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Singular Minimal Surfaces which are Minimal

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Abstract

In the present paper, we discuss the singular minimal surfaces in Euclidean $3-$space $\mathbb{R}^3$ which are minimal. Such a surface is nothing but a plane, a trivial outcome. However, a non-trivial outcome is obtained when we modify the usual condition of singular minimality by using a special semi-symmetric metric connection instead of the Levi-Civita connection on $\mathbb{R}^3$. With this new connection, we prove that, besides planes, the singular minimal surfaces which are minimal are the generalized cylinders, providing their explicit equations. A trivial outcome is observed when we use a special semi-symmetric non-metric connection. Furthermore, our discussion is adapted to the Lorentz-Minkowski 3-space.

1. Introduction

Let $\left(\mathbb{R}^3,\langle\cdot,\cdot\rangle\right)$ be a Euclidean $3-$space and $\mathbf{v}$ a fixed unit vector in $\mathbb{R}^3$. Let $\mathbf{r}: M^2 \to \mathbb{R}^3_+(\mathbf{v})$ be a smooth immersion of an oriented compact surface M^2 into the halfspace

$$\mathbb{R}^3_+(\mathbf{v}) : \left\{p \in \mathbb{R}^3 : \langle p,\mathbf{v}\rangle > 0\right\}.$$

Denote H and $\mathbf{n}$ the mean curvature and unit normal vector field on M^2. Let $\alpha \in \mathbb{R}$. The potential $\alpha-$energy of $\mathbf{r}$ in the direction of $\mathbf{v}$ is defined by [32]

$$E(\mathbf{r}) = \int_{M^2} \langle p,\mathbf{v}\rangle^{\alpha}\, dM^2,$$

where dM^2 is the measure on M^2 with respect to the induced metric tensor from the Euclidean metric $\langle\cdot,\cdot\rangle$ and $p=\mathbf{r}(\tilde{p})$, $\tilde{p}\in M^2$.
Let $\Sigma : M^2 \times (-\theta,\theta) \to \mathbb{R}^3_+(\mathbf{v})$ be a compactly supported variation of $\mathbf{r}$ with variaton vector field ξ. The first variation of E becomes

$$E'(0) = -\int_{M^2} \left(2H\langle \mathbf{r},\mathbf{v}\rangle - \alpha\langle \mathbf{n},\mathbf{v}\rangle\right)\langle \xi,\mathbf{n}\rangle^{\alpha-1}\, dM^2.$$

For all compactly supported variations, the immersion $\mathbf{r}$ is a critical point of E if and only if

$$2H(\tilde{p}) = \alpha\frac{\langle \mathbf{n}(\tilde{p}),\mathbf{v}\rangle}{\langle \mathbf{r}(\tilde{p}),\mathbf{v}\rangle}, \tag{1.1}$$

for some point $\tilde{p}\in M^2$.
A surface M^2 is referred to as a *singular minimal surface* or $\alpha-$*minimal surface* with respect to the vector $\mathbf{v}$, if holds Eq. (1.1) (see [11, 12]). In the particular case $\alpha=1$ and $\mathbf{v}=(0,0,1)$, the surface M^2 represents *two-dimensional analogue of the catenary* which is known as a model for the surfaces with the lowest gravity center, in other words, one has minimal potential energy under gravitational forces [6, 13, 18].

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A *translation surface* M^2 in $\mathbb{R}^3$ is a surface that can be written as the sum of two so-called *translating curves* [9]. When the translating curves lie in orthogonal planes, up to a change of coordinates, the surface M^2 can be locally given in the explicit form $z = p(x) + q(y)$, where (x,y,z) is the rectangular coordinates and p,q smooth functions. In such case, if M^2 is minimal (H vanishes identically [27, p. 17]), it describes a plane or the *Scherk surface* [43]

$$z(x,y) = \frac{1}{\lambda}\log\left|\frac{\cos\lambda x}{\cos\lambda y}\right|,\ \lambda \in \mathbb{R},\ \lambda \neq 0.$$

If the translating curves lie in non-orthogonal planes, the translation surface M^2 is locally given by $z = p(x) + q(y+\mu x)$, $\mu \in \mathbb{R}$, $\mu \neq 0$, and so-called an *affine translation surface* or a *translation graph* [26,45]. A minimal affine translation surface is so-called *affine Scherk surface* and is given in the explicit form

$$z(x,y) = \frac{1}{\lambda}\log\left|\frac{\cos\lambda\sqrt{1+\mu^2}x}{\cos\lambda(y+\mu x)}\right|.$$

López [32] obtained the singular minimal translation surfaces in $\mathbb{R}^3$ of type $z = p(x) + q(y)$ with respect to horizontal and vertical directions. This result was generalized to higher dimensions in [5]. For further study of singular minimal surfaces, we refer to the López's series of interesting papers on the solutions of the Dirichlet problem for the $\alpha-$singular minimal surface equation [33], the Lorentz-Minkowski counterpart of the condition of singular minimality [34], the compact singular minimal surfaces [35] and the singular minimal surfaces with density [36].

In this paper, we approach a singular minimal surface M^2 in $\mathbb{R}^3$ which is minimal. We hereinafter assume that $\alpha \neq 0$ in Eq. (1.1), otherwise any minimal surface obeys our approach, which is trivial. Under this circumstance, Eq. (1.1) gives $\langle \mathbf{n}(\tilde{p}), \mathbf{v}\rangle = 0$, that is, the tangent plane of M^2 at any point $\tilde{p}$ is parallel to $\mathbf{v}$. In such case, the surface M^2 belongs to the class of so-called *constant angle surfaces* and has to be a plane parallel to $\mathbf{v}$ (see [37, Proposition 9]), yielding the following outcome.

Proposition 1.1. *Let M^2 be a singular minimal surface in $\mathbb{R}^3$ with respect to an arbitrary vector $\mathbf{v}$. If M^2 is minimal, then it is a plane parallel to $\mathbf{v}$.*

This result is changed when we modify Eq. (1.1) by using a special semi-symmetric metric connection ∇ (see Eq. (3.1)) on $\mathbb{R}^3$. In Section 3, we prove that, besides planes, the singular minimal surfaces which are minimal with respect to ∇ are the generalized cylinders, providing their explicit equations. It is also observed, in Section 3, that this approach produces only trivial example when a special semi-symmetric non-metric connection D (see Eq. (3.19)) is used.

