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On Z-Symmetric Manifold Admitting Projective Curvature Tensor

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ABSTRACT

The object of the present paper is to study the Z-symmetric manifold with the projective curvature tensor. At first, we study the case of Z-tensor and projective Ricci tensor being of Codazzi type. Next, we consider recurrent Z-tensor and recurrent projective Ricci tensor. We also study the Z-symmetric manifold with projective curvature tensor with divergence-free Z-tensor. Finally, we construct an example of the Z-symmetric manifold with projective curvature tensor.

Keywords: *Projective curvature tensor, Z-symmetric tensor, Codazzi tensor, recurrent tensor.*
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1. Introduction

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be an n-dimensional Riemannian manifold. If there exists an one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. The projective curvature tensor P of type (0,4) is given by [1]

$$P(Y,Z,U,V) = R(Y,Z,U,V) - \frac{1}{n-1}[S(Z,U)g(Y,V) - S(Y,U)g(Z,V)] \tag{1.1}$$

where R is the curvature tensor and S is the Ricci tensor. Let $\{e_i, \quad i = 1,2,...,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then the Ricci tensor S of type (0,2) and the scalar curvature r are given by the following:

$$S(X,Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i) \quad and \quad r = \sum_{i=1}^{n} S(e_i, e_i) = \sum_{i=1}^{n} g(Qe_i, e_i)$$

where Q is the Ricci-operator i.e., $g(QX,Y) = S(X,Y)$.

Now, from (1.1), we have the following [2]:

$$\begin{aligned}
&(i) \sum_{i=1}^{n} P(e_i, Z, U, e_i) = \sum_{i=1}^{n} P(e_i, e_i, U, V) = \sum_{i=1}^{n} P(Y, Z, e_i, e_i) = 0 \\
&(ii) P(Y,Z,U,V) = -P(Z,Y,U,V) \\
&(iii) P(Y,Z,U,V) \neq -P(Y,Z,V,U) \\
&(iv) P(Y,Z,U,V) \neq P(U,V,Y,Z) \\
&(v) P(Y,Z,U,V) + P(Z,U,Y,V) + P(U,Y,Z,V) = 0.
\end{aligned} \tag{1.2}$$

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Also from (1.1), the projective Ricci tensor can be obtained as [3]

$$\bar{P}(Y,V)=\sum_{i=1}^{n}P(Y,e_i,e_i,V)=\frac{n}{n-1}\big[S(Y,V)-\frac{r}{n}g(Y,V)\big]. \tag{1.3}$$

By the aid of (1.3), we get the divergence of the projective Ricci tensor as the form

$$(div\bar{P})(Y)=\frac{n-2}{2(n-1)}(\nabla r)(Y). \tag{1.4}$$

In fact, M is projectively flat (that is, P = 0) if and only if the manifold is of constant curvature [4]. Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature. A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric [5-7] if R(X, Y).R = 0, where R is the Riemannian curvature tensor and R(X,Y) is considered as a derivation of the tensor algebra at each point of the manifold for the tangent vectors X,Y. If a Riemannian manifold satisfies R(X,Y).P = 0, then the manifold is said to be a projectively semi-symmetric manifold. In [5], it is proved that the projectively semi-symmetric spaces are also semi-symmetric.

It is interesting to note that the some curvatures restricted geometric structures formed by imposing a condition on P are equivalent to the similar structures formed by R. For example, (i) locally symmetric ($\nabla R = 0$) $\Leftrightarrow$ projectively symmetric ($\nabla P = 0$), (ii) recurrent ($\nabla R = \lambda R$) $\Leftrightarrow$ projectively recurrent ($\nabla P = \lambda P$), (iii) semisymmetric ($RR = 0$) $\Leftrightarrow$ projectively semisymmetric ($RP = 0$), (iv) pseudosymmetric ($RR = LQ(g,R)$) $\Leftrightarrow$ projectively pseudosymmetric ($RP = LQ(g,P)$), etc. For details about these facts we refer the reader to [8] and references therein. As R · P = 0 is equivalent to R · R = 0, the study of R · P = 0 is meaningless, however, many researchers (e.g. see [9-11], etc.) studied this condition in the context of the contact geometry.

It is noteworthy to mention that the different curvatures restricted geometric structures for P along with some additional assumptions were studied by several authors (see [10-13]).

