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New metrics for deltoidal hexacontahedron and pentakis dodecahedron

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ABSTRACT

There are only five regular convex polyhedra known as platonic solids. Semi-regular convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. These solids are called the Archimedian solids. Archimedean solids' s duals are known as the Catalan solids which are only thirteen. It has been shown that deltoidal icositetrahedron which is Chinese Checker' s unit sphere [1]. In this study, we introduce new metrics which their spheres are pentakis dodecahedron and deltoidal hexacontahedron

Keywords:pentakis dodecahedron, deltoidal hexacontahedron, metric, chinese checker metric, catalan solid.

Deltoidal hexacontahedron ve pentakis dodecahedron için yeni metrikler

ÖZ

Platonik cisimler olarak tanımlanan sadece beş tane düzgün konveks çokyüzlü vardır. Yarı-düzgün konveks çokyüzlülerin köşe noktalarında iki veya daha fazla tipten düzgün çokgen birleşir. Bu cisimlere Arşimet cisimleri adı verilir. Arşimet cisimlerinin dualleri Catalan cisimler olarak bilinirler ve sadece onüç tanedir. Son yıllardaki çalışmalarda Çin Dama metriğinin birim küresinin deltoidal icositetrahedron olduğu gösterildi [1]. Bu çalışmada birim küreleri deltoidal hexacontahedron ve pentakis dodecahedron olan metrikleri vereceğiz.

Anahtar Kelimeler: pentakis dodecahedron, deltoidal hexacontahedron, metrik, çin dama metriği, katalan cisim.

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1. INTRODUCTION

A polyhedron is a three dimensional solid which consists of a collection of polygons, always joined at their edges. There are many thinkers that worked on polyhedra among the ancient Greeks. Early civilizations worked out mathematics as problems and their solutions. Polyhedrons have been studied by mathematicians, scientists during many years, because of their symmetries [2-4].

A polyhedron is called regular if all its faces are equal and regular polygons. It is called semi-regular if all its faces are regular polygons and all its vertices are equal. An irregular polyhedron is defined by polygons that are composed of elements that are not all equal. A regular polyhedron is called Platonic solid, a semi-regular polyhedron is called Archimedean solid and an irregular polyhedron is called Catalan solid.

Platonic solids have been studied by mathematicians, geometers during many years. Nowadays some mathematicians study new metrics of which spheres are Platonic solids. The Archimedean solids take their name from Archimedes, who discussed them in a now-lost work. Pappus refers to it, stating that Archimedes listed 13 polyhedra. The Archimedean solids are distinguished by having very high symmetry. They are distinct from the Platonic which are composed of only one type of polygon meeting in identical vertices, whoseregular polygonal faces do not meet in identical vertices. The dual polyhedra of the Archimedean solids are called Catalan solids. The Catalan solids are named for the Belgian mathematician, Eugène Catalan, who first described them in 1865. The Catalan solids are all convex and ırregular polyhedra. The number of Catalan solids is thirteen.

Minkowski geometry is non-Euclidean geometry in a finite number of dimensions. Instead of the usual sphere in Euclidean space, the unit ball is symmetric closed convex set [7]. Some mathematicians have been studied and improved metric space geometry. The Chinese Checker metric plane and space geometry have been studied and developed by some mathematicians. O. Gelişgen, R. Kaya, M. Ozcan have defined CC-metric of which sphere is Deltoidal Icositetrahedron that is a Catalan solid (See [1]). In the 3-dimensional analytical space the CC-metric is defined by

$$d_C(A,B)=\max\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\}+(\sqrt{2}-1)\min\{|x_1-x_2|+|y_1-y_2|,|x_1-x_2|+|z_1-z_2|,|y_1-y_2|+|z_1-z_2|\} \quad (1)$$

where $A=(x_1,y_1,z_1)$, $B=(x_2,y_2,z_2)$ are two points in $\mathbb{R}^3$.

This influence us to the question "Are there some metrics of which unit spheres are the Catalan Solids?". For this goal, firstly we put up the solid to coordinate system to be its center the origin and some of solid's surfaces distance from the origin are 1. And then, we can have the metric which provide plane equation related with solid's surface. In this work, we introduce that new metrics of which spheres are Deltoidal Hexacontahedron and Pentakis Dodecahedron.

2. DELTOIDAL HEXACONTAHEDRON

A deltoidal hexecontahedron (also sometimes called a trapezoidal hexecontahedron, a strombic hexecontahedron, or a tetragonal hexacontahedron) is a catalan solid which looks a bit like either an overinflated dodecahedron or icosahedron. It is sometimes also called the trapezoidal hexecontahedron or strombic hexecontahedron. Its dual polyhedron is the rhombicosidodecahedron. The 60 faces are deltoids or kites (not trapezoidal). The short and long edges of each kite are in the ratio 1.00/1.54. The Deltoidal Hexacontahedron has 60 faces, 120 edges and 62 = 12 + 20 + 30 vertices [8].