We find the motivation in Wang's paper [44] whose minimal translation surfaces were obtained with respect to the connections ∇ and D. The notion of a semi-symmetric metric (resp. non-metric) connection on a Riemannian manifold were defined by Hayden [22] (resp. Agashe [1]) and since then has been studied by many authors. Without giving a complete list, we may refer to [2–4,7,10,14,15,19,25,38–42,47–50]. The present authors also obtained singular minimal translation surfaces in $\mathbb{R}^3$ with respect to the connections ∇ and D [16].

Let $\mathbb{R}^3_1$ be a Lorentz-Minkowski 3−space endowed with the canonical Lorentzian metric $\langle\cdot,\cdot\rangle_L = dx^2 + dy^2 - dz^2$. Then we have [34, Definition 1.1]

Definition 1.1. *Let $\mathbf{r}$ be a smooth immersion of a spacelike surface M^2 in the halfspace $z > 0$ of $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H the mean curvature. M^2 is called $\alpha-$singular maximal surface if satisfies*

$$H = -\alpha\frac{\langle \mathbf{n}, (0,0,1)\rangle_L}{z},\ \alpha \neq 0. \tag{1.2}$$

Due to the fact that the $z-$coordinate represents the time coordinate, the concept of gravity has no meaning. Therefore, unlike the Riemannian case, Eq. (1.2) describes only spacelike surfaces with prescribed angle between $\mathbf{n}$ and the $z-$axis. Point out that H is non-vanishing in Eq. (1.2) if $\alpha \neq 0$ because $\langle \mathbf{n}, (0,0,1)\rangle_L \neq 0$ for timelike vectors $\mathbf{n}$ and $(0,0,1)$ and so we can not adapt Eq. (1.2) to our study as is. For this reason, we modify the concept as follows:

Definition 1.2. *Let $\mathbf{r}$ be a smooth immersion of an oriented timelike surface M^2 in $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H the mean curvature. Let $\mathbf{v} \in \mathbb{R}^3_1$, $\mathbf{v} \neq \mathbf{0}$, a spacelike vector non-parallel to $\mathbf{n}$ such that $\mathbf{n}$ and $\mathbf{v}$ span a spacelike 2-space. Then M^2 is called singular minimal surface with respect to $\mathbf{v}$ if satisfies*

$$2H = \alpha\frac{\langle \mathbf{n}, \mathbf{v}\rangle_L}{\langle \mathbf{r}, \mathbf{v}\rangle_L},\ \alpha \in \mathbb{R},\ \alpha \neq 0. \tag{1.3}$$

With Definition 1.2, we may view the singular minimal surface M^2 as a timelike surface in $\mathbb{R}^3_1$ with prescribed Lorentz spacelike angle between $\mathbf{n}$ and $\mathbf{v}$. If M^2 is minimal, it follows from Eq. (1.3) that $\langle \mathbf{n}, \mathbf{v}\rangle_L = 0$, namely the angle is $\frac{\pi}{2}$, and, as in Riemannian case, M^2 becomes a timelike constant angle surface which has to be a plane (see [21, Theorem 3.1]), yielding the following trivial outcome.

Proposition 1.2. *Let M^2 be a singular minimal surface in $\mathbb{R}^3_1$ with respect to a spacelike vector $\mathbf{v}$. If M^2 is minimal, then it is a plane parallel to $\mathbf{v}$.*

In Section 4, we also state non-trivial results in $\mathbb{R}^3_1$ for singular minimal surfaces which are minimal with respect to the connections ∇ and D given by Eqs. (4.1) and (4.19), respectively.

2. Preliminaries

Most of following notions can be found [8,40,46].
Let $(\bar{M},\bar{g})$ be a semi-Riemannian manifold and $\bar{\nabla}$ a linear connection on $\bar{M}$. The *torsion tensor field* T of $\bar{\nabla}$ is defined by

$$T(\bar{\mathbf{x}},\bar{\mathbf{y}}) = \bar{\nabla}_{\bar{\mathbf{x}}}\bar{\mathbf{y}} - \bar{\nabla}_{\bar{\mathbf{x}}}\bar{\mathbf{y}} - [\bar{\mathbf{x}},\bar{\mathbf{y}}],$$

where $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are vector fields on $\bar{M}$. A linear connection is called a *semi-symmetric* (resp. *non-*) *metric connection* if there exist a $1-$form π such that

$$T(\bar{\mathbf{x}},\bar{\mathbf{y}}) = \pi(\bar{\mathbf{y}})\bar{\mathbf{x}} - \pi(\bar{\mathbf{x}})\bar{\mathbf{y}},\ \bar{\nabla}\bar{g} = 0 \text{ (resp. } \bar{\nabla}\bar{g} \neq 0).$$

The linear connection $\bar{\nabla}$ is called *Levi-Civita connection* if $T=0$ and $\bar{\nabla}\bar{g}=0$. We denote the Levi-Civita connection by $\bar{\nabla}^L$.
Let M be a semi-Riemannian submanifold of $\bar{M}$ and ∇^L and g the induced Levi-Civita connection and metric tensor, respectively. Then the *Gauss formula* follows

$$\bar{\nabla}^L_{\mathbf{x}}\mathbf{y} = \nabla^L_{\mathbf{x}}\mathbf{y} + h(\mathbf{x},\mathbf{y}),$$

where h is so-called *second fundamental form* of M and $\mathbf{x}$ and $\mathbf{y}$ tangent vector fields to M. Let $\{\mathbf{f}_1,...,\mathbf{f}_n\}$ be an orthonormal frame on M at any point $p\in M$. Then the *mean curvature vector* $\mathbf{H}(p)$ at p is defined by

$$\mathbf{H}(p) = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i h(\mathbf{f}_i,\mathbf{f}_i),$$

where $\varepsilon_i = g(\mathbf{f}_i,\mathbf{f}_i)$ and $n=\dim M$. The length of mean curvature vector is called *mean curvature*. A semi-Riemannian submanifold is called *minimal* if its mean curvature vanishes identically.
Let $\bar{M}=\mathbb{R}^3_1$ be the Lorentz-Minkowski $3-$space and $\bar{g}=\langle\cdot,\cdot\rangle_L = dx^2+dy^2-dz^2$. A vector field $\mathbf{x}$ on $\mathbb{R}^3_1$ is said to be *spacelike* (resp. *timelike*) if $\mathbf{x}=0$ or $\langle\mathbf{x},\mathbf{x}\rangle_L>0$ (resp. $\langle\mathbf{x},\mathbf{x}\rangle_L<0$). A vector field $\mathbf{x}$ is said to be *null* if $\langle\mathbf{x},\mathbf{x}\rangle_L=0$ and $\mathbf{x}\neq 0$. A timelike vector $\mathbf{x}=(a,b,c)$ is said to be *future pointing* (resp. *past pointing*) if $c>0$ (resp. $c<0$). A *Lorentz timelike angle* θ between two future (past) pointing timelike vectors $\mathbf{x}$ and $\mathbf{y}$ is associated with [17]

$$|\langle\mathbf{x},\mathbf{y}\rangle_L| = \sqrt{|\langle\mathbf{x},\mathbf{x}\rangle_L|}\sqrt{|\langle\mathbf{y},\mathbf{y}\rangle_L|}\cosh\theta.$$