A Riemannian manifold is said to have Codazzi type of Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfy the following condition, [14]

$$(\nabla_X S)(Y,W)=(\nabla_Y S)(X,W). \tag{1.5}$$

A non-flat Riemannian manifold is said to be generalized Ricci-recurrent manifold if its Ricci tensor is non-vanishing and satisfies the following condition, [15]

$$(\nabla_X S)(Y,Z)=\lambda(X)S(Y,Z)+\beta(X)g(Y,Z) \tag{1.6}$$

where λ and β are two non-zero 1-forms.

A non-flat Riemannian manifold is called a recurrent manifold [16] if the curvature tensor of this manifold satisfies the relation

$$(\nabla_W R)(X,Y,Z,U)=A(W)R(X,Y,Z,U) \tag{1.7}$$

where A is a non-zero 1-form. A non-flat Riemannian manifold is called a Ricci-recurrent manifold if the Ricci tensor of this manifold satisfies the relation([17-19])

$$(\nabla_X S)(Y,Z)=A(X)S(Y,Z) \tag{1.8}$$

where A is a non-zero 1-form.

This paper is organized as follows: In the first section of this paper, the definition of the projective curvature tensor of a Riemannian manifold is given. In the second section, the properties of the Z-symmetric tensor are considered. The third section deals with some theorems about the Z-symmetric tensor admitting projective curvature tensor. In the last section, an example for the existence of these manifolds is constructed.

2. The Z-Tensor on a Riemannian Manifold

The notion of the Z-tensor was introduced by Mantica and Molinari [20]. A $(0,2)$ symmetric tensor is generalized Z-tensor if it satisfies

$$Z_{kl}=S_{kl}+\phi g_{kl}, \tag{2.1}$$

where ϕ is an arbitrary scalar function. The scalar $\bar{Z}$ is the trace of the Z-tensor and from (2.1), it can be written as

$$\bar{Z} = g^{kl} Z_{kl} = r + n\phi. \tag{2.2}$$

The classical Z tensor is obtained with the choice $\phi = -\frac{1}{n}r$. Shortly, the generalized Z-tensor is called as the Z-tensor. In some cases, the Z-tensor gives the several well known structures on Riemannian manifolds. For example, i) If $Z_{kl} = 0$ (i.e, Z-flat) then this manifold reduces to an Einstein manifold, [21]; ii) If $\nabla_j Z_{kl} = \lambda_j Z_{kl}$ (Z-recurrent) then this manifold reduces to a generalized Ricci recurrent manifold [22]; iii) If $\nabla_j Z_{kl} = \nabla_k Z_{jl}$, (Codazzi tensor) then we find $\nabla_j S_{kl} - \nabla_k S_{jl} = \frac{1}{2(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)r$, [23]. This result gives us that this manifold is a nearly conformal symmetric manifold ($(NCS)_n$), [24]; iv) The relation between the Z-tensor and the energy-stress tensor of Einstein' s equations, [25], with cosmological constant Λ is $Z_{jl} = kT_{jl}$ where $\phi = -\frac{1}{2}r + \Lambda$ and k is the gravitational constant. In this case, the Z-tensor may be considered as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ. The vacuum solution (Z=0) determines an Einstein space with $\Lambda = \left(\frac{n-2}{2n}\right)r$; the conservation of total energy-momentum ($\nabla^l T_{kl} = 0$) gives $\nabla_j Z_{kl} = 0$ then this spacetime gives the conserved enery-momentum density.

This manifold has received a great deal of attention and is studied in considerable detail by many authors [26-33]. Motivated by the above studies, in the present, we examine the properties of the Z-tensor of a Riemannian manifold admitting the projective curvature tensor.

3. The Z-Symmetric Manifold Admitting the Projective Curvature Tensor

In this section, we consider the Z-symmetric manifold admitting the projective curvature tensor. In the local coordinates, from (1.1) and (2.1), the relation between the Z-tensor and the projective curvature tensor can be found as

$$P_{hijk} = R_{hijk} - \frac{1}{n-1}\big(Z_{ij}g_{hk} - Z_{hj}g_{ik} - \phi(g_{ij}g_{hk} - g_{hj}g_{ik})\big) \tag{3.1}$$

By taking the covariant derivative of (3.1), we can find

$$P_{hijk,l} = R_{hijk,l} - \frac{1}{n-1}\big(Z_{ij,l}g_{hk} - Z_{hj,l}g_{ik} - \phi_l(g_{ij}g_{hk} - g_{hj}g_{ik})\big). \tag{3.2}$$

