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Sabit Katsayılı Pantograph Denkleminin Çözümü için Chebyshev Yaklaşım Metodu

Gamze YÜKSEL*, Mehmet SEZER
Muğla Üniversitesi, Fen Fakültesi, Matematik Bölümü / MUĞLA
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Özet: Bu çalışmada sabit katsayılı genelleştirilmiş pantograph denklemlerinin Chebyshev polinomlarını baz alarak yaklaşık çözümlerini bulmak için polinom yaklaşımına dayalı bir numeric metod sunulmuştur. Bu metod Chebyshev matris metodunun geliştirimiş bir halidir. Başlangıç koşullarına dayalı bazı problemler metodun doğruluğu ve etkinliği için verilmiştir. Ayrıca bulunan sonuçlar bilinen problemlerle karşılaştırılmıç ve çözümlerin doğruluğu ve hata analizi üzerine çalışmalar yapılmışyır.

Keywords: Pantograph Denklemleri; Chebyshev Matris Metodu; Chebyshev Polinomları.

A Chebyshev Approximate Method for Solving Constant Coefficients Pantograph Equations

Abstract: In this paper, a numerical method based on polynomial approximation, using Chebyshev polynomial basis, to obtain the approximate solution of generalized pantograph equations with constant coefficients is presented. The technique we have used is an improved Chebyshev matrix method. Some numerical examples, which consist of initial conditions, are given to illustrate the accuracy and efficiency of the method. Also, the results obtained are compared by the known results; the accuracy of solutions and the error analysis are performed.

Keywords: Pantograph Equations; Chebyshev Matrix Method; Chebyshev Polynomials.

Introduction

Functional-differential equations with proportional delays are usually referred to as pantograph equations or generalized equations. The name *pantograph* originated from the work of J.R. Ockendon and A.B. Tayler (Ockendon and Taylor, 1971) on the collection of current by the pantograph head of an electric locomotive. These equations arise in many applications such as number theory, electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics, cell growth, industrial applications and in studies based on biology, economy and electro-dynamics (Derfel and Iserles, 1997, Morris, Feldstein and Bowen, 1972, Ajello, Freedman and Wu, 1992, Mayers, Ockendon and Tayler ,1971, Buhmann and Iserles, 1993) Properties of the analytic solution of these equations as well as numerical methods have been studied by several authors. For example, equations with variable coefficients are treated in (Derfel and Iserles, 1997, Morris, Feldstein and Bowen, 1972, Derfel and Iserles , 1997, Feldstein and Liu , 1998). On the other hand, the Taylor matrix methods based on Taylor polynomials have been given to find approximate solutions of pantograph equations by Sezer et al (Sezer and Akyüz-Daşçıoğlu , 2007, Sezer, Yalçınbaş and Şahin, 2008, Sezer, Yalçınbaş and Gülsu, 2008).
In recent years, Chebyshev matrix and Chebyshev collocation methods have been given to find polynomial solutions of differential, integral and integro-differential equations by Sezer et al. (Sezer and Kaynak, 1996, Sezer and Doğan, 1996, Akyüz and Sezer, 1999) and Akyuz-Dascioglu et al. (Akyüz-Daşçıoğlu, 2006, Akyüz-Daşçıoğlu, 2004). Our purpose in this study is to develop and to apply the mentioned Chebyshev methods to the high-order pantograph equation with constant coefficients.

$$y^{(m)}(t)=\sum_{j=0}^{J}\sum_{k=0}^{m-1}P_{jk}y^{(k)}(\alpha_j t+\beta_j)+g(t)\ ,\ 0\le t\le 1 \qquad (1)$$

under the initial conditions

$$\sum_{k=0}^{m-1}c_{ik}y^{(k)}(0)=\lambda_i\ \ ,i=0,1,...,m-1 \qquad (2)$$

where $g(t)$ is analytical functions; P_{jk}, c_{ik}, λ_i, α_j and β_j are real or complex constants.

The aim of this study is to get solution as truncated Chebyshev series defined by

$$y(t)=\sum_{n=0}^{N}{}' a_n T_n^*(t)\,,$$

* ngamze@mu.edu.tr

$$T_n^*(t)=\cos(n\theta)\ ,\quad 2t-1=\cos\theta\quad ,\ 0\le t\le 1 \qquad (3)$$

where $T_n^*(t)$ denotes the shifted Chebyshev polynomials of the first kind; $\sum{}'$ denotes a sum whose first term is halved; a_n, $0\le n\le N$, are unknown Chebyshev coefficients, and N is chosen any positive integer such that $N\ge m$.

