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Authors: Hami MALKOÇ, Banu PAZAR VAROL

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Bipolar Fuzzy Supra Topological Spaces

Hami MALKOÇ[1], Banu PAZAR VAROL*[2]

Abstract

In the present work, we introduce bipolar fuzzy supra topological space as a generalization of fuzzy supra topological space, investigate the basic properties, give the concepts of interior and closure and encouraged them by examples and counterexamples. Moreover, we study the concepts of bipolar fuzzy supra continuity and S^* bipolar fuzzy supra continuity and see that composition of two S^* bipolar fuzzy supra continuous functions may not be a S^* bipolar fuzzy supra continuous function. Also, we attempt to define the concept of compactness on bipolar fuzzy supra topology.

Keywords: Bipolar fuzzy set, bipolar fuzzy supra topology, bipolar fuzzy supra continuity, bipolar fuzzy supra compactness.

1. INTRODUCTION

The fundamentals of theory of fuzzy sets were given by L. Zadeh [1] in 1965. Fuzzy generalizations of different mathematical concepts were introduced and studied such as interval valued fuzzy sets [2], intuitionistic fuzzy sets [3], fuzzy soft sets [4] etc. In 1994, Zhang [5] introduced the notion of a bipolar fuzzy set and in 2004, Lee [6] gave the definition of bipolar fuzzy sets as an extension of fuzzy sets. In bipolar fuzzy sets membership degree space is extended from the interval [0, 1] to [−1, 1]. In this set, the membership degree 0 produces that elements are irrelevant to the corresponding property, the membership degrees on (0, 1] assign that element somewhat supply the property and the membership degrees on [−1, 0) assign that elements somewhat supply the implicit counter property. At present, studies on bipolar fuzzy theory are very popular and have been applied in various fields. In 2013, M. S. Anitha et. al. [7] defined the concept of bipolar fuzzy subgroup and worked some properties. S.P. Subbian et. al. [8] studied on bipolar valued fuzzy ideals of ring in 2018. B. Pazar Varol [9] gave the definition of bipolar fuzzy submodule of a given classical module and investigated some fundamental properties in 2021. The concept of bipolar fuzzy topology was introduced by M. Azhagappan and M. Kamaraj [10] in 2016. Then, J. H. Kim et. al. [11] introduced the bipolar fuzzy neighborhood structure, base and subbase in 2019.

In 1983, Mashhour et al. [12] defined the concept of supra topology, which is the weaker version of classical topology and they also studied supra closed (open) sets and supra continuous functions.

* Corresponding author: banupazar@kocaeli.edu.tr
[1] Kocaeli University, Institute of Natural Sciences
E-mail: hami.malkoc.4161@gmail.com
ORCID: https://orcid.org/0000-0002-0321-0316
Kocaeli University, Faculty of Science and Literature, Department of Mathematics
ORCID: https://orcid.org/0000-0002-8627-7910

They called a family $\tau \subset \wp(X)$ a supra topology on X if it contains X and $\emptyset$ and is closed under arbitrary union. In 1987, Abd El-Monsef et al. [13] defined fuzzy supra topological space as a generalization of the concept of supra topological space and studied fundamental concepts of fuzzy supra topological spaces. Then, many authors studied supra topology and its extensions such as supra soft topology and fuzzy supra (soft) topology ([14], [15], [16], [17], [18], [19]). In this paper, we have given the definition of bipolar fuzzy supra topology and have investigated some concepts on bipolar fuzzy supra topology such as interior, closure, continuity and compactness. In bipolar fuzzy supra topological spaces we have lost some of the properties in bipolar fuzzy topological spaces, for example, the concept of closure (interior) does not have distributive property on union (intersection). We have showed these by the examples. We have defined S^* bipolar fuzzy supra continuity and seen that composition of two S^* bipolar fuzzy supra continuous functions may not be a S^* bipolar fuzzy supra continuous function.

2. PRELIMINARIES

Definition 2.1: [6] Let U be a nonempty set. A bipolar fuzzy set X in U is defined as

$X = \{< u, \mu_X^+(u), \mu_X^-(u) > \mid u \in U\}$

where $\mu_X^+ : U \longrightarrow [0,1]$ and $\mu_X^- : U \longrightarrow [-1,0]$ are two functions. $\mu_X^+(u)$ is called the positive membership degree of u and it denotes the satisfaction degree of u is to be the element of X. $\mu_X^-(u)$ is called the negative membership degree and it denotes the satisfaction degree of u to some implicit counter property of bipolar valued fuzzy set X.

The family of all bipolar fuzzy set in U is denoted by $BPF(U)$.

Example 2.2: $X = \{< a, 0.6, -0.4 >, < b, 0.8, -0.3 >, < c, 0.5, -0.5 >\}$ is a bipolar fuzzy set in $U = \{a, b, c\}$.

Definition 2.3: [10] 1. Universal bipolar fuzzy set is a bipolar fuzzy set on U and denoted by $\mathbf{1}_{BP} = (\mathbf{1}_{BP}^+, \mathbf{1}_{BP}^-)$ where for each $u \in U$, $\mathbf{1}_{BP}^+(u) = 1$ and $\mathbf{1}_{BP}^-(u) = -1$.

2. Bipolar fuzzy empty set is a bipolar fuzzy set on U and denoted by $\mathbf{0}_{BP} = (\mathbf{0}_{BP}^+, \mathbf{0}_{BP}^-)$ where for each $u \in U$, $\mathbf{0}_{BP}^+(u) = \mathbf{0}_{BP} = \mathbf{0}_{BP}^-(u)$.