Figure 1. Deltoidal hexacontahedron

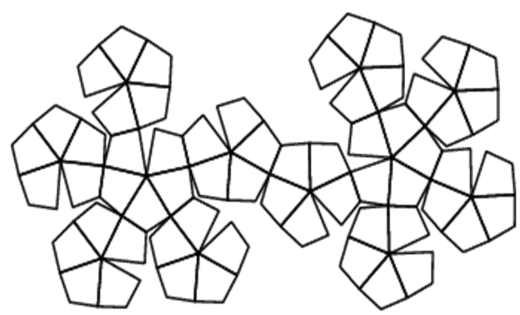

Figure 2. Net for deltoidal hexacontahedron

We describe the metric that unit sphere is deltoidal hexacontahedron as following:

Definition 2. 1:Let $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ be distinct two points in $\mathbb{R}^3$. The distance function $d_{DH}:\mathbb{R}^3 x\ \mathbb{R}^3\rightarrow[0,\infty)$ deltoidal hexacontahedron distance between P_1 and P_2 is defined by

$$d_{DH}(P_1,P_2) = \max\left\{\begin{array}{c} |x_1-x_2| + (2\varphi-3) \max\left\{\begin{array}{c} |y_1-y_2| + |z_1-z_2|, \\ (1-\varphi)|x_1-x_2| + \\ (1+\varphi)|z_1-z_2|, \\ -|x_1-x_2| + \varphi|z_1-z_2| + \\ (1+\varphi)|y_1-y_2| \end{array}\right\} \\ |y_1-y_2| + (2\varphi-3) \max\left\{\begin{array}{c} |z_1-z_2| + |x_1-x_2|, \\ (1-\varphi)|y_1-y_2| + \\ (1+\varphi)|x_1-x_2|, \\ -|y_1-y_2| + \varphi|x_1-x_2| + \\ (1+\varphi)|z_1-z_2| \end{array}\right\} \\ |z_1-z_2| + (2\varphi-3) \max\left\{\begin{array}{c} |y_1-y_2| + |x_1-x_2|, \\ (1-\varphi)|z_1-z_2| + \\ (1+\varphi)|y_1-y_2|, \\ -|z_1-z_2| + \varphi|y_1-y_2| + \\ (1+\varphi)|x_1-x_2| \end{array}\right\} \end{array}\right\} \quad (2)$$

where $\varphi = (1+\sqrt{5})/2$is the golden ratio.

Deltoidal hexacontahedron distance function may seem a bit complicated. In fact there is an orientation in d_{DH}. Let $a = |x_1 - x_2|, b = |x_1 - x_2|, c = |z_1 - z_2|$. This orientation is a − b − c − a. According to orientation, if one can put *b, c, a* instead of *a, b, c*, respectively, in first term of distance function, then it is obtained second term. Similarly, if one can put *c,a, b* instead of *a, b, c*, respectively, in first term of distance function, then it is obtained third term.

Lemma 2. 2: Let $P_1=(x_1,y_1z_1)$ and $P_2=(x_2,y_2,z_2)$ be any distinct two points in $\mathbb{R}^3$. Then

$$d_{DH}(P_1,P_2) \geq |x_1-x_2| + (2\varphi-3) \max\left\{\begin{array}{l} |y_1-y_2| + |z_1-z_2|, \\ (1-\varphi)|x_1-x_2| + (1+\varphi)|z_1-z_2|, \\ -|x_1-x_2| + \varphi|z_1-z_2| + (1+\varphi)|y_1-y_2| \end{array}\right\}$$

$$d_{DH}(P_1,P_2) \geq |y_1-y_2| + (2\varphi-3) \max\left\{\begin{array}{l} |z_1-z_2| + |x_1-x_2|, \\ (1-\varphi)|y_1-y_2| + (1+\varphi)|x_1-x_2|, \\ -|y_1-y_2| + \varphi|x_1-x_2| + (1+\varphi)|z_1-z_2| \end{array}\right\}$$

$$d_{DH}(P_1,P_2) \geq |z_1-z_2| + (2\varphi-3) \max\left\{\begin{array}{l} |y_1-y_2| + |x_1-x_2|, \\ (1-\varphi)|z_1-z_2| + (1+\varphi)|y_1-y_2|, \\ -|z_1-z_2| + \varphi|y_1-y_2| + (1+\varphi)|x_1-x_2| \end{array}\right\}$$

where $\varphi = (1+\sqrt{5})/2$is the golden ratio.

Proof: Proof is trivial by definition of maximum function.

Theorem 2. 3: The distance function d_{DH} is a metric of which unit sphere is a deltoidal hexacontahedron in $\mathbb{R}^3$.