Fundamental Relations

Let us consider the pantograph equation (1) and find the matrix forms of each term in the equation. First we can convert the solution $y(t)$ defined by a truncated series (3) and its derivative $y^{(k)}(t)$ to matrix forms

$$y(t)=\mathbf{T}^*(t)\mathbf{A}$$

and

$$y^{(k)}(t)=\mathbf{T}^{*(k)}(t)\mathbf{A} \qquad (4)$$

where

$$\mathbf{T}^*(t)=\left[T_0^*(t)\,T_1^*(t)\ldots T_N^{\ *}(t)\right]$$

$$\mathbf{A}=[1/2a_0\ a_1\ldots a_N]^T$$

On the other hand, it is well known (Snyder,1966) that the relation between the powers t^n and the shifted Chebyshev polynomials $T_n^*(t)$ is

$$t^n=2^{-2n+1}\sum_{k=0}^{n}{}'\binom{2n}{k}T_{n-k}^*(t)\ ,\ 0\le t\le 1 \qquad (5)$$

By using the expression (5) and taking $n=0,1,\ldots,N$ we find the corresponding matrix relation as follows

$$\left(\mathbf{X}(t)\right)^T=\mathbf{D}\left(\mathbf{T}^*(t)\right)^T\Rightarrow \mathbf{X}(t)=\mathbf{T}^*(t)\mathbf{D}^{\mathbf{T}}$$

or

$$\mathbf{T}^*(t)=\mathbf{X}(t)\left(\mathbf{D}^{-1}\right)^{\mathbf{T}} \qquad (6)$$

where

$$\mathbf{X}(t)=\begin{bmatrix}1 & t & \cdots & t^N\end{bmatrix}$$

$$\mathbf{D}=\begin{bmatrix}
2^0\binom{0}{0} & 0 & 0 & 0 & \cdots & 0\\
2^{-2}\binom{2}{1} & 2^{-1}\binom{2}{2} & 0 & 0 & \cdots & 0\\
2^{-4}\binom{4}{2} & 2^{-3}\binom{4}{3} & 2^{-3}\binom{4}{4} & 0 & \cdots & 0\\
2^{-6}\binom{6}{3} & 2^{-5}\binom{6}{4} & 2^{-5}\binom{6}{5} & 2^{-5}\binom{6}{6} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & & \vdots\\
2^{-2N}\binom{2N}{N} & 2^{-2N+1}\binom{2N}{N+1} & 2^{-2N+1}\binom{2N}{N+2} & 2^{-2N+1}\binom{2N}{N+3} & \cdots & 2^{-2N+1}\binom{2N}{2N}
\end{bmatrix}$$

Also it follows from (6) that

$$\mathbf{T}^*(t)=\mathbf{X}(t)\left(\mathbf{D}^{-1}\right)^{\mathbf{T}}$$

and

$$\left(\mathbf{T}^*(t)\right)^{(k)}=\mathbf{X}^{(k)}(t)\left(\mathbf{D}^{-1}\right)^{\mathbf{T}}\ ,\ k=0,1,2,\ldots \qquad (7)$$

To obtain the matrix $X^{(k)}(t)$ in terms of the matrix $X(t)$, we can use the following procedure:

$$\begin{aligned}
&\mathbf{X}^{(1)}(t)=\mathbf{X}(t)\mathbf{B}^{\mathbf{T}}\\
&\mathbf{X}^{(2)}(t)=\mathbf{X}^{(1)}(t)\mathbf{B}^{\mathbf{T}}=\mathbf{X}(t)(\mathbf{B}^{\mathbf{T}})^2\\
&\mathbf{X}^{(3)}(t)=\mathbf{X}^{(2)}(t)\mathbf{B}^{\mathbf{T}}=\mathbf{X}(t)(\mathbf{B}^{\mathbf{T}})^3\\
&\vdots\\
&\mathbf{X}^{(k)}(t)=\mathbf{X}(t)(\mathbf{B}^{\mathbf{T}})^{\mathbf{k}}\ ,\ k=0,1,2,\ldots
\end{aligned} \qquad (8)$$

where

$$\mathbf{B}=\begin{bmatrix}
0 & 0 & 0 & . & . & . & 0\\
1 & 0 & 0 & . & . & . & 0\\
0 & 2 & 0 & . & . & . & 0\\
. & . & . & . & & & .\\
. & . & . & & . & & .\\
. & . & . & & & . & .\\
0 & 0 & 0 & . & . & \mathrm{N} & 0
\end{bmatrix}$$