Now, we have the following theorems:

Theorem 3.1. *Let (M,g) be of Codazzi type Z-symmetric tensor. A necessary and sufficient condition for the projective Ricci tensor to be divergence-free is that the scalar function ϕ of (M,g) be constant.*

Proof. Assume that (M,g) is of Codazzi type Z-symmetric tensor, then, we have from (1.5) and (2.1),

$$S_{ij,l} - S_{il,j} = \phi_j g_{il} - \phi_l g_{ij}. \tag{3.3}$$

Multiplying (3.3) by g^{ij}, we get

$$r_{,l} = 2(1-n)\phi_l. \tag{3.4}$$

Differentiating covariantly of the projective Ricci tensor given as the equation (1.3), we obtain

$$\bar{P}_{ij,l} = \frac{n}{n-2}(S_{ij,l} - \frac{r_{,l}}{n}g_{ij}). \tag{3.5}$$

Multiplying (3.5) by g^{ij}, it can be found

$$\bar{P}^j_{l,j} = \frac{n-2}{2(n-1)}r_{,l}. \tag{3.6}$$

Thus, if we use the equations (3.4) and (3.6), we can easily see that

$$\bar{P}^j_{l,j} = (2-n)\phi_l. \tag{3.7}$$

In this case, if the projective Ricci tensor is divergence-free, the scalar function ϕ must be constant. The converse is also true. Thus, the proof is completed. □

Theorem 3.2. *Let (M,g) be of Codazzi type projective Ricci tensor. A necessary and sufficient condition for the Z-symmetric tensor to be Codazzi type is that the scalar function ϕ of (M,g) be constant.*

Proof. Assume that the projective Ricci tensor of (M,g) is Codazzi type. Thus, we have from the equations (1.3) and (1.5)

$$S_{ij,k} - \frac{r_{,k}}{n} g_{ij} - S_{ik,j} + \frac{r_{,j}}{n} g_{ik} = 0. \tag{3.8}$$

Multiplying (3.8) by g^{ij}, we get

$$S^j_{k,j} = \frac{1}{n} r_{,k}. \tag{3.9}$$

Now, using the expression $S^j_{k,j} = \frac{1}{2} r_{,k}$, known as Ricci Identity, in (3.9), we find

$$r_{,k} = 0. \tag{3.10}$$

By the aid of the equations (1.5) and (2.1), if the Z-tensor is Codazzi type, we have

$$0 = Z_{ij,k} - Z_{ik,j} = S_{ij,k} - S_{ik,j} + \phi_k g_{ij} - \phi_j g_{ik}. \tag{3.11}$$

Thus, multiplying (3.11) by g^{ij}, the equation (3.11) reduces to

$$r_{,k} - S^j_{k,j} = (1-n)\phi_k. \tag{3.12}$$

In this case, comparing the equations (3.10) and (3.12), one can obtain

$$\phi_k = 0. \tag{3.13}$$

Conversely, from the equations (3.8), (3.10) and (3.11), if the equation (3.13) is satisfied then it can be obtained that the Z-tensor is Codazzi type. Thus, the proof is completed. □

Theorem 3.3. *Let the projective Ricci tensor of (M,g) be a Codazzi type tensor. The trace function of the Z-tensor is harmonic if and only if the 1-from ϕ_l generated by the scalar function ϕ is divergence-free.*

Proof. Assume that the projective Ricci tensor of (M,g) is Codazzi type. In this case, we have from (2.2) and (3.10)

$$\bar{Z}_{,k} = n\phi_k. \tag{3.14}$$

Hence, by taking the covariant derivative of (3.14), we find

$$\bar{Z}_{,kl} = n\phi_{k,l}. \tag{3.15}$$

Now, multiplying (3.15) by g^{kl}, we get

$$g^{kl}\bar{Z}_{,kl} = \Delta\bar{Z} = n\phi^l_{,l}. \tag{3.16}$$

In this case, if the trace of the Z-tensor is harmonic, then the vector field ϕ_l is divergence-free. The converse is also true. Thus, the proof is completed.

□

Theorem 3.4. *If the projective curvature tensor of (M,g) is recurrent tensor with the recurrence vector field λ_l then the Z-symmetric tensor is generalized recurrent as the form*

$$Z_{ij,l} = \lambda_l Z_{ij} + \beta_l g_{ij}$$

where $\beta_l = \frac{1}{n}(r_{,l} - \lambda_l r) + \phi_l - \lambda_l \phi$.