Definition 2.4: [6] Let $X, Y \in BPF(U)$. Then,

1. $X \subseteq Y \Leftrightarrow \mu_X^+(u) \leq \mu_Y^+(u)$ and

$$\mu_X^-(u) \geq \mu_Y^-(u), \text{ for each } u \in U.$$

2. $X = Y \Leftrightarrow \mu_X^+(u) = \mu_Y^+(u)$ and $\mu_X^-(u) = \mu_Y^-(u)$, for each $u \in U$.

3. $X \cap Y = \{< x, \mu_{X\cap Y}^+(u), \mu_{X\cap Y}^-(u) > | u \in U\}$, where

$$\mu_{X\cap Y}^+(u) = \min\{\mu_X^+(u), \mu_Y^+(u)\}$$

and

$$\mu_{X\cap Y}^-(u) = \max\{\mu_X^-(u), \mu_Y^-(u)\}.$$

4. $X \cup Y = \{< x, \mu_{X\cup Y}^+(u), \mu_{X\cup Y}^-(u) > | u \in U\}$, where

$$\mu_{X\cup Y}^+(u) = \max\{\mu_X^+(u), \mu_Y^+(u)\}$$

and

$$\mu_{X\cup Y}^-(u) = \min\{\mu_X^-(u), \mu_Y^-(u)\}.$$

5. The complement of a bipolar fuzzy set X is defined as,

$X^c = \{< u, 1 - \mu_X^+(u), -1 - \mu_X^-(u) > | u \in U\}$.

Proposition 2.5: [11] Let X, Y and W be bipolar fuzzy sets in common universal set U. Then followings are satisfied.

1. $X \cup X = X$ and $X \cap X = X$.

2. $X \cup Y = X \cup Y$ and $X \cap Y = Y \cap X$.

2. $X \cup (Y \cup Z) = (X \cup Y) \cup Z$ and

$X \cap (Y \cap Z) = (X \cap Y) \cap Z$.

3. $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ and

$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

4. $X \cup (X \cap Y) = X$ and $X \cap (X \cup Y) = X$.

5. $X \cap Y \subset X$ and $X \cap Y \subset Y$.

6. $X \subset X \cup Y$ and $Y \subset X \cup Y$.

7. $(X^c)^c = X$.

8. $(X \cup Y)^c = X^c \cap Y^c$ and $(X \cap Y)^c = X^c \cup Y^c$.

9. If $X \subset Y$ and $Y \subset Z$ then $X \subset Z$.

10. If $X \subset Y$, then

$X \cap Z \subset Y \cap Z$ and $X \cup Z \subset Y \cup Z$.

Definition 2.6: [6] Let U be a nonempty set and $(X_j)_{j\in J}$ be a family of some bipolar fuzzy sets in U.

1. The intersection of $(X_j)_{j\in J}$ is a bipolar fuzzy set in U, denoted by $\bigcap_{j\in J} X_j$ and defined by

$$\left(\bigcap_{j\in J} X_j\right)(u) = \left(\bigwedge_{j\in J} \mu_X^+(u), \bigvee_{j\in J} \mu_X^-(u)\right)$$

for each $u \in U$.

2. The union of $(X_j)_{j\in J}$ is a bipolar fuzzy set in U, is denoted by $\bigcup_{j\in J} X_j$ and defined by

$$\left(\bigcup_{j\in J} X_j\right)(u) = \left(\bigvee_{j\in J} \mu_X^+(u), \bigwedge_{j\in J} \mu_X^-(u)\right)$$

for each $u \in U$.

Corollary 2.7: [11] Let U be a nonemty set, $X \in BPF(U)$ and $(X_j)_{j\in J} \subset BPF(U)$. Then followings are satisfied:

1. (Generalized distributive laws):

$X \cup \left(\bigcap_{j\in J} X_j\right) = \bigcap_{j\in J}(X_j \cup X)$,

$X \cap \left(\bigcup_{j\in J} X_j\right) = \bigcup_{j\in J}(X_j \cap X)$.

2. (Generalized De Morgan's laws):

$\left(\bigcup_{j\in J} X_j\right)^c = \bigcap_{j\in J} {X_j}^c$,

$\left(\bigcap_{j\in J} X_j\right)^c = \bigcup_{j\in J} {X_j}^c$.

Definition 2.8: [11] Let $g: U \longrightarrow V$ be a mapping and $X \in BPF(U)$, $Y \in BPF(V)$.

1. The image of X under g, denoted by $g(X)(v) = \left(\mu_{g(X)}^+(v), \mu_{g(X)}^-(v)\right) = \left(g(\mu_X^+)(v), g(\mu_X^-)(v)\right)$, is a bipolar fuzzy set in V defined as follows.

$$g(\mu_X^+)(v) = \begin{cases} \bigvee \mu_X^+(u), & u \in g^{-1}(v), \\ 0, & other \end{cases}$$

$$g(\mu_X^-)(v) = \begin{cases} \bigwedge \mu_X^-(u), & u \in g^{-1}(v), \\ 0, & other \end{cases}, \ \forall\, v \in V.$$

2. The preimage of Y under g, denoted by $g^{-1}(Y) = \left(g^{-1}(\mu_Y^+),\ g^{-1}(\mu_Y^-)\right)$, is a bipolar fuzzy set in U defined as follows.

$$[g^{-1}(\mu_Y^+)](u) = \mu_Y^+ \circ g(u)$$

and

$$[g^{-1}(\mu_Y^-)](u) = \mu_Y^- \circ g(u),\ \forall\, u \in U.$$

Corollary 2.9: [11] Let $f: U \longrightarrow V$ be a mapping and X, $X_1, X_2 \in BPF(U)$, $(X_j)_{j\in J} \subset BPF(U)$, Y, $Y_1, Y_2 \in BPF(V)$ and $(Y_j)_{j\in J} \subset BPF(V)$. Then the followings are satisfied.