Proof: Let d_{DH}: $\mathbb{R}^3$x $\mathbb{R}^3 \rightarrow \mathbb{R}$ and $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ distinct three points in $\mathbb{R}^3$. To prove that d_{DH} is a metric in $\mathbb{R}^3$, the following axioms can be supplied for all P_1, P_2 and $P_3 \in \mathbb{R}^3$.

M1) $d(P_1,P_2) \geq 0$ ve $d(P_1,P_2)=0 \Leftrightarrow P_1=P_2$

M2) $d(P_1,P_2)=d(P_2,P_1)$

M3) $d(P_1,P_3) \leq d(P_1,P_2)+d(P_2,P_3)$.

M1) Since absolute values is always nonnegative maximum of sums of absolute value is always nonnegative. Thus $d_{DH}(P_1,P_2) \geq 0$. If $d_{DH}(P_1,P_2)=0$ then according to deltoidal hexacontahedron distance function three cases are possible.

Case I: If

$$d_{DH}(P_1,P_2) = |x_1-x_2| + (2\varphi-3) \max\left\{\begin{array}{l} |y_1-y_2| + |z_1-z_2|, \\ (1-\varphi)|x_1-x_2| + (1+\varphi)|z_1-z_2|, \\ -|x_1-x_2| + \varphi|z_1-z_2| + (1+\varphi)|y_1-y_2| \end{array}\right\}$$

the $d_{DH}(P_1,P_2)=0 \Leftrightarrow |x_1-x_2|=0, |y_1 - y_2|=0, |z_1 - z_2|=0 \Leftrightarrow x_1=x_2, y_1=y_2, z_1=z_2 \Leftrightarrow P_1=P_2$.

The other cases can be easily shown by similar way in case i. Thus we get $d_{DH}(P_1,P_2)=0$ iff $P_1=P_2$.

M2) By the definition of absolute value

$|x_i-x_j|=|x_j-x_i|$,
$|y_i-y_j|=|y_j-y_i|$,
$|z_i-z_j|=|z_j-z_i|$

for all x_i, y_i, z_i,x_j, y_j, $z_j \in \mathbb{R}^3$and i,j=1,2,3. Therefore one can get $d_{DH}(P_1,P_2)=d_{DH}(P_2,P_1)$.

M3) Let $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ be any distinct three points in $\mathbb{R}^3$. Then

$$d_{DH}(P_1,P_3) = \max\left\{\begin{array}{c} |x_1-x_3|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y_1-y_3|+|z_1-z_3|, \\ (1-\varphi)|x_1-x_3|+(1+\varphi)|z_1-z_3|, \\ -|x_1-x_3|+\varphi|z_1-z_3|+ \\ (1+\varphi)|y_1-y_3| \end{array}\right\}, \\ |y_1-y_3|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |z_1-z_3|+|x_1-x_3|, \\ (1-\varphi)|y_1-y_3|+(1+\varphi)|x_1-x_3|, \\ -|y_1-y_3|+\varphi|x_1-x_3|+ \\ (1+\varphi)|z_1-z_3| \end{array}\right\}, \\ |z_1-z_3|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y_1-y_3|+|x_1-x_3|, \\ (1-\varphi)|z_1-z_3|+(1+\varphi)|y_1-y_3|, \\ -|z_1-z_3|+\varphi|y_1-y_3|+ \\ (1+\varphi)|x_1-x_3| \end{array}\right\} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} |x_1-x_2+x_2-x_3|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y_1-y_2+y_2-y_3|+|z_1-z_2+z_2-z_3|, \\ (1-\varphi)|x_1-x_2+x_2-x_3|+ \\ (1+\varphi)|z_1-z_2+z_2-z_3|, \\ -|x_1-x_2+x_2-x_3|+\varphi|z_1-z_2+z_2-z_3| \\ +(1+\varphi)|y_1-y_2+y_2-y_3| \end{array}\right\} \\ |y_1-y_2+y_2-y_3|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |z_1-z_2+z_2-z_3|+|x_1-x_2+x_2-x_3|, \\ (1-\varphi)|y_1-y_2+y_2-y_3|+ \\ (1+\varphi)|x_1-x_2+x_2-x_3|, \\ -|y_1-y_2+y_2-y_3|+\varphi|x_1-x_2+x_2-x_3| \\ +(1+\varphi)|z_1-z_2+z_2-z_3| \end{array}\right\} \\ |z_1-z_2+z_2-z_3|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y_1-y_2+y_2-y_3|+|x_1-x_2+x_2-x_3|, \\ (1-\varphi)|z_1-z_2+z_2-z_3|+ \\ (1+\varphi)|y_1-y_2+y_2-y_3|, \\ -|z_1-z_2+z_2-z_3|+\varphi|y_1-y_2+y_2-y_3| \\ +(1+\varphi)|x_1-x_2+x_2-x_3| \end{array}\right\} \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} |x_1-x_2|+|x_2-x_3|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y_1-y_2|+|y_2-y_3|+|z_1-z_2|+|z_2-z_3|, \\ (1-\varphi)(|x_1-x_2|+|x_2-x_3|)+ \\ (1+\varphi)(|z_1-z_2|+|z_2-z_3|), \\ -(|x_1-x_2|+|x_2-x_3|)+\varphi(|z_1-z_2|+|z_2-z_3|) \\ +(1+\varphi)(|y_1-y_2|+|y_2-y_3|) \end{array}\right\} \\ |y_1-y_2|+|y_2-y_3|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |z_1-z_2|+|z_2-z_3|+|x_1-x_2|+|x_2-x_3|, \\ (1-\varphi)(|y_1-y_2|+|y_2-y_3|)+ \\ (1+\varphi)(|x_1-x_2|+|x_2-x_3|), \\ -(|y_1-y_2|+|y_2-y_3|)+\varphi(|x_1-x_2|+|x_2-x_3|) \\ +(1+\varphi)(|z_1-z_2|+|z_2-z_3|) \end{array}\right\} \\ |z_1-z_2|+|z_2-z_3|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y_1-y_2|+|y_2-y_3|+|x_1-x_2|+|x_2-x_3|, \\ (1-\varphi)(|z_1-z_2|+|z_2-z_3|)+ \\ (1+\varphi)(|y_1-y_2|+|y_2-y_3|), \\ -(|z_1-z_2|+|z_2-z_3|)+\varphi(|y_1-y_2|+|y_2-y_3|) \\ +(1+\varphi)(|x_1-x_2|+|x_2-x_3|) \end{array}\right\} \end{array}\right\}$$