Similarly, we have the matrix relations between $X^{(k)}(\alpha_j t+\beta_j)$ and $X(t)$ as follows

$$\begin{aligned}
&\mathbf{X}(t)=[1\ \ t\ \ \cdots\ \ t^N]\\
&\mathbf{X}(\alpha_j t+\beta_j)=\left[1\ \ \alpha_j t+\beta_j\ \ \cdots\ \ (\alpha_j t+\beta_j)^N\right]\\
&\mathbf{X}(\alpha_j t+\beta_j)=\mathbf{X}(t)\mathbf{B}(\alpha_j,\beta_j);
\end{aligned} \qquad (9)$$

$$\begin{aligned}
&\mathbf{X}^{(1)}(\alpha_j t+\beta_j)=\mathbf{X}(\alpha_j t+\beta_j)\mathbf{B}^{\mathbf{T}}\\
&\mathbf{X}^{(2)}(\alpha_j t+\beta_j)=\mathbf{X}(\alpha_j t+\beta_j)(\mathbf{B}^{\mathbf{T}})^2\\
&\vdots\\
&\mathbf{X}^{(k)}(\alpha_j t+\beta_j)=\mathbf{X}(\alpha_j t+\beta_j)(\mathbf{B}^{\mathbf{T}})^{\mathbf{k}}
\end{aligned} \qquad (10)$$

and, from (9) and (10),

$$\mathbf{X}^{(k)}(\alpha_j t+\beta_j)=\mathbf{X}(t)\mathbf{B}(\alpha_j,\beta_j)(\mathbf{B}^{\mathbf{T}})^{\mathbf{k}} \qquad (11)$$

where for $\alpha_j\ne 0$ and $\beta_j\ne 0$

$$\mathbf{B}(\alpha_j,\beta_j)=\begin{bmatrix} \binom{0}{0}(\alpha_j)^0(\beta_j)^0 & \binom{1}{0}(\alpha_j)^0(\beta_j)^1 & \cdots & \binom{N}{0}(\alpha_j)^0(\beta_j)^N \\ 0 & \binom{1}{1}(\alpha_j)^1(\beta_j)^0 & \cdots & \binom{N}{1}(\alpha_j)^1(\beta_j)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\alpha_j)^N(\beta_j)^0 \end{bmatrix}$$

and for $\alpha_j \neq 0, \beta_j = 0$

$$\mathbf{B}(\alpha_j,0)=\begin{bmatrix} (\alpha_j)^0 & 0 & \cdots & 0 \\ 0 & (\alpha_j)^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\alpha_j)^N \end{bmatrix}$$

Consequently, by substituting the matrix forms (7) and (8) into (4), we have the matrix relation

$$y^{(m)}(t)=\mathbf{X}(t)(\mathbf{B}^{\mathbf{T}})^{\mathbf{m}}(\mathbf{D}^{-1})^{\mathbf{T}}\mathbf{A} \quad (12)$$

and by means of (4), (7) and (11), the matrix relation

$$\begin{aligned} y^{(k)}(\alpha_j t+\beta_j) &= \mathbf{T}^{*(k)}(\alpha_j t+\beta_j)\mathbf{A} \\ &= \mathbf{X}^{(k)}(\alpha_j t+\beta_j)(\mathbf{D}^{-1})^{\mathbf{T}}\mathbf{A} \\ &= \mathbf{X}(t)\mathbf{B}(\alpha_j,\beta_j)(\mathbf{B}^{\mathbf{T}})^{\mathbf{k}}(\mathbf{D}^{-1})^{\mathbf{T}}\mathbf{A}\,,k=0,1,2,... \quad (13) \end{aligned}$$

Matrix Representation of the Function g(t)