1. If $X_1 \subset X_2$ then $f(X_1) \subset f(X_2)$.

2. $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$,

$f\left(\bigcup_{j\in J} X_j\right) = \bigcup_{j\in J} f(X_j)$.

3. $f(X_1 \cap X_2) \subset f(X_1) \cap f(X_2)$,

$f\left(\bigcap_{j\in J} X_j\right) \subset \bigcap_{j\in J} f(X_j)$.

4. If f is one to one, then

$f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$,

$f\left(\bigcap_{j\in J} X_j\right) = \bigcap_{j\in J} f(X_j)$.

5. If $Y_1 \subset Y_2$, then $f^{-1}(Y_1) \subset f^{-1}(Y_2)$.

6. $f(X) = \mathbf{0}_{BP} \Leftrightarrow X = \mathbf{0}_{BP}$.

7. $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$,

$f^{-1}\left(\bigcup_{j\in J} Y_j\right) = \bigcup_{j\in J} f^{-1}\left(Y_j\right)$.

8. $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$,

$f^{-1}\left(\bigcap_{j\in J} Y_j\right) = \bigcap_{j\in J} f^{-1}\left(Y_j\right)$.

9. $f^{-1}(Y) = \mathbf{0}_{BP} \Leftrightarrow Y \cap f(\mathbf{1}_{BP}) = \mathbf{0}_{BP}$.

10. $X \subset (f^{-1}of)(X)$, the equality holds if

f is injective.

11. $(fof^{-1})(Y) \subset Y$, the equality holds if

f is surjective.

12. $f^{-1}(Y^c) = \left(f^{-1}(Y)\right)^c$.

13. If f: $U \longrightarrow V$ and g: $V \longrightarrow W$ are two functions, then $(gof)(X) = (g(f(X))$ for every $X \in BPF(U)$.

14. If $f: U \longrightarrow V$ and $g: V \longrightarrow W$ are two functions, then $(gof)^{-1}(Y) = f^{-1}(g^{-1}(Y))$ for every $Y \in BPF(W)$.

Definition 2.10: [10] Let U be a nonempty set and $\tau \subset BPF(U)$. Then τ is called a bipolar fuzzy topology on U if it satisfies the following conditions.

1. $\mathbf{0}_{BP}$, $\mathbf{1}_{BP} \in \tau$.

2. $X \cap Y \in \tau$, for $X, Y \in \tau$.

3. $\left(\bigcup_{j\in J} X_j\right) \in \tau$, for every $\left(X_j\right)_{j\in J} \subset \tau$.

3. BIPOLAR FUZZY SUPRA TOPOLOGICAL SPACES

Definition 3.1: Let U be a nonempty set and $\tau \subset BPF(U)$. Then τ is called a bipolar fuzzy supra topology on U if it satisfies the following conditions.

1. $\mathbf{0}_{BP}$, $\mathbf{1}_{BP} \in \tau$.

2. τ is closed under arbitrary union.

In this case the pair of (U, τ) is called bipolar fuzzy supra topological space. Members of τ are named bipolar fuzzy supra open sets and members whose complements are belong to τ are called bipolar fuzzy supra closed sets. We will denote the family of all bipolar fuzzy supra topologies on U as $BPFST(U)$.

Let τ be a bipolar fuzzy supra topology on U. Then τ^* is called associated bipolar fuzzy topology with the $\tau :\Leftrightarrow \tau^* \subset \tau$.

From definition of bipolar fuzzy supra topological spaces it is clear that every bipolar fuzzy topological space is a bipolar fuzzy supra topological space.

Remark 3.2: There is no need to satisfy the arbitrary intersection property in bipolar fuzzy supra topological spaces. For example, let $U = \{a, b\}$, X, $Y \in BPF(U)$ and $\tau = \{\mathbf{0}_{BP}, \mathbf{1}_{BP}, X, Y\}$. Here,

$X(a) = (1; -0{,}1)$, $X(b) = (0{,}5; -1)$ and

$Y(a) = (0{,}6; -1)$, $Y(b) = (1; -0{,}5)$.

Therefore

$(X \cup Y)(a) = (1; -1) \in \tau$ and

$(X \cup Y)(b) = (1; -1) \in \tau$

but since

$(X \cap Y)(a) = (0{,}6; -0{,}1) \notin \tau$ and

$(X \cap Y)(b) = (0{,}5; -0{,}5) \notin \tau$

then we have what we desire.

Definition 3.3: Let τ and σ be two bipolar fuzzy supra topological spaces on U. If $\tau \subset \sigma$, then we say that τ is coarser than σ or σ is finer than τ and it's denoted by $\tau \preccurlyeq \sigma$.