$$= I$$

One can easily find that $I \leq d_{DH}(P_1,P_2) + d_{DH}(P_2,P_3)$ from Lemma 2. 1.

So $d_{DH}(P_1,P_3) \leq d_{DH}(P_1,P_2)+d_{DH}(P_2,P_3)$. That is, d_{DH} distance function satisfies the triangle inequality. Consequently, the set

$$S_{DH} = \left\{\begin{array}{c} (x,y,z): d_{DH}(X,O) = \\ \max\left\{\begin{array}{c} |x|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y|+|z|, (1-\varphi)|x|+(1+\varphi)|z|, \\ -|x|+\varphi|z|+(1+\varphi)|y| \end{array}\right\} \\ |y|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |z|+|x|, (1-\varphi)|y|+(1+\varphi)|x|, \\ -|y|+\varphi|x|+(1+\varphi)|z| \end{array}\right\} \\ |z|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y|+|x|, (1-\varphi)|z|+(1+\varphi)|y|, \\ -|z|+\varphi|y|+(1+\varphi)|x| \end{array}\right\} \end{array}\right\} \\ = 1 \end{array}\right\}$$

is the set of all points $X=(x,y,z)\in\mathbb{R}^3$ deltoidal hexacontahedron distance is 1 from $O=(0,0,0)$. Thus the graph of S_{DH} is as in the figure

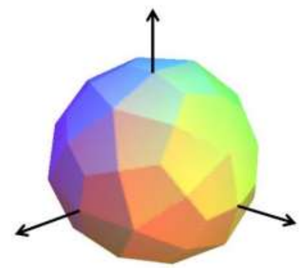

Figure 3. Deltoidal hexacontahedron

Corollary 2. 4: The equation of the deltoidal hexacontahedron with center $C=(x_0,y_0,z_0)$ and radius r is

$$\max\left\{\begin{array}{c} |x-x_0|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y-y_0|+|z-z_0|, \\ (1-\varphi)|x-x_0|+ \\ (1+\varphi)|z-z_0|, \\ -|x-x_0|+\varphi|z-z_0| \\ +(1+\varphi)|y-y_0| \end{array}\right\} \\ |y-y_0|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |z-z_0|+|x-x_0|, \\ (1-\varphi)|y-y_0|+ \\ (1+\varphi)|x_1-x_3|, \\ -|y-y_0|+\varphi|x-x_0| \\ +(1+\varphi)|z-z_0| \end{array}\right\} \\ |z-z_0|+(2\varphi-3) \\ \max\left\{\begin{array}{c} |y-y_0|+|x-x_0|, \\ (1-\varphi)|z_1-z_3|+ \\ (1+\varphi)|y-y_0|, \\ -|z-z_0|+\varphi|y-y_0| \\ +(1+\varphi)|x-x_0| \end{array}\right\} \end{array}\right\} =$$