Proof. Let (M,g) be a Riemannian manifold with the recurrent projective curvature tensor admitting the recurrence vector field λ_l. By the aid of the equation (1.7), we get

$$P_{hijk,l} = \lambda_l P_{hijk}. \tag{3.17}$$

Thus, multiplying (3.17) by g^{ij} and using the equation (1.3), we have

$$S_{ij,l} = \lambda_l S_{ij} + \frac{1}{n}(r_{,l} - \lambda_l r)g_{ij}. \tag{3.18}$$

Now, putting the equation (3.18) in the equation (2.1), we find

$$Z_{ij,l} = \lambda_l S_{ij} + \frac{1}{n}(r_{,l} - \lambda_l r)g_{ij} + \phi_l g_{ij}. \tag{3.19}$$

Thus, we conclude from (3.19) that

$$Z_{ij,l} = \lambda_l Z_{ij} + \big(\frac{1}{n}(r_{,l} - \lambda_l r) + \phi_l - \lambda_l \phi\big)g_{ij}. \tag{3.20}$$

Hence, if we take $\beta_l = \frac{1}{n}(r_{,l} - \lambda_l r) + \phi_l - \lambda_l\phi$ in the equation (3.20) and if we use (1.6), then the Z-tensor reduces to a generalized recurrent tensor in the form $Z_{ij,l} = \lambda_l Z_{ij} + \beta_l g_{ij}$. Thus, the proof is completed. □

Theorem 3.5. *If the Z-tensor of (M,g) is a recurrent tensor with the recurrence vector field λ_l then the projective Ricci tensor is a recurrent tensor with the recurrence vector field λ_l.*

Proof. Assume that (M,g) is of the recurrent Z-tensor admitting the recurrence vector field λ_l. Thus, from (1.7) it can be written as

$$Z_{ij,k} = \lambda_k Z_{ij}. \tag{3.21}$$

Now, if we use the equations (3.2) and (3.21), we obtain

$$P_{hijk,l} = R_{hijk,l} - \frac{1}{n-1}\big(\lambda_l(Z_{ij}g_{hk} - Z_{hj}g_{ik}) - \phi_l(g_{ij}g_{hk} - g_{hj}g_{ik})\big). \tag{3.22}$$

Hence, multiplying the equation (3.22) by g^{ij}, we find

$$\bar{P}_{hk,l} = \frac{n}{n-1}\lambda_l(S_{hk} - \frac{r}{n}g_{hk}) = \lambda_l \bar{P}_{hk}. \tag{3.23}$$

It is clear from (3.23) that the projective Ricci tensor is also recurrent tensor. This completes the proof. □

Theorem 3.6. *Let the Z-tensor of (M,g) be divergence-free. A necessary and sufficient condition the projective Ricci tensor to be divergence-free is that the scalar function ϕ to be a constant.*

Proof. Let (M,g) be of divergence-free Z-tensor generated by the scalar function ϕ. Thus, we have

$$Z^l_{j,l} = 0. \tag{3.24}$$

Using the Ricci identity and the equation (2.1), the equation (3.24) takes the form

$$r_{,j} = -2\phi_j. \tag{3.25}$$

By the aid of (1.4) and (3.25), the divergence of the projective Ricci tensor is found as

$$\bar{P}^l_{j,l} = \frac{2-n}{n-1}\phi_j. \tag{3.26}$$

Now, we assume that the projective Ricci tensor is divergence-free, i.e., the condition

$$\bar{P}^l_{j,l} = 0 \tag{3.27}$$

is satisfied. Thus, from (3.26) and (3.27), we obtain

$$\phi_j = 0. \tag{3.28}$$

Conversely, if the equation (3.28) holds then it can be obtained from (1.4) that the projective Ricci tensor is divergence-free. Thus, the proof is completed. □

Theorem 3.7. *Assume that a manifold (M,g) with a constant scalar curvature admits the recurrent projective curvature tensor with the recurrence vector field* λ_l*. If the Z-symmetric tensor is recurrent with the same recurrence vector field* λ_l *then the vector fields* ϕ_l *and* λ_l *are parallel and they satisfy the following form*

$$\phi_l = (\frac{r}{n} + \phi)\lambda_l$$

.