Example 3.4:

1. Let U be a nonempty set and $\tau^0 = \{\mathbf{0}_{BP}, \mathbf{1}_{BP}\}$. In this case, τ^0 is a bipolar fuzzy supra topology on U. τ^0 is called indiscrete bipolar fuzzy supra topology and (U, τ^0) is called indiscrete bipolar fuzzy supra topological space.

2. Let U be a nonempty set and $\tau^1 = BPF(U)$. In this case, τ^1 is a bipolar fuzzy supra topology on X. τ^1 is called discrete bipolar fuzzy supra topology and (U, τ^1) is called discrete bipolar fuzzy supra topological space.

3. Let (U, τ) be a bipolar fuzzy supra topological space. The families

$\tau^+ = \{\mu_X^+ \in I^U | X \in \tau\}$ and

$\tau^- = \{-\mu_X^- \in I^U | X \in \tau\}$

are two fuzzy supra topologies in the sense of Abd El Monsef and Ramadan [13].

Proposition 3.5: Let (U, τ) be a bipolar fuzzy supra topological space and $X \in BPF(U)$. Then the set $\tau_X = \{X \cap O | O \in \tau\}$ is a bipolar fuzzy supra topology on X.

Proof: Since $\mathbf{0}_{BP}, \mathbf{1}_{BP} \in \tau$ then we have

$X \cap \mathbf{0}_{BP} = \mathbf{0}_{BP}$ and $X \cap \mathbf{1}_{BP} = X$.

2. Let J be an index set and $Y_j \in \tau_X$ for each $j \in J$. Then by the definition of τ_X there exists a $Z_j \in \tau$ such that $Y_j = X \cap Z_j$ for each $j \in J$. Since τ is a bipolar fuzzy supra topology, we have $\bigcup_{j\in J} Z_j \in \tau$. Thus

$\left(\bigcup_{j\in J} Z_j\right) \cap X = \bigcup_{j\in J}\left(Z_j \cap X\right) = \bigcup_{j\in J} Y_j$.

By the definition of τ_X, $\bigcup_{j\in J} Y_j \in \tau_X$.

The set of τ_X is called bipolar fuzzy supra topology induced by X and the pair of (X, τ_X) is called bipolar fuzzy supra subspace.

Theorem 3.6: Let (U, τ) be a bipolar fuzzy supra topological space and $X \in BPF(U)$. Then, the family

$$\eta_\tau = \{X \in BPF(U) | X \cap Y \in \tau,\ for\ \forall\ Y \in \tau\}$$

is a bipolar fuzzy topology on U and $\eta_\tau \subseteq \tau$.

Proof:

(T1) For $X \in \tau$ we have $X \cap \mathbf{1}_{BP} = X \in \tau$ and $X \cap \mathbf{0}_{BP} = \mathbf{0}_{BP} \in \tau$. Therefore $\mathbf{0}_{BP}, \mathbf{1}_{BP} \in \eta_\tau$.

(T2) Let X_1, $X_2 \in \eta_\tau$ and $Y \in \tau$. We have $(X_1 \cap X_2) \cap X = X_1 \cap (X_2 \cap Y)$. Since $X_1 \in \eta_\tau$ and $Y \in \tau$, $X_1 \cap Y \in \tau$ and since $X_2 \in \eta_\tau$ and $Y \in \tau$, $X_2 \cap Y \in \tau$. So, $X_1 \cap (X_2 \cap Y) \in \tau$. Hence, $X_1 \cap X_2 \in \eta_\tau$.

(T3) Let $\{X_i | i \in J\} \subset \eta_\tau$ and $Y \in \tau$. Then for each $i \in J$, we have $(\bigcup_{i\in J} X_i) \cap Y = \bigcup_{i\in J}(X_i \cap Y)$.

Since $\{X_i | i \in J\} \subset \eta_\tau$ and $Y \in \tau$, we get $X_i \cap Y \in \tau$. τ is bipolar fuzzy supra topology, so

$\bigcup_{i\in J}(X_i \cap Y) = (\bigcup_{i\in J} X_i) \cap Y \in \tau$.

Hence $\bigcup_{i\in J} X_i \in \eta_\tau$.

Let $K \in \eta_\tau$. Since $\mathbf{1}_{BP} \in \tau$, $K \cap \mathbf{1}_{BP} = K \in \tau$ then $K \in \tau$. Therefore $\eta_\tau \subseteq \tau$.

Theorem 3.7: Let $\mathscr{F}$ be the family of all bipolar fuzzy supra closed sets in the bipolar fuzzy supra topological space (U, τ). Then the followings are true.

i. $\mathbf{0}_{BP}, \mathbf{1}_{BP} \in \mathscr{F}$.

ii. Let $X_i \in \mathscr{F}$ for each $i \in I$. Then $\bigcap_{i\in I} X_i \in \mathscr{F}$.

Proof: Straightforward.

Theorem 3.8: Let (U, τ_1) and (U, τ_2) be two bipolar fuzzy topological spaces. Then $(U, \tau_1 \cap \tau_2)$ is a bipolar fuzzy supra topological space.

Proof: Straightforward.

Definition 3.9: Let (U, τ) be a bipolar fuzzy supra topological space and $X \in BPF(U)$.

i. The union of all bipolar fuzzy supra open sets that contained in X is called the bipolar fuzzy supra interior of X and denoted by $int_\tau(X)$. Then, $int_\tau(X) = \bigcup\{O \subseteq X | O \in \tau\}$.

ii. The intersection of all bipolar fuzzy supra closed sets that contains X is called the bipolar fuzzy supra closure of X and denoted by $cl_\tau(X)$. Then, $cl_\tau(X) = \bigcap\{X \subseteq K | K^c \in \tau\}$.