Lemma 2. 5: Let l be the line through the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in theanalytical 3-dimensional space and d_E denotes the Euclidean metric. If l has direction vector(p, q, r), then $d_{DH}(P_1,P_2) = \mu(P_1P_2)d_E(P_1,P_2)$, where $\mu(P_1P_2)$ is equal to

$$\frac{max\left\{\begin{array}{c}|p|+(2\varphi-3)\\ max\left\{\begin{array}{c}|q|+|r|,(1-\varphi)|p|+(1+\varphi)|r|,\\ -|p|+\varphi|r|+(1+\varphi)|q|\end{array}\right\}\\ |q|+(2\varphi-3)\\ max\left\{\begin{array}{c}|r|+|p|,(1-\varphi)|q|+(1+\varphi)|p|,\\ -|q|+\varphi|p|+(1+\varphi)|r|\end{array}\right\}\\ |r|+(2\varphi-3)\\ max\left\{\begin{array}{c}|p|+|q|,(1-\varphi)|r|+(1+\varphi)|q|,\\ -|r|+\varphi|q|+(1+\varphi)|p|\end{array}\right\}\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}$$

Proof: Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R}$. Thus,

$$d_{DH}(P_1,P_2) = \lambda max\left\{\begin{array}{c}|p|+(2\varphi-3)\\ max\left\{\begin{array}{c}|q|+|r|,(1-\varphi)|p|+(1+\varphi)|r|,\\ -|p|+\varphi|r|+(1+\varphi)|q|\end{array}\right\}\\ |q|+(2\varphi-3)\\ max\left\{\begin{array}{c}|r|+|p|,(1-\varphi)|q|+(1+\varphi)|p|,\\ -|q|+\varphi|p|+(1+\varphi)|r|\end{array}\right\}\\ |r|+(2\varphi-3)\\ max\left\{\begin{array}{c}|p|+|q|,(1-\varphi)|r|+(1+\varphi)|q|,\\ -|r|+\varphi|q|+(1+\varphi)|p|\end{array}\right\}\end{array}\right\}$$

and $d_E(P_1,P_2) = \lambda\sqrt{p^2+q^2+r^2}$ which implies the required result.

The above lemma says that d_{DH}- distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 2. 6: If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then

$d_E(P_1,X) = d_E(P_2,X)$ if and only if $d_{DH}(P_1,X) = d_{DH}(P_2,X)$

Corollary 2. 7: If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensionalspace, then

$$d_{DH}(X, P_1) / d_{DH}(X, P_2) = d_E(X, P_1) / d_E(X, P_2) .$$

That is, the ratios of the Euclidean and d_{DH} distances along a line are the same.

3. PENTAKIS DODECAHEDRON

A pentakis dodecahedron is a Catalan solid. Its dual is the truncated icosahedron, an Archimedean solid. It can be seen as a dodecahedron with a pentagonal pyramid covering each face; that is, it is the Kleetope of the dodecahedron. A pentakis dodecahedron has 60 faces, 90 edges and 62 = 12 + 20 + 32 vertices([9]).

Figure 4. Pentakis dodecahedron

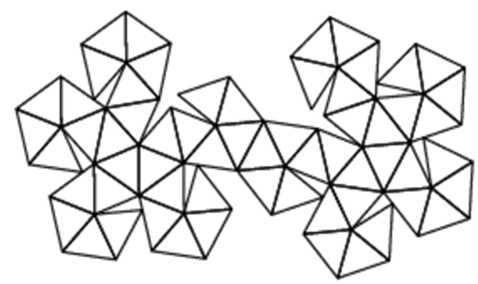

Figure 5. Net for pentakis dodecahedron

We describe the metric that unit sphere is Pentakis Dodecahedron metric as following:

Definition 3. 1:Let $P_1=(x_1,y_1z_1)$ and $P_2=(x_2,y_2,z_2)$ be distinct two points in $\mathbb{R}^3$. The distance function d_{PD}: $\mathbb{R}^3 x \mathbb{R}^3 \to [0,\infty)$Pentakis Dodecahedron metricdistance between P_1 and P_2 is defined by

$$d_{PD}(P_1,P_2) = max\left\{\begin{array}{c}|x_1-x_2|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|y_1-y_2|,2|y_1-y_2|+|z_1-z_2|+\\ \omega(|z_1-z_2|-|x_1-x_2|),\\ |y_1-y_2|+2|z_1-z_2|+\\ 2\omega(|z_1-z_2|-|x_1-x_2|)\end{array}\right\}\\ |y_1-y_2|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|z_1-z_2|,2|z_1-z_2|+|x_1-x_2|\\ +\omega(|x_1-x_2|-|y_1-y_2|),\\ |z_1-z_2|+2|x_1-x_2|+\\ 2\omega(|x_1-x_2|-|y_1-y_2|)\end{array}\right\}\\ |z_1-z_2|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|x_1-x_2|,2|x_1-x_2|+|y_1-y_2|\\ +\omega(|y_1-y_2|-|z_1-z_2|),\\ |x_1-x_2|+2|y_1-y_2|+\\ 2\omega(|y_1-y_2|-|z_1-z_2|)\end{array}\right\}\end{array}\right\} \quad (3)$$

where $\omega = (\sqrt{5}-1)/2$.