The Taylor matrix representations of the non-homogenous term $g(t)$ of Eq.(1), respectively, can be written in the forms

$$[g(t)]=\sum_{n=0}^{N} g_n t^n=\mathbf{X}(t)\boldsymbol{G}\,, \quad g_n=\frac{g^{(n)}(0)}{n!}\,,$$

$$\boldsymbol{G}=\begin{bmatrix} g_o & g_1 & \cdots & g_N \end{bmatrix}^T$$

and therefore

$$[\boldsymbol{g}(\boldsymbol{t})]=\boldsymbol{X}(t)\boldsymbol{G} \quad (14)$$

Method of Solution

We now ready to construct the fundamental matrix equation corresponding to Eq. (1). For this purpose, substituting the matrix relations (12) , (13) and (14) into Eq. (1) and then simplifying, we obtain the fundamental matrix equation

$$(\mathbf{B}^{\mathbf{T}})^{\mathbf{m}}(\mathbf{D}^{-1})^{\mathbf{T}}\mathbf{A}=\sum_{j=0}^{J}\sum_{k=0}^{m-1}P_{jk}\mathbf{B}(\alpha_j,\beta_j)(\mathbf{B}^{\mathbf{T}})^{k}(\mathbf{D}^{-1})^{\mathbf{T}}\mathbf{A}+\boldsymbol{G} \quad (15)$$

Hence, the fundamental matrix equation (15) corresponding to Eq. (1) can be written in the form

$$\mathbf{WA}=\mathbf{G} \text{ or } [\mathbf{W};\mathbf{G}],$$

$$\mathbf{W}=[w_{nh}];\ n,h=0,\ 1,\ ...,\ N. \quad (16)$$

where

$$\mathbf{W}=(\mathbf{B}^T)^m(\mathbf{D}^{-1})^{\mathbf{T}}-\sum_{j=0}^{J}\sum_{k=0}^{m-1}\mathbf{P}_{jk}\mathbf{B}(\alpha_j,\beta_j)(\mathbf{B}^T)^m(\mathbf{D}^{-1})^{\mathbf{T}}$$

Here, Eq. (16) corresponds to a system of $(N+1)$ linear algebraic equations with unknown Chebyshev coefficients $a_0, a_1, \ldots, a_N$.

We can obtain the corresponding matrix forms for the conditions (2), by means of the relation (12),

$$\sum_{k=0}^{m-1} c_{ik}\mathbf{X}(0)(\mathbf{B}^T)^{(k)}(\mathbf{D}^{-1})^{\mathbf{T}}\mathbf{A}=[\lambda_i],\ i=0,1,2,...,m-1$$

On the other hand, the matrix form for conditions can be written as

$$\mathbf{U}_i\mathbf{A}=[\lambda_i] \text{ or } [\mathbf{U}_i;\lambda_i],\ i=0,1,2,...,m-1 \quad (17)$$

where

$$\mathbf{U}_i=\sum_{k=0}^{m-1}c_{ik}\mathbf{X}(0)(\mathbf{B}^T)^{(k)}(\mathbf{D}^{-1})^{\mathbf{T}}$$

$$\mathbf{U_i}=\begin{bmatrix} u_{i\,0} & u_{i\,1} & u_{i\,2} & \ldots & u_{i\,N} \end{bmatrix},$$

$$i=0,1,2,...,m-1$$

To obtain the solution of Eq. (1) under conditions (2), by replacing the row matrices (17) by the last m rows of the matrix (16), we have the new augmented matrix (Akyüz and Sezer, 1999, Akyüz-Daşçıoğlu, 2006, Sezer and Gülsu, 2005)

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{bmatrix} w_{0\,0} & w_{0\,1} & w_{0\,2} & \cdots & w_{0\,N} & ; & g_0 \\ w_{1\,0} & w_{1\,1} & w_{1\,2} & \cdots & w_{1\,N} & ; & g_1 \\ w_{2\,0} & w_{2\,1} & w_{2\,2} & \cdots & w_{2\,N} & ; & g_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{N-m\,0} & w_{N-m\,1} & w_{N-m\,2} & \cdots & w_{N-m\,N} & ; & g_{N-m} \\ u_{0\,0} & u_{0\,1} & u_{0\,2} & \cdots & u_{0\,N} & ; & \lambda_0 \\ u_{1\,0} & u_{1\,1} & u_{1\,2} & \cdots & u_{1\,N} & ; & \lambda_1 \\ u_{2\,0} & u_{2\,1} & u_{2\,2} & \cdots & u_{2\,N} & ; & \lambda_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{m-1\,0} & u_{m-1\,1} & u_{m-1\,2} & \cdots & u_{m-1\,N} & ; & \lambda_{m-1} \end{bmatrix} \quad (18)$$

If $\text{rank}\,\tilde{\mathbf{W}}=\text{rank}\,[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=N+1$, then we can write

$$\mathbf{A}=(\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}$$

Thus the matrix **A** (thereby the coefficients $a_0, a_1, \ldots, a_N$) is uniquely determined. Also the Eq.(1) with conditions (2) has a unique solution. This solution is given by truncated Chebyshev series (3).

