\{(\lambda r)^{p_n/H}:\left(\sum_k a_{nk}\left(M_k\left(\frac{|\lambda x_k|}{\rho}\right)\right)^{p_k}\right)^{\frac{1}{H}}\le 1,\ n=1,2,3,\ldots\right\}$$

where $r=\rho/\lambda$.

Since $|\lambda|^{p_k}\le\max(1,|\lambda|^H)$ therefore

$|\lambda|^{\frac{p_k}{H}}\le\max(1,|\lambda|^H)^{1/H}$.

Hence

$G(\lambda x)\le\max(1,|\lambda|^H)^{1/H}$

$$\inf\left\{(r)^{p_n/H}:\left(\sum_k a_{nk}\left(M_k\left(\frac{|\lambda x_k|}{\rho}\right)\right)^{p_k}\right)^{\frac{1}{H}}\le 1,\ n=1,2,3,\ldots\right\}$$

Which converges to zero as G(x) converges to zero in $W_0(A,M,p)$ where

$W_0(A,M,p)=$

$$\left\{x\in\omega:\sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}\to 0,\text{ as } n\to\infty \text{ for some } \rho>0\right\}$$

For arbitrary $\varepsilon>0$, let N be a pozitive integer such that $\sum_{k=N+1}^{\infty} a_{nk}\left(M_k\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}<\frac{\varepsilon}{2}$ for some ρ>0. This implies that

$$\left(\sum_{k=N+1}^{\infty} a_{nk}\left(M_k\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}\right)^{1/H}<\frac{\varepsilon}{2}$$

Let $0<|\lambda|<1$, using convexity of M we get

$$\sum_{k=N+1}^{\infty} a_{nk}\left(|\lambda|M_k\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}<\sum_{k=N+1}^{\infty} a_{nk}\left(M_k\left(\frac{|x_k|}{\rho}\right)\right)^{p_k}<\left(\frac{\varepsilon}{2}\right)^H.$$

Since M is continuous every where in $[0,\infty)$, then

$f(t)=\sum_{k=1}^{N} a_{nk}\left(M_k\left(\frac{|t x_k|}{\rho}\right)\right)$ is continuous at 0. So there is 1>δ>0 such that $|f(t)|<\frac{\varepsilon}{2}$ for 0<t<δ. Let K be such that $|\lambda_n|<\delta$ for n>K, then for n>K,

$$\left(\sum_{k=1}^{N} a_{nk}\left(M_k\left(\frac{|\lambda_n x_k|}{\rho}\right)\right)^{p_k}\right)^{1/H}<\frac{\varepsilon}{2}$$

Thus

$$\left(\sum_{k=1}^{\infty} a_{nk}\left(M_k\left(\frac{|\lambda_n x_k|}{\rho}\right)\right)^{p_k}\right)^{1/H}<\varepsilon \quad \text{for } n>K.$$

Remark: It can be easily verified that when $M_k(x)=x$, then the paranorm defined in $W_0(A,M,p)$ and paranorm defined in $[A,p]_0$ are same.

Definition(Krasnoselskii and Rutitsky[4],25): An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists, constant K>0, such that $M(2u)\le KM(u)$ $(u\ge 0)$.

The Δ_2-condition is equivalent to the satisfaction of inequality $M(lu)\le K.lM(u)$ for all values of u and for $l\ge 1$.

Theorem3: Let A be a nonnegative regular matrix, and $M=(M_k)$ be a sequence of Orlicz functions which satisfies Δ_2-condition for all k. Then

i) $[A,p]_0\subset W_0(A,M,p)$
ii) $[A,p]\subset W(A,M,p)$
iii) $[A,p]_\infty\subset W_\infty(A,M,p)$

Where

$$[A,p]_0=\left\{x\in\omega:\sum_k a_{nk}|x_k|^{p_k}\to 0,\ as\ n\to\infty\right\}$$

$$[A,p]=\left\{x\in\omega:\sum_k a_{nk}|x_k-l|^{p_k}\to 0,\ as\ n\to\infty\right\},$$

$$[A,p]_\infty=\left\{x\in\omega:\sup_n\sum_k a_{nk}|x_k|^{p_k}<\infty\right\}$$

Proof: (ii) Let $x\in[A,p]$, then

$S_n=\sum_k a_{nk}|x_k-l|^{p_k}\to 0$, as $n\to\infty$.

Let $\varepsilon>0$ and choose δ with 0<δ<1 such that $M_k(t)<\varepsilon$ for $0\le t\le\delta$ and for all k.