Pentakis dodecahedron distance function may seem a bit complicated. In fact there is an orientation in d_{PD} just as in Deltoidal hexacontahedron metric. Let $a =$

$|x_1 - x_2|, b = |x_1 - x_2|, c = |z_1 - z_2|$. This orientation is a − b − c − a. According to orientation, if one can put *b, c, a* instead of *a, b, c*, respectively, in first term of distance function, then it is obtained second term. Similarly, if one can put *c,a, b* instead of *a, b, c*, respectively, in first term of distance function, then it is obtained third term.

Lemma 3. 2: Let $P_1=(x_1,y_1z_1)$ and $P_2=(x_2,y_2,z_2)$ be any distinct two points in $\mathbb{R}^3$. Then

$$d_{PD}(P_1,P_2) \geq |x_1 - x_2| + \frac{\omega}{3} \max\left\{\begin{matrix} |y_1 - y_2|, \\ 2|y_1 - y_2| + |z_1 - z_2| + \omega(|z_1 - z_2| - |x_1 - x_2|), \\ |y_1 - y_2| + 2|z_1 - z_2| + 2\omega(|z_1 - z_2| - |x_1 - x_2|) \end{matrix}\right\}$$

$$d_{PD}(P_1,P_2) \geq |y_1 - y_2| + \frac{\omega}{3} \max\left\{\begin{matrix} |z_1 - z_2|, \\ 2|z_1 - z_2| + |x_1 - x_2| + \omega(|x_1 - x_2| - |y_1 - y_2|), \\ |z_1 - z_2| + 2|x_1 - x_2| + 2\omega(|x_1 - x_2| - |y_1 - y_2|) \end{matrix}\right\}$$

$$d_{PD}(P_1,P_2) \geq |z_1 - z_2| + \frac{\omega}{3} \max\left\{\begin{matrix} |y_1 - y_2|, 2|y_1 - y_2| + |z_1 - z_2| \\ +\omega(|z_1 - z_2| - |x_1 - x_2|), \\ |y_1 - y_2| + 2|z_1 - z_2| + \\ 2\omega(|z_1 - z_2| - |x_1 - x_2|) \end{matrix}\right\}$$

where $\omega = (\sqrt{5} - 1)/2$.

Proof: Proof is trivial by definition of maximum function.

Theorem 3. 3: The distance function d_{PD} is a metric of which unit sphere is a deltoidal hexacontahedron in $\mathbb{R}^3$.

Proof: Let d_{PD}: $\mathbb{R}^3 \mathrm{x}\ \mathbb{R}^3 \to \mathbb{R}$ and $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ distinct three points in $\mathbb{R}^3$. To prove that d_{PD} is a metric in $\mathbb{R}^3$, the following axioms can be supplied for all P_1, P_2 and $P_3 \in \mathbb{R}^3$.

M1) $d(P_1,P_2) \geq 0$ ve $d(P_1,P_2)=0 \Leftrightarrow P_1=P_2$

M2) $d(P_1,P_2)=d(P_2,P_1)$

M3) $d(P_1,P_3) \leq d(P_1,P_2)+d(P_2,P_3)$.

M1) Since absolute values is always nonnegative maximum of sums of absolute value is always nonnegative. Thus $d_{PD}(P_1,P_2) \geq 0$. If $d_{PD}(P_1,P_2)=0$ then three cases are possible.

Case I : If

$$d_{PD}(P_1,P_2) = |x_1 - x_2| + \frac{\omega}{3} \max\left\{\begin{matrix} |y_1 - y_2|, \\ 2|y_1 - y_2| + |z_1 - z_2| + \omega(|z_1 - z_2| - |x_1 - x_2|), \\ |y_1 - y_2| + 2|z_1 - z_2| + 2\omega(|z_1 - z_2| - |x_1 - x_2|) \end{matrix}\right\}$$

then,

$$d_{PD}(P_1,P_2)=0 \Leftrightarrow |x_1-x_2|=0, |y_1-y_2|=0, |z_1-z_2|=0 \Leftrightarrow x_1=x_2, y_1=y_2, z_1=z_2 \Leftrightarrow P_1=P_2.$$

The other cases can be easily shown by similar way in case i. Thus we get $d_{PD}(P_1,P_2)=0$ iff $P_1=P_2$.