A *Lorentz spacelike angle* θ between two spacelike vectors $\mathbf{x}$ and $\mathbf{y}$ spanning a spacelike vector subspace ($\mathbb{R}^3_1$ induces a Riemannian metric on it) is associated with [17]

$$|\langle\mathbf{x},\mathbf{y}\rangle_L| = \sqrt{|\langle\mathbf{x},\mathbf{x}\rangle_L|}\sqrt{|\langle\mathbf{y},\mathbf{y}\rangle_L|}\cos\theta.$$

Let M^2 be an immersed surface into $\mathbb{R}^3_1$. The surface M^2 is said to be *spacelike* (resp. *timelike*) if all tangent planes of M^2 are spacelike (resp. timelike). For such a spacelike (resp. timelike) surface, we have the decomposition $\mathbb{R}^3_1 = T_pM^2\oplus\left(T_pM^2\right)^{\perp}$, where T_pM^2 is the tangent plane of M^2 at the point p. Notice that $\left(T_pM^2\right)^{\perp}$ is a timelike (resp. spacelike) $1-$space of $\mathbb{R}^3_1$. A *Gauss map* $\mathbf{n}$ of M^2 is a smooth map $\mathbf{n}: M^2\to\mathbb{R}^3_1$, $|\langle\mathbf{n},\mathbf{n}\rangle_L|=1$.
We finish this section remarking that a spacelike (resp. timelike) surface in $\mathbb{R}^3_1$ is locally a graph of a smooth function $u(x,y)$ (resp. $u(x,z)$ or $u(y,z)$) [28, Proposition 3.3].

3. Singular minimal surfaces in $\mathbb{R}^3$

3.1. $\nabla-$Singular minimal surfaces

Let ∇^L be the Levi-Civita connection on $\mathbb{R}^3$ and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ the standard basis on $\mathbb{R}^3$ and $\mathbf{x},\mathbf{y}$ tangent vector fields to $\mathbb{R}^3$. Consider the following semi-symmetric metric connection on $\mathbb{R}^3$ [44]

$$\nabla_{\mathbf{x}}\mathbf{y} = \nabla^L_{\mathbf{x}}\mathbf{y} + \langle\mathbf{y},\mathbf{e}_3\rangle\mathbf{x} - \langle\mathbf{x},\mathbf{y}\rangle\mathbf{e}_3. \tag{3.1}$$

The nonzero derivatives are

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\mathbf{e}_3,\ \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1,\ \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3,\ \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2.$$

Definition 3.1. *Let $\mathbf{r}$ be a smooth immersion of an oriented surface M^2 into $\mathbb{R}^3$ and $\mathbf{n}$ unit normal vector field on M^2 and H^{∇} the mean curvature with respect to ∇. Let $\mathbf{v}\in\mathbb{R}^3$, $\mathbf{v}\neq\mathbf{0}$, a unit fixed vector non-parallel to $\mathbf{n}$. The surface M^2 is called $\nabla-$singular minimal surface with respect to $\mathbf{v}$ if holds*

$$2H^{\nabla} = \alpha\frac{\langle\mathbf{n},\mathbf{v}\rangle}{\langle\mathbf{r},\mathbf{v}\rangle},\ \alpha\in\mathbb{R},\ \alpha\neq 0. \tag{3.2}$$

In particular, the surface M^2 is said to be $\nabla-$*minimal* if $H^{\nabla}=0$. With Definition 3.1, we first observe the $\nabla-$singular minimal surfaces of type $z=u(x,y)$ which are $\nabla-$minimal.

Theorem 3.1. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}^3$ of type $z=u(x,y)$ with respect to a unit vector $\mathbf{v}=(a,b,c)$, $a^2+b^2\neq 0$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. $\mathbf{v}=(0,b\neq 0,c)$ *and*

$$u(x,y)=\frac{c}{b}y+\frac{1}{2b^2}\ln\left[\cos\left(2bx+\lambda_1\right)\right]+\lambda_2;$$

2. $\mathbf{v}=(a\neq 0,0,c)$ *and*

$$u(x,y)=\frac{c}{a}x+\frac{1}{2a^2}\ln\left[\cos\left(2ay+\lambda_3\right)\right]+\lambda_4;$$

3. $\mathbf{v}=(a,b,c)\,,\ ab\neq 0$, *and*

$$u(x,y)=\frac{c}{a}x-\frac{1}{2\left(a^2+b^2\right)}\ln\left[\cos\left(-2\left|a\right|\left(y-\frac{b}{a}x\right)+\lambda_5\right)\right]+\frac{bc}{a^2+b^2}\left(y-\frac{b}{a}x\right)+\lambda_6,$$

where $\lambda_1,\ldots,\lambda_6\in\mathbb{R}$.

Proof. The unit normal vector field on M^2 is

$$\mathbf{n}=\frac{-u_x\mathbf{e}_1-u_y\mathbf{e}_2+\mathbf{e}_3}{\sqrt{1+\left(u_x\right)^2+\left(u_y\right)^2}},$$

where $u_x=\frac{\partial u}{\partial x}$ and so. Suppose that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eq. (3.2) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and

$$au_x+bu_y=c. \tag{3.3}$$

The condition of $\nabla-$minimality yields

$$\left[1+\left(u_y\right)^2\right]u_{xx}-2u_xu_yu_{xy}+\left[1+\left(u_x\right)^2\right]u_{yy}-2\left[1+\left(u_x\right)^2+\left(u_y\right)^2\right]=0. \tag{3.4}$$