Theorem 3.10: Let $X, Y \in BPF(U)$ and (U, τ) be a bipolar fuzzy supra topological space. Then followings are satisfied.

1. X is a bipolar fuzzy supra open (closed) set $\Leftrightarrow$ $X = int_\tau(X)$ ($X = cl_\tau(X)$).

2. If $X \subseteq Y$, then $int_\tau(X) \subseteq int_\tau(Y)$ and $cl_\tau(X) \subseteq cl_\tau(Y)$.

3. $cl_\tau(X) \cup cl_\tau(Y) \subseteq cl_\tau(X \cup Y)$.

4. $int_\tau(X) \cup int_\tau(Y) \subseteq int_\tau(X \cup Y)$.

5. $int_\tau(X \cap Y) \subseteq int_\tau(X) \cap int_\tau(Y)$.

6. $cl_\tau(X \cap Y) \subseteq cl_\tau(X) \cap cl_\tau(Y)$.

7. $int_\tau(\mathbf{1_{BP}} - X) = \mathbf{1_{BP}} - cl_\tau(X)$.

8. $cl_\tau(\mathbf{1_{BP}}) = \mathbf{1_{BP}} = int_\tau(\mathbf{1_{BP}})$ and $cl_\tau(\mathbf{0_{BP}}) = \mathbf{0_{BP}} = int_\tau(\mathbf{0_{BP}})$.

9. $int_\tau(int_\tau(X)) = int_\tau(X)$, $cl_\tau(cl_\tau(X)) = cl_\tau(X)$.

Proof: We only give here proofs of (3.), (5.) and (7.). The others can be proved in the same way.

3. Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$ and from (2.) we have $cl_\tau(X) \subseteq cl_\tau(X) \cup cl_\tau(Y)$ and $cl_\tau(Y) \subseteq cl_\tau(X) \cup cl_\tau(Y)$. Therefore $cl_\tau(X) \cup cl_\tau(Y) \subseteq cl_\tau(X \cup Y)$.

5. Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$ and from (2.) we have $int_\tau(X \cap Y) \subseteq int_\tau(X)$ and $int_\tau(X \cap Y) \subseteq int_\tau(Y)$. Therefore, we have $int_\tau(X \cap Y) \subseteq int_\tau(X) \cap int_\tau(Y)$.

7. $int_\tau(\mathbf{1_{BP}} - X)$

$= \bigcup\{K^c | K^c \; is \; closed \; in \; U \; and \; K^c \subseteq X^c\}$

$= \mathbf{1_{BP}} - cl_\tau(X)$.

Remark 3.11: In bipolar fuzzy topological spaces the followings are true.

$int_\tau(A \cap B) = int_\tau(A) \cap int_\tau(B)$ and

$cl_\tau(A) \cup cl_\tau(B) = cl_\tau(A \cup B)$.

But in bipolar fuzzy supra topological spaces there is no need to be true these two properties.

Example 3.12:

Let $U = \{a, b\}$ and $\tau = \{\mathbf{0_{BP}}, \mathbf{1_{BP}}, X, Y\}$. Here X and Y are two bipolar fuzzy sets on U such that;

$X=\{\langle a, 1, -0.4\rangle, \langle b, 0.3, -1\rangle\}$,

$Y = \{\langle a, 0.7, -1\rangle, \langle b, 1, -0.7\rangle\}$.

Therefore, the complements of these two bipolar fuzzy sets are;

$X^c=\{\langle a, 0, -0.6\rangle, \langle b, 0.7, 0\rangle\}$,

$Y^c = \{\langle a, 0.3, 0\rangle, \langle b, 0, -0.3\rangle\}$.

The family of all bipolar fuzzy supra closed sets is; $\tau^c = \{\mathbf{0_{BP}}, \mathbf{1_{BP}}, X^c, Y^c\}$.

Let us define two bipolar fuzzy sets such that;

$Z=\{\langle a, 0, -0.5\rangle, \langle b, 0.1, 0\rangle\}$,

$T = \{\langle a, 0.2, 0\rangle, \langle b, 0, -0.2\rangle\}$.

Therefore

$Z \cup T = \{\langle a, 0.2, -0.5\rangle, \langle b, 0.1, -0.2\rangle\}$.

Since $Z \subseteq \mathbf{1_{BP}}, X^c$ and $T \subseteq \mathbf{1_{BP}}, Y^c$ then

$cl_\tau(Z) = \mathbf{1_{BP}} \cap X^c = \{\langle a, 0, -0.6\rangle, \langle b, 0.7, 0\rangle\}$,
$cl_\tau(T) = \mathbf{1_{BP}} \cap Y^c = \{\langle a, 0.3, 0\rangle, \langle b, 0, -0.3\rangle\}$.

and

$cl_\tau(Z) \cup cl_\tau(T) = \{\langle a, 0.3, -0.6\rangle, \langle b, 0.7, -0.3\rangle\}$.

Since $Z \cup T \subseteq \mathbf{1_{BP}}$ then $cl_\tau(Z \cup T) = \mathbf{1_{BP}} = \{\langle a, 1, -1\rangle, \langle b, 1, -1\rangle\}$.

Thus we clearly have $cl_\tau(Z \cup T) \neq cl_\tau(Z) \cup cl_\tau(T)$.