M2) By the definition of absolute value

$|x_i-x_j|=|x_j-x_i|$,
$|y_i-y_j|=|y_j-y_i|$,
$|z_i-z_j|=|z_j-z_i|$

for all x_i, y_i, z_i,x_j, y_j, $z_j \in \mathbb{R}^3$and i,j=1,2,3. Therefore one can get $d_{PD}(P_1,P_2) = d_{PD}(P_2,P_1)$.

M3) Let $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ be any distinct three points in $\mathbb{R}^3$. Then by using well known property $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$,we have

$$d_{PD}(P_1,P_3) = \max\left\{\begin{matrix} |x_1 - x_3| + \frac{\omega}{3} \max\left\{\begin{matrix} |y_1 - y_3|, 2|y_1 - y_3| + |z_1 - z_3| \\ +\omega(|z_1 - z_3| - |x_1 - x_3|), \\ |y_1 - y_3| + 2|z_1 - z_3| \\ +2\omega(|z_1 - z_3| - |x_1 - x_3|) \end{matrix}\right\}, \\ |y_1 - y_3| + \frac{\omega}{3} \max\left\{\begin{matrix} |z_1 - z_3|, 2|z_1 - z_3| + |x_1 - x_3| \\ +\omega(|x_1 - x_3| - |y_1 - y_3|), \\ |z_1 - z_2| + 2|x_1 - x_2| \\ +2\omega(|x_1 - x_2| - |y_1 - y_2|) \end{matrix}\right\}, \\ |z_1 - z_3| + \frac{\omega}{3} \max\left\{\begin{matrix} |x_1 - x_3|, 2|x_1 - x_3| + |y_1 - y_3| \\ +\omega(|y_1 - y_3| - |z_1 - z_3|), \\ |x_1 - x_3| + 2|y_1 - y_3| \\ +2\omega(|y_1 - y_3| - |z_1 - z_3|) \end{matrix}\right\} \end{matrix}\right\}$$

$$\leq \max\left\{\begin{matrix} |x_1 - x_2| + |x_2 - x_3| + \frac{\omega}{3} \max\left\{\begin{matrix} |y_1 - y_2| + |y_2 - y_3|, \\ 2(|y_1 - y_2| + |y_2 - y_3|) + |z_1 - z_2| \\ +\omega(|z_1 - z_2| + |z_2 - z_3| - |x_1 - x_2| - |x_2 - x_3|), \\ |y_1 - y_2| + |y_2 - y_3| + 2(|z_1 - z_2| + |z_2 - z_3|) + \\ 2\omega(|z_1 - z_2| + |z_2 - z_3| - |x_1 - x_2| - |x_2 - x_3|) \end{matrix}\right\} \\ |y_1 - y_2| + |y_2 - y_3| + \frac{\omega}{3} \max\left\{\begin{matrix} |z_1 - z_2| + |z_2 - z_3|, \\ 2(|z_1 - z_2| + |z_2 - z_3|) + |x_1 - x_2| + |x_2 - x_3| + \\ \omega(|x_1 - x_2| - |y_1 - y_2| - |y_2 - y_3|), \\ |z_1 - z_2| + |z_2 - z_3| + 2(|x_1 - x_2| + |x_2 - x_3|) + \\ 2\omega(|x_1 - x_2| - |y_1 - y_2| - |y_2 - y_3|) \end{matrix}\right\} \\ |z_1 - z_2| + |z_2 - z_3| + \frac{\omega}{3} \max\left\{\begin{matrix} |x_1 - x_2| + |x_2 - x_3|, \\ 2(|x_1 - x_2| + |x_2 - x_3|) + |y_1 - y_2| + |y_2 - y_3| + \\ \omega(|y_1 - y_2| + |y_2 - y_3| - |z_1 - z_2| - |z_2 - z_3|), \\ |x_1 - x_2| + |x_2 - x_3| + 2(|y_1 - y_2| + |y_2 - y_3|) + \\ 2\omega(|y_1 - y_2| + |y_2 - y_3| - |z_1 - z_2| - |z_2 - z_3|) \end{matrix}\right\} \end{matrix}\right\}$$

$= I$

One can easily find that $I \leq d_{PD}(P_1,P_2) + d_{PD}(P_2,P_3)$ from Lemma 3. 2.

So $d_{PD}(P_1,P_3) \leq d_{PD}(P_1,P_2)+d_{PD}(P_2,P_3)$. That is, d_{DH} distance function satisfies the triangle inequality.