We distinguish several cases: the first case is that $a=0$. Then Eq. (3.3) follows $u(x,y)=f(x)+\frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (3.4) leads to

$$\frac{bf''}{1+\left(bf'\right)^2}=2b, \tag{3.5}$$

where $f'=\frac{df}{dx}$ and so. The first statement of the theorem is obtained by integrating Eq. (3.5). The roles of x and y in Eq. (3.4) are symmetric and hence we may conclude the second statement of the theorem by similar steps when $a\neq 0$ and $b=0$. The last case is that $ab\neq 0$. Then the solution to Eq. (3.3) is given by

$$u(x,y)=\frac{c}{a}x+g\left(y-\frac{b}{a}x\right), \tag{3.6}$$

for a smooth function g. Substituting Eq. (3.6) into Eq. (3.4) follows

$$g''-2\left[a^2+\left(c-bg'\right)^2+\left(ag'\right)^2\right]=0, \tag{3.7}$$

for $g'=\frac{dg}{d\tilde{y}}$, $g''=\frac{d^2g}{d\tilde{y}^2}$, $\tilde{y}=y-\frac{b}{a}x$. Eq. (3.7) can be rewritten as

$$\frac{\left(a^2+b^2\right)g''}{a^2+\left(bc-\left(a^2+b^2\right)g'\right)^2}=2. \tag{3.8}$$

The proof is completed by integrating Eq. (3.8). □

Remark 3.1. *The surface given in the first statement of Theorem 3.1 is a generalized cylinder (see [20, p. 439]) and may be written parametrically*

$$\mathbf{r}(x,y)=\left(x,0,\frac{1}{2b^2}\ln\left[\cos\left(2bx+\lambda_1\right)\right]+\lambda_2\right)+y\left(0,1,\frac{c}{b}\right).$$

This is a $\nabla-$minimal translation surface of type $z=p(x)+q(y)$ which was already found by Wang [44]. The same may be concluded for the above second statement. However, the surface described in the last statement of Theorem 3.1 is the generalized cylinder parametrically written by

$$\mathbf{r}(x,\tilde{y})=x\left(1,\frac{b}{a},\frac{c}{a}\right)+\left(0,\tilde{y},g\left(\tilde{y}\right)\right),$$

where $\tilde{y}=y-\frac{b}{a}x$. Due to $b\neq 0$, it belongs to the class of affine translation surfaces and a new example of $\nabla-$minimal surfaces.

In the following we classify $\nabla-$singular minimal surfaces in $\mathbb{R}^3$ of type $y=u(x,z)$ which are $\nabla-$minimal.

Theorem 3.2. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}^3$ of type $y=u(x,z)$ with respect to a unit vector $\mathbf{v}=(a,b,c)\,,\ a^2+c^2\neq 0$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. *M^2 is a plane parallel to the vector $(0,0,1)$;*
2. $\mathbf{v}=(0,b,c)$, $bc\neq 0$, *and*
$$u(x,z)=\frac{b}{c}z+\frac{1}{2bc}\ln\left[\cos\left(2bx+\lambda_1\right)\right]+\lambda_2;$$
3. $\mathbf{v}=(a,b,0)$, $a\neq 0$, *and*
$$u(x,z)=\frac{b}{a}x\pm\frac{1}{2|a|}\arctan\left(\frac{1}{|a\lambda_2|}\sqrt{e^{4z}-a^2}\right)+\lambda_3;$$
4. $\mathbf{v}=(a,0,c)$, $ac\neq 0$, *and*
$$u(x,z)=\pm\frac{1}{2|a|}\arctan\left(\frac{1}{|\lambda_4|}\sqrt{e^{4a^2\left(z-\frac{c}{a}x\right)}-\lambda_4^2}\right)+\lambda_5,\ \lambda_4\neq 0;$$
5. $\mathbf{v}=(a,b,c)$, $ac\neq 0$, *and*
$$u(x,z)=\frac{b}{a}x+h\left(z-\frac{c}{a}x\right),$$
where h is a smooth function satisfying
$$z-\tfrac{c}{a}x=\tfrac{1}{2|a|(a^2+c^2)(a^2+b^2c^2)}\left\{bc\left(2|a|h+\lambda_6\right)-\right.$$
$$\left.-|a|\ln\left[bc\cos\left(2|a|h+\lambda_6\right)-|a|\sin\left(2|a|h+\lambda_6\right)\right]\right\}+\lambda_7,$$
for $\lambda_1,\ldots,\lambda_7\in\mathbb{R}$.

Proof. Let M^2 be locally given by
$$(x,z)\longmapsto\mathbf{r}(x,z)=(x,u(x,z),z),$$
for a smooth function $u=u(x,z)$. The normal vector field on M^2 is
$$\mathbf{n}=\frac{u_x\mathbf{e}_1-\mathbf{e}_2+u_z\mathbf{e}_3}{\sqrt{1+(u_x)^2+(u_z)^2}}. \tag{3.9}$$
Because M^2 is $\nabla-$singular minimal, we get Eq. (3.2). Assume that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eqs. (3.2) and (3.9) follow $\langle\mathbf{n},\mathbf{v}\rangle=0$ and
$$au_x+cu_z=b. \tag{3.10}$$
Remark also that we may write $\mathbf{v}=a\mathbf{r}_x+c\mathbf{r}_z$, which means that the tangent plane of M at any point is parallel to $\mathbf{v}$. The condition of $\nabla-$minimality leads to
$$\left[1+(u_z)^2\right]u_{xx}-2u_xu_zu_{xz}+\left[1+(u_x)^2\right]u_{zz}+2\left[1+(u_x)^2+(u_z)^2\right]u_z=0. \tag{3.11}$$
We distinguish several cases:

1. $a=0$, $c\neq 0$. Then Eq. (3.10) gives $u_z=\frac{b}{c}$ and so Eq. (3.11) turns M^2 to a plane parallel to $\mathbf{v}$ if $b=0$. Otherwise, $b\neq 0$, the solution to Eq. (3.10) is given by $u(x,z)=\frac{b}{c}z+f(x)$, for an arbitrary smooth function f. Hence Eq. (3.11) reduces to
$$\frac{cf''}{1+(cf')^2}=-2b, \tag{3.12}$$
where $f'=\frac{df}{dx}$, etc. The second statement of the theorem is obtained by integrating Eq. (3.12).
2. $a\neq 0$, $c=0$. Then Eq. (3.10) gives $u(x,z)=\frac{b}{a}x+g(z)$ for an arbitrary smooth function g and so Eq. (3.11) may be written as
$$\frac{g''}{g'}-\frac{a^2g'g''}{1+(ag')^2}=-2, \tag{3.13}$$
where $g'=\frac{dg}{dz}$, etc. Integrating Eq. (3.13), we obtain the third statement of the theorem.
3. $ac\neq 0$. The solution to Eq. (3.10) is
$$u(x,z)=\frac{b}{a}x+h\left(z-\frac{c}{a}x\right), \tag{3.14}$$
for an arbitrary smooth function h. By plugging the partial derivatives of Eq. (3.14) into Eq. (3.11), we write
$$h''+2\left[a^2+\left(b-ch'\right)^2+\left(ah'\right)^2\right]h'=0, \tag{3.15}$$
where $h'=\frac{dh}{d\tilde{z}}$, $h''=\frac{d^2h}{d\tilde{z}^2}$, $\tilde{z}=z-\frac{c}{a}x$. We have two subcases: the first subcase is that $b=0$. Then Eq. (3.15) may be rewritten as
$$\frac{h''}{h'}-\frac{h'h''}{a^2+(h')^2}=-2a^2. \tag{3.16}$$