Write $y_k=|x_k-l|$ and consider

$$\sum_k a_{nk}\left(M_k\left(|y_k|\right)\right)^{p_k}=\sum_1+\sum_2$$

where the first summation is over $y_k\leq\delta$ and the second summation over $y_k>\delta$. Since M_k is continuous for all k

$\sum_1 <\varepsilon^{H}\sum_k a_{nk}$ and for $y_k>\delta$ we use the fact that

$$y_k<\frac{y_k}{\delta}<1+\left(\frac{y_k}{\delta}\right).$$

Since M_k is non decreasing and convex for all k, it follows that

$$M_k(y_k)<M_k\left[1+\left(\frac{y_k}{\delta}\right)\right]$$
$$<\frac{1}{2}M_k(2)+\frac{1}{2}M_k\left[2\left(\frac{y_k}{\delta}\right)\right]$$

Since M_k satisfies Δ_2-condition for all k, therefore

$$M_k(y_k)<\frac{1}{2}k\frac{y_k}{\delta}M_k(2)+\frac{1}{2}k\frac{y_k}{\delta}M_k(2)=k\frac{y_k}{\delta}M_k(2).$$

Hence

$\sum_2 M_k(y_k)\leq k\delta^{-1}M_k(2).n.S_n$, which together with

$\sum_1 <\varepsilon^{H}\sum_k a_{nk}$ yields $[A,p]\subset W(A,M,p)$.

Following similar arguments we can prove that $[A,p]_0\subset W_0(A,M,p)$ and $[A,p]_\infty\subset W_\infty(A,M,p)$.

Theorem 4: ***i)*** Let $0<\inf p_k\leq p_k\leq 1$, then $W(A,M,p)\subset W(A,M)$

ii) Let $1\leq p_k\leq\sup p_k<\infty$, then $W(A,M)\subset W(A,M,p)$.

iii) Let $0<p_k<q_k$ and (q_k/p_k) be bounded. Then $W(A,M,q)\subset W(A,M,p)$.

Proof: (i) Let $x\in W(A,M,p)$.

Since $0<\inf p_k\leq p_k\leq 1$., we get

$$\sum_k a_{nk}\left(M_k\left(\frac{|x_k-l|}{\rho}\right)\right)\leq\sum_k a_{nk}\left(M_k\left(\frac{|x_k-l|}{\rho}\right)\right)^{p_k}$$

and hence $x\in W(A,M)$.

(ii): Let $1\leq p_k\leq\sup p_k<\infty$ for each k. Let $x\in W(A,M)$. Then for each $1>\varepsilon>0$ there exists a pozitive integer N such that

$$\sum_k a_{nk}\left(M_k\left(\frac{|x_k-l|}{\rho}\right)\right)\leq\varepsilon\leq 1$$

for all $n\geq N$. This implies that

$$\sum_k a_{nk}\left(M_k\left(\frac{|x_k-l|}{\rho}\right)\right)^{p_k}\leq\sum_k a_{nk}\left(M_k\left(\frac{|x_k-l|}{\rho}\right)\right)$$

Therefore $x\in W(A,M,p)$.

(iii): Let $x\in W(A,M,q)$.

Write $t_k=a_{nk}\left(M_k\left(\frac{|x_k-l|}{\rho}\right)\right)^{q_k}$ and $\lambda_k=\frac{p_k}{q_k}$. Since $p_k\leq q_k$ therefore $0<\lambda_k\leq 1$.

Take $0<\lambda<\lambda_k$. Define $u_k=t_k\ (t_k\geq 1)$, $=0\ (t_k<1)$ and $v_k=0\ (t_k\geq 1)$, $=t_k\ (t_k<1)$. So $t_k=u_k+v_k$ and $t_k^{\lambda_k}=u_k^{\lambda_k}+v_k^{\lambda_k}$.

Now it follows that $u_k^{\lambda_k}\leq u_k\leq t_k$ and $v_k^{\lambda_k}\leq v_k^{\lambda}$.

Therefore

$$\sum_k t_k^{\lambda_k}\leq\sum_k t_k+\left(\sum_k v_k\right)^{\lambda}$$

and hence $x\in W(A,M,p)$.

Corollary: Let $A=(C,1)$ Cesaro matrix and $M=(M_k)$ a sequence of Orlicz functions. Then

i) If $M=(M_k)$ satisfies Δ_2-condition for all k, then $W_1\subset W(M,p), W_0\subset W_0(M,p), W_\infty\subset W_\infty(M,p)$

where

$$W_1=\left\{x\in\omega:\frac{1}{n}\sum_{k=1}^{n}|x_k|^{p_k}\to 0\text{ , as }n\to\infty\right\}$$

$$W_0=\left\{x\in\omega:\frac{1}{n}\sum_{k=1}^{n}|x_k-l|^{p_k}\to 0\text{ , as }n\to\infty\right\}$$

$$W_\infty=\left\{x\in\omega:\ \sup_n\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^{p_k}\right)<\infty\right\}$$

ii) Let $0<\inf p_k\leq p_k\leq 1$, then $W(M,p)\subset W(M)$

iii) Let $1\leq p_k\leq\sup p_k<\infty$, then $W(M)\subset W(M,p)$.

iv) Let $0<p_k<q_k$ and (q_k/p_k) be bounded, then $W(M,q)\subset W(M,p)$.

Proof: It is trivial.

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