Example 3.13: Let us consider the same topology and the same set U in the previous example and define two bipolar fuzzy sets as follows:

$G = \{\langle a, 1, -0.5\rangle, \langle b, 0.4, -1\rangle\}$,

$H = \{\langle a, 0.8, -1\rangle, \langle b, 1, -0.9\rangle\}$.

Then the intersection of these two sets is;

$G \cap H = \{\langle a, 0.8, -1\rangle, \langle b, 0.4, -0.9\rangle\}$.

Since $\mathbf{0}_{BP}, A \subseteq G$ and $\mathbf{0}_{BP}, B \subseteq H$ then

$int_\tau(G) = \{\langle a, 1, -0.4\rangle, \langle b, 0.3, -1\rangle\}$,

$int_\tau(H) = \{\langle a, 0.7, -1\rangle, \langle b, 1, -0.7\rangle\}$

and since $\mathbf{0}_{BP} \subseteq G \cap H$ then $int_\tau(G) \cap int_\tau(H) = \{\langle a, 0.7, -0.4\rangle, \langle b, 0.3, -0.7\rangle\}$.

Thus we clearly have

$int_\tau(G \cap H) \neq int_\tau(G) \cap int_\tau(H)$.

4. CONTINUOUS MAPPINGS ON BIPOLAR FUZZY SUPRA TOPOLOGICAL SPACES

Definition 4.1: Let (U, τ) and (V, σ) be two bipolar fuzzy supra topological spaces and

$f: (U, \tau) \longrightarrow (V, \sigma)$ be a mapping.

1. For every $X \in \tau$ if $f(X) \in \sigma$, then f is called a bipolar fuzzy supra open mapping.

2. For every $Y^c \in \tau$ if $[f(X)]^c \in \sigma$, then f is called a bipolar fuzzy supra closed mapping.

3. For every $X \in \sigma$ if $f^{-1}(X) \in \tau$, then f is called a bipolar fuzzy supra continuous function. Moreover, $f: (U, \tau) \longrightarrow (V, \sigma)$ is a bipolar fuzzy supra continuous: $\Leftrightarrow f^{-1}(\sigma) \subset \tau$.

Definition 4.2: Let (U, τ) and (V, σ) be two bipolar fuzzy supra topological spaces, τ^* and σ^* be two associated bipolar fuzzy topologies with τ and σ, respectively. Let $f: (U, \tau^*) \longrightarrow (V, \sigma^*)$ be a mapping.

1. For every $X \in \tau^*$ if $f(X) \in \sigma$, then f is called a S^* bipolar fuzzy supra open mapping.

2. For every $Y^c \in \tau^*$ if $\left(f(Y)\right)^c \in \sigma$, then f is called a S^* bipolar fuzzy supra closed mapping.

3. For every $X \in \sigma^*$ if $f^{-1}(X) \in \tau$, then f is called a S^* bipolar fuzzy supra continuous mapping. Moreover, $f: (U, \tau) \longrightarrow (V, \sigma)$ is a S^* bipolar fuzzy supra continuous: $\Leftrightarrow f^{-1}(\sigma^*) \subset \tau$.

Remark 4.3: From the Definition.4.2 it is clear that every S^* bipolar fuzzy supra continuous function is a bipolar fuzzy supra continuous mapping.

Theorem 4.4: Let (U, τ) and (V, σ) be two bipolar fuzzy supra topological spaces, τ^* and σ^* be two associated with bipolar fuzzy topologies with τ and σ, respectively.

Let $f: (U, \tau^* \subseteq \tau) \longrightarrow (V, \sigma^* \subseteq \sigma)$ be a mapping. Then the followings are equivalent.

1. The mapping f is S^* bipolar fuzzy supra continuous.

2. For every bipolar fuzzy supra closed set Y in (V, σ^*), $\left(f^{-1}(Y)\right)^c \in \tau$.

3. For every bipolar fuzzy supra Y in (V, σ^*), $cl_\tau\left(f^{-1}(Y)\right) \subseteq f^{-1}\left(cl_{\sigma^*}(Y)\right)$.

4. For every bipolar fuzzy set X in (U, τ^*), $f\left(cl_\tau(X)\right) \subseteq cl_{\sigma^*}\left(f(X)\right)$.

5. For every bipolar fuzzy set Y in (V, σ^*), $f^{-1}\left(int_{\sigma^*}(Y)\right) \subseteq int_\tau\left(f^{-1}(Y)\right)$.

Proof: (1⟹2) Let $f: (U, \tau^* \subseteq \tau) \longrightarrow (V, \sigma^* \subseteq \sigma)$ be a S^* bipolar fuzzy supra continuous mapping and $Y^c \in \sigma$. Hence $f^{-1}(Y^c) = f^{-1}(\mathbf{1}_{BP} - Y) = \mathbf{1}_{BP} - f^{-1}(Y) \in \tau$. This means that $f^{-1}(Y)$ is bipolar fuzzy supra closed in U.

(2⟹3) Since $cl_{\sigma^*}(Y)$ is a bipolar fuzzy closed set for any $Y \in \sigma^*$, then $f^{-1}\left(cl_{\sigma^*}(Y)\right)$ is a bipolar fuzzy supra closed in τ. Hence

$$f^{-1}\left(cl_{\sigma^*}(Y)\right) = cl_\tau\left(f^{-1}\left(cl_{\sigma^*}(Y)\right)\right) \overset{B \subseteq cl_{\sigma^*}(B)}{\supseteq} cl_\tau\left(f^{-1}(Y)\right)$$

and we get $cl_\tau(f^{-1}(Y)) \subseteq f^{-1}(cl_{\sigma^*}(Y))$.