The fourth statement of the theorem is proved by integrating Eq. (3.16). The second subcase is $b \neq 0$. Hence, we may write Eq. (3.15) as

$$\frac{-\left(a^2+c^2\right)h''}{a^2+\left(bc-\left(a^2+c^2\right)h'\right)^2}=2h'. \tag{3.17}$$

A first integration of Eq. (3.17) yields

$$\frac{\left(a^2+c^2\right)dh}{-|a|\tan\left(2|a|h+\lambda\right)+bc}=d\tilde{z}, \tag{3.18}$$

for $\lambda \in \mathbb{R}$. By a first integration of Eq. (3.18), we finish the proof.

□

Remark 3.2. *The surfaces given in the second and third statements of Theorem 3.2 are $\nabla-$minimal generalized cylinders and are examples of $\nabla-$minimal translation surfaces of type $y=p(x)+q(z)$, which was found by Wang [44]. However, the surfaces given in the last two statements of Theorem 3.2 are a $\nabla-$minimal affine translation surface.*

Lastly, we deal with a surface M^2 of type $x=u(y,z)$. The unit normal vector field on M^2 is

$$\mathbf{n}=\frac{\mathbf{e}_1-u_y\mathbf{e}_2-u_z\mathbf{e}_3}{\sqrt{1+\left(u_y\right)^2+\left(u_z\right)^2}}.$$

Suppose that M^2 is $\nabla-$singular minimal with respect to the vector $\mathbf{v}=(a,b,c)$. The mean curvature is same as that of the surface of type $y=u(x,z)$. If M^2 is also $\nabla-$minimal, then Eq. (1.3) gives

$$bu_y+cu_z=a,$$

where $b^2+c^2\neq 0$. Therefore, without giving a proof, we may state a similar result for those surfaces of type $x=u(y,z)$ to Theorem 3.2 by replacing x with y and a with b.

3.2. $D-$Singular minimal surfaces

Let D be the semi-symmetric non-metric connection on $\mathbb{R}^3$ given by [44]

$$D_{\mathbf{x}}\mathbf{y}=\nabla^L_{\mathbf{x}}\mathbf{y}+\langle\mathbf{y},\mathbf{e}_3\rangle\mathbf{x}, \tag{3.19}$$

where $\mathbf{x},\mathbf{y}$ are tangent vector fields to $\mathbb{R}^3$. The nonzero derivatives are

$$D_{\mathbf{e}_1}\mathbf{e}_3=\mathbf{e}_1,\ D_{\mathbf{e}_2}\mathbf{e}_3=\mathbf{e}_2,\ D_{\mathbf{e}_3}\mathbf{e}_3=\mathbf{e}_3.$$

Definition 3.2. *Let $\mathbf{r}$ be a smooth immersion of an oriented surface M^2 into $\mathbb{R}^3$ and $\mathbf{n}$ unit normal vector field on M^2 and H^D denote the mean curvature with respect to D. Let $\mathbf{v}\in\mathbb{R}^3$, $\mathbf{v}\neq\mathbf{0}$, a unit fixed vector non-parallel to $\mathbf{n}$. The surface M^2 is called $D-$singular minimal surface with respect to $\mathbf{v}$ if holds*

$$2H^D=\alpha\frac{\langle\mathbf{n},\mathbf{v}\rangle}{\langle\mathbf{r},\mathbf{v}\rangle},\ \alpha\in\mathbb{R},\ \alpha\neq 0. \tag{3.20}$$

In particular, the surface M^2 is said to be $D-$*minimal* if $H^D=0$. We first consider the $D-$singular minimal surfaces of type $z=u(x,y)$ which are $D-$minimal. Hence Eq. (3.20) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and

$$au_x+bu_y=c, \tag{3.21}$$

where $\mathbf{v}=(a,b,c)$ and $a^2+b^2\neq 0$. Morever the condition of $D-$minimality yields

$$\left[1+\left(u_y\right)^2\right]u_{xx}-2u_xu_yu_{xy}+\left[1+\left(u_x\right)^2\right]u_{yy}=0, \tag{3.22}$$

where the roles of x and y are symmetric. If $a=0$, then Eq. (3.21) follows $u(x,y)=f(x)+\frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (3.22) yields $\frac{1}{b^2}\frac{d^2f}{dx^2}=0$, which leads M^2 to be a plane parallel to $\mathbf{v}$. By symmetry, we may obtain same obtain when $a\neq 0$ and $b=0$. Let $ab\neq 0$. Then the solution to Eq. (3.21) is $u(x,y)=\frac{c}{a}x+g\left(y-\frac{b}{a}x\right)$, for an arbitrary smooth function f. After substituting its partial derivatives into Eq. (3.22), we conclude $\frac{1}{a^2}\frac{d^2g}{d\tilde{y}^2}=0$, $\tilde{y}=y-\frac{b}{a}x$, yielding that M is a plane parallel to $\mathbf{v}$.
Therefore we state the following

Theorem 3.3. *Let M^2 be a $D-$singular minimal surface in $\mathbb{R}^3$ of type $z=u(x,y)$ with respect to a unit vector $\mathbf{v}=(a,b,c)$, $a^2+b^2\neq 0$. If M^2 is $D-$minimal, then it is a plane parallel to $\mathbf{v}$.*

When we take surfaces of type $y=u(x,z)$ and $x=u(y,z)$, we get similar equations to Eqs. (3.21) and (3.22) and thus the above result remains true for those surfaces as well.