Consequently, the set

$$S_{PD}=\left\{\begin{array}{c}(x,y,z): d_{PD}(X,O)=\\ \left\{\begin{array}{c}|x|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|y|,2|y|+|z|+\omega(|z|-|x|),\\ |y|+2|z|+2\omega(|z|-|x|)\end{array}\right\}\\ |y|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|z|,2|z|+|x|+\omega(|x|-|y|),\\ |z|+2|x|+2\omega(|x|-|y|)\end{array}\right\}\\ |z|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|x|,2|x|+|y|+\omega(|y|-|z|),\\ |x|+2|y|+2\omega(|y|-|z|)\end{array}\right\}\end{array}\right\}\\ =1\end{array}\right\}$$

is the set of all points $X=(x,y,z)\in\mathbb{R}^3$ that pentakis dodecahedron distance is 1 from $O=(0,0,0)$. Thus the graph of S_{PD} is as in the figure 6:

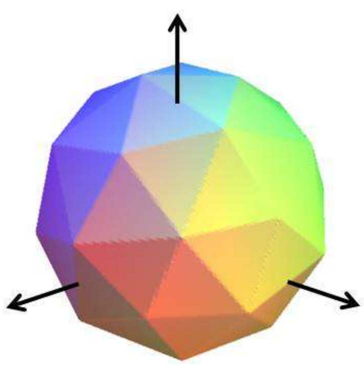

Figure 6. Pentakis dodecahedron

Corollary 3. 4: The equation of the pentakis dodecahedron with center $C=(x_0,y_0,z_0)$ and radius r is

$$max\left\{\begin{array}{c}\left\{\begin{array}{c}|x-x_0|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|y-y_0|,2|y-y_0|+|z-z_0|\\ +\omega(|z-z_0|-|x-x_0|),\\ |y-y_0|+2|z-z_0|\\ +2\omega(|z-z_0|-|x-x_0|)\end{array}\right\}\end{array}\right\},\\ \left\{\begin{array}{c}|y-y_0|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|z-z_0|,2|z-z_0|+|x-x_0|\\ +\omega(|x-x_0|-|y-y_0|),\\ |z-z_0|+2|x-x_0|\\ +2\omega(|x-x_0|-|y-y_0|)\end{array}\right\}\end{array}\right\},\\ \left\{\begin{array}{c}|z-z_0|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|x-x_0|,2|x-x_0|+|y-y_0|\\ +\omega(|y-y_0|-|z-z_0|),\\ |x-x_0|+2|y-y_0|\\ +2\omega(|y-y_0|-|z-z_0|)\end{array}\right\}\end{array}\right\}\end{array}\right\}=r$$

Lemma 3. 5: Let l be the line through the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in the

analytical 3-dimensional space and d_E denotes the Euclidean metric. If l has direction vector

(p, q, r), then $d_{PD}(P_1,P_2) = \mu(P_1P_2)d_E(P_1,P_2)$, where $\mu(P_1P_2)$ is equal to

$$\frac{max\left\{\begin{array}{c}\left\{\begin{array}{c}|p|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|q|,2|q|+|q|+\omega(|r|-|p|),\\ |q|+2|r|+2\omega(|r|-|p|)\end{array}\right\}\end{array}\right\},\\ \left\{\begin{array}{c}|q|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|r|,2|r|+|p|+\omega(|p|-|q|),\\ |r|+2|p|+2\omega(|p|-|q|)\end{array}\right\}\end{array}\right\},\\ \left\{\begin{array}{c}|r|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|p|,2|p|+|r|+\omega(|q|-|r|),\\ |p|+2|q|+2\omega(|q|-|r|)\end{array}\right\}\end{array}\right\}\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}$$

Proof: Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda\in\mathbb{R}$. Thus,

$$d_{PD}(P_1,P_2)=$$

$$\lambda max\left\{\begin{array}{c}\left\{\begin{array}{c}|p|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|q|,2|q|+|q|+\omega(|r|-|p|),\\ |q|+2|r|+2\omega(|r|-|p|)\end{array}\right\}\end{array}\right\},\\ \left\{\begin{array}{c}|q|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|r|,2|r|+|p|+\omega(|p|-|q|),\\ |r|+2|p|+2\omega(|p|-|q|)\end{array}\right\}\end{array}\right\},\\ \left\{\begin{array}{c}|r|+\frac{\omega}{3}\\ max\left\{\begin{array}{c}|p|,2|p|+|r|+\omega(|q|-|r|),\\ |p|+2|q|+2\omega(|q|-|r|)\end{array}\right\}\end{array}\right\}\end{array}\right\}$$

and $d_E(P_1, P_2) = \lambda\sqrt{p^2 + q^2 + r^2}$ which implies the required result.

The above lemma says that d_{PD}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 3. 6: If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then

$d_E(P_1,X) = d_E(P_2,X)$ if and only if $d_{PD}(P_1,X) = d_{PD}(P_2,X)$

Corollary 3. 7: If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then

$$d_{PD}(X, P_1) / d_{PD}(X, P_2) = d_E(X, P_1) / d_E(X, P_2) .$$

That is, the ratios of the Euclidean and d_{PD} distances along a line are the same.

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