(3⟹4) Let $f(X) = Y$. Then $cl_\tau(f^{-1}(Y)) \subseteq f^{-1}(cl_{\sigma^*}(Y))$, therefore

$$f^{-1}\left(cl_{\sigma^*}(f(X))\right) \supseteq cl_\tau\left(f^{-1}(f(X))\right) \supseteq cl_\sigma(Y)$$

and $cl_{\sigma^*}(f(X)) \supseteq f\left(f^{-1}\left(cl_{\sigma^*}(f(X))\right)\right) \supseteq f(cl_\tau(X))$. Hence, we have the inclusion $f(cl_\tau(X)) \subseteq cl_{\sigma^*}(f(X))$.

(4⟹2) Let Y be a bipolar fuzzy closed in (Y,σ^*) and $X = f^{-1}(Y)$. Then,

$$f(cl_\tau(X)) \subseteq cl_{\sigma^*}(f(X)) \subseteq cl_{\sigma^*}\left(f(f^{-1}(Y))\right) \subseteq cl_{\sigma^*}(Y) = Y$$

and

$$cl_\tau(X) \subseteq f^{-1}\left(f(cl_\tau(X))\right) \subseteq f^{-1}(Y) = X.$$

Since the closure of a bipolar fuzzy supra set always includes the set itself and we have

$cl_\tau(X) = X$ and then X is a bipolar fuzzy supra closed set. Therefore $f^{-1}(Y)$ is a bipolar fuzzy supra closed set in U.

(5⟹1) Let Y be a bipolar fuzzy open set in V. Then the complement set $\mathbf{1}_{BP} - Y$ is a bipolar fuzzy closed set and $f^{-1}(\mathbf{1}_{BP} - Y) = \mathbf{1}_{BP} - f^{-1}(Y)$ is a bipolar fuzzy supra closed set in X. Therefore $f^{-1}(Y)$ is a bipolar fuzzy supra open set in U. Hence f is a S^* bipolar fuzzy supra continuous mapping.

(1⟹5) Let $Y \in \sigma^*$, since f is a S^* bipolar fuzzy supra continuous mapping then $f^{-1}(int_{\sigma^*}(Y)) \in \tau$. But since $f^{-1}(int_{\sigma^*}(Y))$ is bipolar fuzzy open it is the same set as bipolar fuzzy supra interior of itself so $f^{-1}(int_{\sigma^*}(Y)) = int_\tau\left(f^{-1}(int_{\sigma^*}(Y))\right)$. Since $int_{\sigma^*}(Y) \subseteq Y$, $f^{-1}(int_{\sigma^*}(Y)) = int_\tau\left(f^{-1}(int_{\sigma^*}(Y))\right) \subseteq int_\tau(f^{-1}(Y))$ and then

$$f^{-1}(int_{\sigma^*}(Y)) \subseteq int_\tau(f^{-1}(Y)).$$

Theorem 4.5: Let (U,τ^*) and (V,σ^*) be two bipolar fuzzy topological spaces, τ^* and σ^* be two associated bipolar fuzzy topologies with τ and σ, respectively. Let $f: (U,\tau^*) \longrightarrow (V,\sigma^*)$ be a mapping. Then the followings are equivalent.

1. The mapping f is bipolar fuzzy supra continuous.

2. The inverse image of every bipolar fuzzy supra closed set in (V,σ), is bipolar fuzzy supra closed in (U,τ).

3. For every bipolar fuzzy set Y in V,

$$cl_\tau(f^{-1}(Y)) \subseteq f^{-1}(cl_\sigma(Y)) \subseteq f^{-1}(cl_{\sigma^*}(Y)).$$

4. For every bipolar fuzzy set X in U,

$$f(cl_\tau(X)) \subseteq cl_\sigma(f(X)) \subseteq cl_{\sigma^*}(f(X)).$$

5. For every bipolar fuzzy set Y in V,

$$int_\tau(f^{-1}(Y)) \supseteq f^{-1}(int_\sigma(Y)) \supseteq f^{-1}(int_{\sigma^*}(Y)).$$

Proof: It can easily be shown by the previous theorem.

Theorem 4.6: If $f: (U,\tau) \longrightarrow (V,\sigma)$ is a bipolar fuzzy supra continuous and $g: (V,\sigma) \longrightarrow (W,\delta)$ is a S^* bipolar fuzzy supra continuous, then the function $g \circ f$: $(U,\tau) \longrightarrow (W,\delta)$ is a S^* bipolar fuzzy supra continuous.

Proof: Let $Z \in \delta^* \subset \delta$. Since g is S^* bipolar fuzzy supra continuous then $g^{-1}(Z) \in \sigma$. From the bipolar fuzzy supra continuity of f we have $f^{-1}(g^{-1}(Z)) = (g \circ f)^{-1}(Z) \in \tau$ and therefore, $g \circ f$: $(U,\tau) \longrightarrow (W,\delta)$ is bipolar fuzzy supra continuous mapping.

Theorem 4.7: If $f: (U,\tau) \longrightarrow (V,\sigma)$ and $g: (V,\sigma) \longrightarrow (W,\delta)$ are two bipolar fuzzy supra continuous functions, then the function $g \circ f$: $(U,\tau) \longrightarrow (W,\delta)$ is bipolar fuzzy supra continuous.