4. Singular minimal surfaces in $\mathbb{R}^3_1$

4.1. $\nabla-$Singular minimal surfaces

Let ∇^L be the Levi-Civita connection $\mathbb{R}^3_1$ and $\mathbf{x},\mathbf{y}$ tangent vector fields to $\mathbb{R}^3_1$. Consider the following semi-symmetric metric connection on $\mathbb{R}^3_1$ [44]

$$\nabla_{\mathbf{x}}\mathbf{y} = \nabla^L_{\mathbf{x}}\mathbf{y} + \langle \mathbf{y},\mathbf{e}_3\rangle_L \mathbf{x} - \langle \mathbf{x},\mathbf{y}\rangle_L \mathbf{e}_3. \tag{4.1}$$

The nonzero derivatives are

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\mathbf{e}_3,\ \nabla_{\mathbf{e}_1}\mathbf{e}_3 = -\mathbf{e}_1,\ \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3,\ \nabla_{\mathbf{e}_2}\mathbf{e}_3 = -\mathbf{e}_2.$$

Definition 4.1. *Let $\mathbf{r}$ be a smooth immersion of an oriented timelike surface M^2 in $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H^∇ the mean curvature of M^2 with respect to ∇. Let $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^3_1$ a unit fixed spacelike vector non-parallel to $\mathbf{n}$ such that $\mathbf{n}$ and $\mathbf{v}$ span a spacelike 2-space. M^2 is called $\nabla-$singular minimal surface with respect to $\mathbf{v}$ if satisfies*

$$2H^\nabla = \alpha \frac{\langle \mathbf{n},\mathbf{v}\rangle_L}{\langle \mathbf{r},\mathbf{v}\rangle_L},\ \alpha\in\mathbb{R},\ \alpha\neq 0. \tag{4.2}$$

The surface M^2 is called $\nabla-$*minimal* if $H^\nabla = 0$. With Definition 4.1, we classify the $\nabla-$singular minimal surfaces of type $z = u(x,y)$, which are $\nabla-$minimal.

Theorem 4.1. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}^3_1$ of type $z = u(x,y)$ with respect to a unit spacelike vector $\mathbf{v} = (a,b,c)$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. $\mathbf{v} = (0, b\neq 0, c)$ *and*
$$u(x,y) = \frac{c}{b}y + \frac{1}{2b^2}\ln[\cosh(2bx+\lambda_1)] + \lambda_2;$$
2. $\mathbf{v} = (a\neq 0, 0, c)$ *and*
$$u(x,y) = \frac{c}{a}x + \frac{1}{2a^2}\ln[\cosh(2ay+\lambda_3)] + \lambda_4;$$
3. $\mathbf{v} = (a,b,c)$, $ab\neq 0$, *and*
$$u(x,y) = \frac{c}{a}x + \frac{bc}{a^2+b^2}\left(y-\frac{b}{a}x\right) + \frac{1}{2(a^2+b^2)}\ln\left[\cosh\left(-2|a|\left\{y-\frac{b}{a}x\right\}+\lambda_5\right)\right] + \lambda_6,$$
where $\lambda_1,\dots,\lambda_6\in\mathbb{R}$, $\lambda_5\neq 0$.

Proof. The unit spacelike normal vector field on M^2 is

$$\mathbf{n} = \frac{-u_x\mathbf{e}_1 - u_y\mathbf{e}_2 - \mathbf{e}_3}{\sqrt{-1+(u_x)^2+(u_y)^2}},$$

Suppose that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eq. (4.2) gives $\langle \mathbf{n},\mathbf{v}\rangle = 0$ and

$$au_x + bu_y = c. \tag{4.3}$$

The condition of $\nabla-$minimality yields

$$\left[1-(u_y)^2\right]u_{xx} + 2u_xu_yu_{xy} + \left[1-(u_x)^2\right]u_{yy} + 2\left[-1+(u_x)^2+(u_y)^2\right] = 0. \tag{4.4}$$

We distinguish several cases: the first case is that $a = 0$. Then Eq. (4.3) follows $u(x,y) = f(x) + \frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (4.4) yields

$$\frac{bf''}{1-(bf')^2} = 2b, \tag{4.5}$$

where $f' = \frac{df}{dx}$ and so. The first statement of the theorem is derived by integrating Eq. (4.5). The roles of x and y in Eq. (4.4) are symmetric and hence we may conclude the second statement of the theorem by similar steps when $a\neq 0$ and $b = 0$. The last case is that $ab\neq 0$. Then the solution to Eq. (4.3) is given by

$$u(x,y) = \frac{c}{a}x + g\left(y-\frac{b}{a}x\right), \tag{4.6}$$

for a smooth function g. Substituting Eq. (4.6) into Eq. (4.4) follows

$$g'' + 2\left[-a^2 + (c-bg')^2 + (ag')^2\right] = 0, \tag{4.7}$$

where $g' = \frac{dg}{d\tilde{y}}$, $g'' = \frac{d^2g}{d\tilde{y}^2}$, $\tilde{y} = y - \frac{b}{a}x$. Eq. (4.7) may be rewritten as

$$\frac{-(a^2+b^2)g''}{a^2-\left[bc-(a^2+b^2)g'\right]^2} = -2. \tag{4.8}$$

The proof is completed by integrating Eq. (4.8). $\square$

Remark 4.1. *The last statement of Theorem 4.1 is a new example in $\mathbb{R}^3_1$ of $\nabla-$minimal surfaces while the first two statements are $\nabla-$minimal translation surfaces, introduced by Wang [44].*

In the following we classify $\nabla-$singular minimal surfaces in $\mathbb{R}^3_1$ of type $y=u(x,z)$ which are $\nabla-$minimal.