On the other hand, when $|\tilde{\mathbf{W}}|=0$, if $\text{rank}\,\tilde{\mathbf{W}}=\text{rank}\,[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]<N+1$, then we may find a particular solution. Otherwise if $\text{rank}\tilde{\text{W}}\neq\text{rank}\left[\tilde{\text{W}};\tilde{\text{G}}\right]$, then it is not a solution.

Accuracy of Solution and Error Analysis

We can easily check the accuracy of the method. Since the truncated Chebyshev series (3) is an approximate solution of Eq.(1), when the solution $y_N(t)$ and its derivatives are substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for

$$t = t_q \in [0,1],\ q = 0,1,2,\ldots$$

$$E(t_q) = \left| y_N^{(m)}(t_q) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk} y_N^{(k)}(\alpha_j t_q + \beta_j) - g(t_q) \right| \cong 0 \quad (19)$$

and $E(t_q) \le 10^{-k_q}$ (k_q positive integer).

If max $10^{-k_q} = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E(t_q)$ at each of the points becomes smaller than the prescribed 10^{-k}. On the other hand, the error can be estimated by the function

$$E_N(t) = y_N^{(m)}(t) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk} y_N^{(k)}(\alpha_j t + \beta_j) - g(t) \quad (20)$$

If $E_N(t) \to 0$, when N is sufficiently large enough, then the error decreases.

Illustrations

The method of this study is useful in finding the solutions of pantograph equation with constant coefficients in terms of Chebyshev polynomials. We illustrate it by the following examples. For all the calculation and graphics, we have used Maple 12 and Matlab 7.7

Example 1: Consider the pantograph equation of second order (Evens and Raslan, 2005)

$$y''(t) = \frac{3}{4}y(t) + y(\frac{t}{2}) - t^2 + 2\ ,\ \ y(0) = 0\ ,\ \ y'(0) = 0\ ,\ \ 0 \le t \le 1$$

After the ordinary operations for N=4, the fundamental matrix equation and augmented matrix for the problem are gained as, respectively,

$$\left\{(\mathbf{B}^{\mathrm{T}})^2(\mathbf{D}^{-1})^{\mathrm{T}} - \mathbf{P}_{0,0}\mathbf{B}(1,0)(\mathbf{D}^{-1})^{\mathrm{T}} - \mathbf{P}_{1,0}\mathbf{B}(1/2,0)(\mathbf{D}^{-1})^{\mathrm{T}}\right\}\mathbf{A} = \mathbf{G}$$

and

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right] = \begin{bmatrix} -\frac{7}{4} & \frac{7}{4} & \frac{57}{4} & -\frac{377}{4} & \frac{1273}{4} & ; & 2 \\ 0 & -\frac{5}{2} & 10 & \frac{339}{2} & -1496 & ; & 0 \\ 0 & 0 & -8 & 48 & 1376 & ; & -1 \\ 1 & -1 & 1 & -1 & 1 & ; & 0 \\ 0 & 2 & -8 & 18 & -32 & ; & 0 \end{bmatrix}$$

where the last two rows indicates the augmented matrix of conditions $[\mathbf{U}_i;\lambda_i]$. Solving this system, we get Chebyshev coefficients

$$\mathbf{A} = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 \end{bmatrix}^T.$$

Hence , we obtain the approximate solution

$$y(t) = \sum_{n=0}^{4}{}' a_n T_n^*(t) = \frac{1}{2}a_0T_0^*(t) + a_1T_1^*(t) + a_2T_2^*(t) + a_3T_3^*(t) + a_4T_4^*(t)$$

$$= \frac{3}{8}(1) + \frac{1}{2}(2t-1) + \frac{1}{8}(8t^2 - 8t + 1) = t^2$$

and this is the exact solution.