Proof. Let us assume that (M,g) with a constant scalar curvature is of the recurrent projective curvature tensor with the recurrence vector field λ_l. Then, we have the equation (3.18). Also, assuming that the Z-symmetric tensor is a recurrent tensor with the recurrence vector field λ_l, from the equation (2.1) and (3.21), we obtain

$$S_{ij,l} + \phi_l g_{ij} = \lambda_l(S_{ij} + \phi g_{ij}). \tag{3.29}$$

Since (M,g) has a constant scalar curvature then we get from (3.18)

$$S_{ij,l} = \lambda_l S_{ij} - \frac{r}{n}\lambda_l g_{ij}. \tag{3.30}$$

So, comparing the equations (3.29) and (3.30), we find

$$\phi_l = (\frac{r}{n} + \phi)\lambda_l. \tag{3.31}$$

In this case, the proof is completed. □

4. An Example for the Existence of These Manifolds

We define a Riemannian metric on the 4-dimensional real number space $\mathbb{R}^4$ by the formula

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + e^{x^1}[e^{x^2}(dx^2)^2 + e^{x^3}(dx^3)^2 + e^{x^4}(dx^4)^2]. \tag{4.1}$$

Then, the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature are found as, respectively,

$$\Gamma^1_{22} = -\frac{e^{x^1+x^2}}{2}, \quad \Gamma^1_{33} = -\frac{e^{x^1+x^3}}{2}, \quad \Gamma^1_{44} = -\frac{e^{x^1+x^4}}{2}$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \Gamma^2_{22} = \Gamma^3_{33} = \Gamma^4_{44} = \frac{1}{2},$$

$$R_{1221} = \frac{e^{x^1} + x^2}{4}, \quad R_{1331} = \frac{e^{x^1} + x^3}{4}, \quad R_{1441} = \frac{e^{x^1} + x^4}{4},$$

$$R_{2332} = \frac{e^{2x^1} + x^2 + x^3}{4}, \quad R_{2442} = \frac{e^{2x^1} + x^2 + x^4}{4}, \quad R_{3443} = \frac{e^{2x^1} + x^3 + x^4}{4},$$

$$S_{11} = \frac{3}{4}, \quad S_{22} = \frac{3e^{x^1+x^2}}{4}, \quad S_{33} = \frac{3e^{x^1+x^3}}{4}, \quad S_{44} = \frac{3e^{x^1+x^4}}{4},$$

$$r = 3 \tag{4.2}$$

and the components obtained by the symmetry properties. Then, by using the equation (4.2), the only non-zero components of the Z-symmetric tensor are found as

$$Z_{11} = \frac{3}{4} + \phi, \quad Z_{22} = e^{x^1+x^2}(\frac{3}{4} + \phi),$$
$$Z_{33} = e^{x^1+x^3}(\frac{3}{4} + \phi), \quad Z_{44} = e^{x^1+x^4}(\frac{3}{4} + \phi) \tag{4.3}$$

By, using the equations (3.6) and (4.2), it can be obtained that the projective Ricci tensor satisfies the relation

$$\bar{P}_{ij,k} - \bar{P}_{ik,j} = 0 \qquad i = 1, 2, 3, 4 \tag{4.4}$$

Now, we will show that $\mathbb{R}^4$ given by the metric (4.1) satisfies the condition of Theorem 3.2. To verify the relation (3.8), it is sufficient to check that the following equations

$$Z_{ij,k} - Z_{ik,j} = 0 \qquad i = 1, 2, 3, 4 \tag{4.5}$$

are satisfied. Hence, by taking the covariant derivative of the Z-tensor in the form (4.3) and putting them in the equation (4.5), we find

$$Z_{11,j} = \phi_j = 0 \qquad j = 2, 3, 4$$

$$Z_{22,j} = e^{x^1+x^2}\phi_j = 0 \qquad j = 1, 3, 4$$

$$Z_{33,j} = e^{x^1+x^3}\phi_j = 0 \qquad j = 1, 2, 4$$

$$Z_{44,j} = e^{x^1+x^4}\phi_j = 0 \qquad j = 1, 2, 3 \tag{4.6}$$

It, it can be seen that the scalar function ϕ is independent of the coordinates x^1, x^2, x^3, x^4. Thus, the scalar function ϕ must be constant. Finally, we can say that this manifold endowed by the metric (4.1) is an example to satisfy Theorem 3.2.

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