Theorem 4.2. *Let M^2 be a $\nabla-$singular minimal surface in $\mathbb{R}^3_1$ of type $y=u(x,z)$ with respect to a unit spacelike vector $\mathbf{v}=(a,b,c)$, $a^2+c^2\neq 0$. If M^2 is $\nabla-$minimal, then one of the following happens*

1. *M^2 is a plane parallel to $\mathbf{v}=(a,b,0)$, $a\neq 0$;*
2. *$\mathbf{v}=(0,b,c)$, $bc\neq 0$ and*
$$u(x,y)=\frac{b}{c}z+\frac{1}{2bc}\ln\left[\cosh\left(2bx+\lambda_1\right)\right]+\lambda_2;$$
3. *$\mathbf{v}=(a,b,0)$, $a\neq 0$, and*
$$u(x,z)=\frac{b}{a}x\pm\frac{1}{2|a|}\sinh^{-1}\left(\lambda_3 e^{2z}\right)+\lambda_4,\ \lambda_3\neq 0;$$
4. *$\mathbf{v}=(a,0,c)$, $ac\neq 0$, and*
$$u(x,z)=\pm\frac{1}{2|a|}\sinh^{-1}\left[|\lambda_5|e^{2a^2\left(z-\frac{c}{a}x\right)}\right]+\lambda_6,\ \lambda_5\neq 0;$$
5. *$\mathbf{v}=(a,\pm1,c)$, $a=\pm c$, $c\neq 0$, and*
$$u(x,z)=\frac{\pm1}{c}x\pm\frac{1}{4c}\ln\left[1\pm 2\lambda_7 e^{2\left(1+c^2\right)(z\pm x)}\right]+\lambda_8,\ \lambda_7\neq 0;$$
6. *$\mathbf{v}=(a,b,c)$, $abc\neq 0$, and*
$$u(x,z)=\frac{b}{a}x+h\left(z-\frac{c}{a}x\right),$$
where h is a smooth function satisfying
$$z-\frac{c}{a}x=\frac{-bc\left(c^2-a^2\right)}{2|a|\left(a^2-b^2c^2\right)}\left(2|a|h+\lambda_9\right)-$$
$$-\frac{c^2-a^2}{2\left(a^2-b^2c^2\right)}\ln\left[bc\cosh\left(2|a|h+\lambda_9\right)-|a|\sinh\left(2|a|h+\lambda_9\right)\right]+\lambda_{10},$$
for $\lambda_1,\ldots,\lambda_{10}\in\mathbb{R}$.

Proof. Let M^2 be locally given by
$$(x,z)\longmapsto \mathbf{r}(x,z)=(x,u(x,z),z),$$
for a smooth function $u=u(x,z)$. The normal vector field on M^2 is
$$\mathbf{n}=\frac{u_x\mathbf{e}_1-\mathbf{e}_2-u_z\mathbf{e}_3}{\sqrt{1+(u_x)^2-(u_z)^2}}.$$

Because M^2 is $\nabla-$singular minimal, we get Eq. (4.1). Assume that M^2 is $\nabla-$minimal. Due to $\alpha\neq 0$, Eq. (4.1) gives $\langle\mathbf{n},\mathbf{v}\rangle=0$ and
$$au_x+cu_z=b. \tag{4.9}$$

Remark also that we may write $\mathbf{v}=a\mathbf{r}_x+c\mathbf{r}_z$, implying the tangent plane of M at any point is parallel to $\mathbf{v}$. The condition of $\nabla-$minimality yields
$$\left[(u_z)^2-1\right]u_{xx}-2u_xu_zu_{xz}+\left[1+(u_x)^2\right]u_{zz}-2\left[1+(u_x)^2-(u_z)^2\right]u_z=0. \tag{4.10}$$

We distinguish several cases:

1. $a=0$, $c\neq 0$. Then $b\neq 0$ because $\mathbf{v}$ is spacelike. The solution to Eq. (4.9) is given by $u(x,z)=\frac{b}{c}z+f(x)$, for an arbitrary smooth function f. Hence Eq. (4.10) turns to
$$\frac{cf''}{1-(cf')^2}=-2b, \tag{4.11}$$
where $f'=\frac{df}{dx}$, etc. Because M^2 is non-degenerate, $1-(cf')^2\neq 0$. Therefore the second statement of the theorem is proved by integrating Eq. (4.11).

2. $a \neq 0$, $c = 0$. Then Eq. (4.9) gives $u(x,z) = \frac{b}{a}x + g(z)$ for an arbitrary smooth function g and so Eq. (4.10) leads to

$$g'' - 2\left[1 - (ag')^2\right] g' = 0, \tag{4.12}$$

where $g' = \frac{dg}{dz}$, etc. That $g' = 0$ is a trivial solution to Eq. (4.12), implying the first statement of the theorem. Otherwise, $g' \neq 0$, Eq. (4.12) may be rewritten as

$$\frac{g''}{g'} + \frac{a}{2}\left(\frac{g''}{1 - ag'} - \frac{g''}{1 + ag'}\right) = 2. \tag{4.13}$$

The third statement of the theorem is obtained by integrating Eq. (4.13).
3. $ac \neq 0$. The solution to Eq. (4.9) is

$$u(x,z) = \frac{b}{a}x + h\left(z - \frac{c}{a}x\right),$$

for an arbitrary smooth function h. Therefore Eq. (4.10) reduces to

$$h'' - 2\left[a^2 + (b - ch')^2 - (ah')^2\right] h' = 0, \tag{4.14}$$

where $h' = \frac{dh}{d\tilde{z}}$, $h' = \frac{d^2h}{d\tilde{z}^2}$, $\tilde{z} = z - \frac{c}{a}x$. We have three subcases: the first one is that $b = 0$. Then Eq. (4.14) may be rewritten as

$$\frac{h''}{h'} + \frac{h''}{2(a - h')} - \frac{h''}{2(a + h')} = 2a^2. \tag{4.15}$$

Integrating Eq. (4.15) gives the fourth statement of the theorem. The second subcase is that $a^2 = c^2$ and $b = \pm 1$. Then Eq. (4.14) may be rewritten as

$$\frac{\pm 2ch''}{1 + c^2 \mp 2ch'} + \frac{h''}{h'} = 2\left(1 + c^2\right). \tag{4.16}$$

After integrating Eq. (4.16), we obtain the fifth statement of the theorem. The third subcase is that $a^2 \neq c^2$. Then Eq. (4.14) may be rewritten as

$$\frac{-\left(c^2 - a^2\right) h''}{a^2 - \left[bc - \left(c^2 - a^2\right) h'\right]^2} = 2h'. \tag{4.17}$$

A first integration of Eq. (4.17) yields

$$\frac{\left(c^2 - a^2\right) dh}{-|a| \tanh\left(2|a|h + \lambda\right) + bc} = d\tilde{z}, \tag{4.18}$$

for $\lambda \in \mathbb{R}$. The proof is completed by a first integration of Eq. (4.18).