Example 2: Consider the multi pantograph equation with constant coefficients (Muroya, Ishiwata and Brunner, 2003)

$$y'(t) = -y(t) + \frac{q}{2}y(qt) - \frac{q}{2}e^{-qt}\ ,\ \ y(0) = 1$$

which has exact solution $y(t) = e^{-t}$. In Table 1, it is compared the approximate solutions obtained by the present method for different values of q. In Table 2, the error analysis is performed and results are compared.

Example 3: Let us now consider the pantograph equation of third order (Liu and Li, 2004)

$$y'''(t) = -y(t) - y(t-0.3) + e^{-t+0.3}\ ,\ \ 0 \le t \le 1$$

$$y(0) = 1\ , y'(0) = -1,\ y''(0) = 1\ , y(t) = e^{-t}$$

In Table 3, we make a comparison between Adomian method (Evens and Raslan, 2005), Taylor series method and Chebyshev series method. It is seen from Table 3 that the Taylor and Chebyshev methods are not good as Adomian method for small N; but increasing N, the Taylor method and Chebyshev method are better than Adomian method. And also Chebyshev method is better than Taylor method throughout the interval $[0,1]$.

Table 1. *Comparison error functions for different values of Example 2*

t_i	Present Method N=7	
	$q=0.2$ E(t_i)	$q=0.5$ E(t_i)
0	0	0
0.2	2.5 E-9	3.0 E-9
0.4	3.251 E-7	3.25 E-7
0.6	5.5543 E-6	5.553 E-6
0.8	4.161 E-5	4.1606 E-5
1	1.9841 E-4	1.9839 E-4

t_i	$q=0.8$ E(t_i)	$q=0.9$ E(t_i)
0.1	1.0 E-9	0
0.3	4.4 E-8	4.3 E-8
0.5	1.545 E-6	1.532 E-6
0.7	1.625 E-5	1.6081 E-5
0.9	9.4237 E-5	9.3003 E-5

Table 2. *Error analysis of Example 2*

		$N=10$ $(q=0.2)$	
t_i	Muroya [20] $(q=0.2)$	Present Method	Sezer-Yalçınbaş-Gülsu Method [11]
2^{-1}	0.219 E-4	0.27 E-9	0.200 E-9
2^{-2}	0.108 E-5	0	0.100 E-9
2^{-3}	0.381 E-7	0	0.100 E-9
2^{-4}	0.126 E-8	0	0.100 E-9
2^{-5}	0.409 E-10	0	0.100 E-9
2^{-6}	0.120 E-11	0	0

Table 3. *Comparison of the absolute errors for Example 3*

t_i	Adomian method with six terms [19]	Taylor series method [9]		Chebyshev method (Present Method)	
		$N=5$	$N=17$	$N=5$	$N=17$
0	8.52 E-14	0.00	0.00	0	0
0.2	3.83 E-14	8.54 E-8	0.00	8.54 E-8	1.2 E-19
0.4	1.68 E-13	5.36 E-6	2.22 E-16	5.36 E-6	9.2 E-19
0.6	6.00 E-14	5.95 E-5	1.11 E-16	5.95 E-5	3.19 E-18
0.8	6.66 E-15	3.26 E-4	0.00	3.25 E-4	1.019 E-17
1	4.57 E-14	1.21 E-3	5.55 E-17	1.21 E-3	1.6252 E-16

Conclusions

A new Chebyshev method based on the truncated Chebyshev series is developed to numerically solve higher order pantograph equations with the initial conditions. This method is modified of the technique in refs. [12-16]. Besides, the obtained results in this paper are more better than results obtained by the other methods in the references throughout interval. Moreover, this method is applicable for the approximate solution of the pantograph-type Volterra functional integro-differential equations with variable delays, higher-order differential-difference and integro-differential-difference equations.

It is observed that the method has the best advantage when the known functions in equation can be expanded to Chebyshev series. In addition, generally throughout the interval [0,1], better results are obtained. To get the best approximation, we take more terms from the Chebyshev expansion of functions; that is, the truncation limit N must be chosen large enough.

Another considerable advantage of the method is that our N th order approximation gives the exact solution when the solution is polynomial of degree equal to or less than N. If the solution is not polynomial, Chebyshev series approximation converges to the exact solution as N increases.

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