□

Remark 4.2. *The last three statements of Theorem 4.1 are new examples in $\mathbb{R}^3_1$ of $\nabla-$minimal surfaces while the second and third statements are $\nabla-$minimal translation surfaces, introduced by Wang [44].*

Let M^2 be a timelike surface $\mathbb{R}^3_1$ of type $x = u(y,z)$. The spacelike unit normal vector field on M^2 is

$$\mathbf{n} = \frac{\mathbf{e}_1 - u_y\mathbf{e}_2 + u_z\mathbf{e}_3}{\sqrt{1 + (u_y)^2 - (u_z)^2}}.$$

Suppose that M^2 is $\nabla-$singular minimal with respect to the vector $\mathbf{v} = (a,b,c)$. If M^2 is also $\nabla-$minimal, then Eq. (4.2) gives

$$bu_y + cu_z = a,$$

where $b^2 + c^2 \neq 0$. Notice that the mean curvature is same as that of the surface of type $y = u(x,z)$. Therefore, without giving a proof, we may state a similar result for those surfaces of type $x = u(y,z)$ to Theorem 4.1 by replacing x with y and a with b.

4.2. $D-$Singular minimal surfaces

Consider the following semi-symmetric non-metric connection on $\mathbb{R}^3_1$ [44]

$$D_{\mathbf{x}}\mathbf{y} = \nabla^L_{\mathbf{x}}\mathbf{y} + \langle \mathbf{y}, \mathbf{e}_3 \rangle \mathbf{x}, \tag{4.19}$$

where $\mathbf{x}, \mathbf{y}$ are tangent vector fields to $\mathbb{R}^3$. The nonzero derivatives are

$$D_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1,\ D_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2,\ D_{\mathbf{e}_3}\mathbf{e}_3 = \mathbf{e}_3.$$

Definition 4.2. *Let $\mathbf{r}$ be a smooth immersion of an oriented timelike surface M^2 in $\mathbb{R}^3_1$ and $\mathbf{n}$ unit normal vector field on M^2 and H^D the mean curvature of M^2 with respect to D. Let $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^3_1$ a unit fixed spacelike vector non-parallel to $\mathbf{n}$ such that $\mathbf{n}$ and $\mathbf{v}$ span a spacelike 2-space. M^2 is called $D-$singular minimal surface with respect to $\mathbf{v}$ if satisfies*

$$2H^D = \alpha \frac{\langle \mathbf{n}, \mathbf{v} \rangle_L}{\langle \mathbf{r}, \mathbf{v} \rangle_L},\ \alpha \in \mathbb{R},\ \alpha \neq 0. \tag{4.20}$$

The surface M^2 is called $D-$*minimal* if $H^D = 0$. With Definition 4.2, we first observe the $D-$singular minimal surfaces of type $z = u(x,y)$ which are $D-$minimal. Hence Eq. (4.20) gives $\langle \mathbf{n}, \mathbf{v} \rangle = 0$ and

$$au_x + bu_y = c, \tag{4.21}$$

where $\mathbf{v} = (a,b,c)$. The condition of $D-$minimality yields

$$\left[1 - (u_y)^2\right] u_{xx} + 2u_x u_y u_{xy} + \left[1 - (u_x)^2\right] u_{yy} = 0, \tag{4.22}$$

where the roles of x and y are symmetric. If $a = 0$, then Eq. (4.21) follows $u(x,y) = f(x) + \frac{c}{b}y$, for an arbitrary smooth function f. Considering this into Eq. (4.22) yields $\frac{1}{b^2}\frac{d^2f}{dx^2} = 0$, which leads M^2 to be a plane parallel to $\mathbf{v}$. By symmetry, we can obtain same result when $a \neq 0$ and $b = 0$. Let $ab \neq 0$. Then the solution to Eq. (4.21) is $u(x,y) = \frac{c}{a}x + g\left(y - \frac{b}{a}x\right)$, for a smooth function g. After substituting its partial derivatives into Eq. (4.22), we conclude $\frac{1}{a^2}\frac{d^2g}{d\tilde{y}^2} = 0$, $\tilde{y} = y - \frac{c}{a}x$, yielding that M is a plane parallel to $\mathbf{v}$.
Therefore, we state the following

Theorem 4.3. *Let M^2 be a $D-$singular minimal surface in $\mathbb{R}^3_1$ of type $z = u(x,y)$ with respect to a unit spacelike vector $\mathbf{v}$. If M^2 is $D-$minimal, then it is a plane parallel to $\mathbf{v}$.*

When we take the surfaces of type $y = u(x,z)$ or $x = u(y,z)$, we may state a similar result to Theorem 4.3.

5. Conclusions and further remarks

In this study, we discussed the singular minimal surfaces in $\mathbb{R}^3$ (resp. $\mathbb{R}^3_1$) which are minimal and expressed a trivial outcome, Proposition 1.1 (resp. Proposition 1.2). Nevertheless, the non-trivial outcomes, Theorems 3.1 and 3.2 (resp. Theorems 4.1 and 4.2), were obtained by using the modified version, Definition 3.1 (resp. Definition 4.1), of singular minimality. With this definition, we observed that the singular minimal surfaces which are minimal are a generalized cylinder. Since the generalized cylinders belong to a subcase of translation surfaces, the $\nabla-$minimal translation surfaces introduced by Wang [44] were presented by some of our results. Still, we also exhibited new examples of $\nabla-$minimal surfaces, as explained in Remarks 3.1 and 3.2 (resp. Remarks 4.1 and 4.2). Morever, a trivial outcome, Theorem 3.3 (resp. Theorem 4.3), was found by using the semi-symmetric non-metric connection D given by Eq. (3.19) (resp. Eq. (4.19)).
On the other hand, let M^2 be locally a graph surface in $\mathbb{R}^3$ of a smooth function $u(x,y)$ and H and H^∇ denote the mean curvatures with respect to the Levi-Civita connection and the semi-symmetric metric connection ∇ given by Eq. (3.1), respectively. Then, the following relation occurs

$$H^\nabla = H - \langle \mathbf{n}, (0,0,1) \rangle, \tag{5.1}$$

where $\mathbf{n}$ is the unit normal vector field on M^2. Notice also that Eq. (5.1) remains true for a graph of the forms $u(x,z)$ or $u(y,z)$ up to a sign. Therefore, $\nabla-$minimal graph surfaces turn to the *translating solitons* whose the mean curvature satisfies

$$H = \langle \mathbf{n}, (0,0,1) \rangle \tag{5.2}$$

Eq. (5.2) appears in the theories of mean curvature flow and manifolds with density, for details see ([23, 24, 29–31]). Eventually, the above discussion imply that $\nabla-$singular minimal surfaces which are $\nabla-$minimal are a cylindrical translating soliton. Such surfaces were considered in [23, 31]. Nevertheless, this paper provides a novel approach.

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Author's contributions

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