Proof: Straightforward.

Theorem 4.8: If $f: (U,\tau) \longrightarrow (V,\sigma)$ and $g: (V,\sigma) \longrightarrow (W,\delta)$ are two S^* bipolar fuzzy supra continuous functions, then the function $g \circ f : (U, \tau) \longrightarrow (W, \delta)$ is bipolar fuzzy supra continuous.

Proof: Since f and g are two S^* bipolar fuzzy supra continuous mapping from Remark 4.3 they are bipolar fuzzy supra continuous mappings and by Theorem 4.7 we get the result.

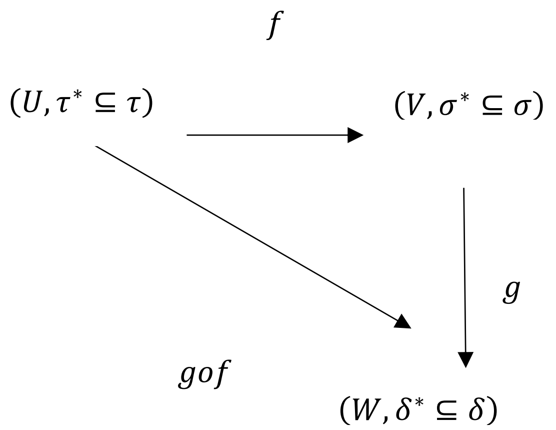
Remark 4.9:

i. Let $f: (U, \tau) \longrightarrow (V, \sigma)$ and $g: (V, \sigma) \longrightarrow (W, \delta)$ be two S^* bipolar fuzzy supra continuous mapping. Thus, the mapping $g \circ f : (U, \tau) \longrightarrow (W, \delta)$ is no need to be S^* bipolar fuzzy supra continuous. Consider the given topologies in the following tables and the diagram:

$U = \{a, b\}$
$\tau^* = \{\mathbf{0}_{BP}, \mathbf{1}_{BP}, A^*\}$
$A^* = \{\langle a, 0.5, -0.4\rangle, \langle b, 0.6, -0.7\rangle\}$
$\tau = \{\mathbf{0}_{BP}, \mathbf{1}_{BP}, A, B\}$
$A = \{\langle a, 1, -0.5\rangle, \langle b, 0.7, -1\rangle\}$
$B = \{\langle a, 0.5, -0.4\rangle, \langle b, 0.6, -0.7\rangle\}$

$V = \{c, d\}$
$\sigma^* = \{\mathbf{0}_{BP}, \mathbf{1}_{BP}, C^*\}$
$C^* = \{\langle c, 0.2, -0.3\rangle, \langle d, 0.3, -0.6\rangle\}$
$\sigma = \{\mathbf{0}_{BP}, \mathbf{1}_{BP}, C, D\}$
$C = \{\langle c, 0.2, -0.3\rangle, \langle d, 0.3, -0.6\rangle\}$
$D = \{\langle c, 1, -0.4\rangle, \langle d, 0.5, -1\rangle\}$

$W = \{e, k\}$
$\delta^* = \{\mathbf{0}_{BP}, \mathbf{1}_{BP}, E^*\}$
$E^* = \{\langle e, 0.4, -0.5\rangle, \langle k, 0.2, -0.8\rangle\}$
$\delta = \{\mathbf{0}_{BP}, \mathbf{1}_{BP}, E, F\}$
$E = \{\langle e, 0.5, -1\rangle, \langle k, 1, -0.9\rangle\}$
$F = \{\langle e, 1, -0.6\rangle, \langle k, 0.3, -1\rangle\}$



If we define $f(a) = c$, $f(b) = d$, $g(c) = e$, $g(d) = k$ then we clearly have

$$(gof)^{-1}(E^*) = \left\{ \begin{array}{l} \left(\mu^+_{E^*}(gof)(a), \mu^-_{E^*}(gof)(a)\right), \\ \left(\mu^+_{E^*}(gof)(b), \mu^-_{E^*}(gof)(b)\right) \end{array} \right\}$$

$$= \left\{\left(\mu^+_{E^*}(e), \mu^-_{E^*}(e)\right), \left(\mu^+_{E^*}(k), \mu^-_{E^*}(k)\right)\right\}$$

$$= \{(0.4, -0.5), (0.2, -0.8)\} \notin \tau.$$

ii. If we take g as a bipolar fuzzy supra continuous mapping in the same remark, then the mapping gof may not be a bipolar fuzzy supra continuous.

5. COMPACTNESS IN BIPOLAR FUZZY SUPRA TOPOLOGICAL SPACES

Definition 5.1: 1. Let (U, τ) be a bipolar fuzzy supra topological space and

$$\{X_i | i \in I\} = \left\{\langle u, \mu^+_{X_i}(u), \mu^-_{X_i}(u)\rangle \middle| i \in I, u \in U\right\}$$

be a family of bipolar fuzzy supra open sets on U.

If $\bigcup_{i \in I} X_i = 1_{BP}$ then $\{X_i | i \in I\}$ is called bipolar fuzzy supra open cover of U. This means that $\bigvee_{i \in I} \mu^+_{X_i}(u) = 1$ and $\bigwedge_{i \in I} \mu^-_{X_i}(u) = -1$.

2. A finite subfamily of a bipolar fuzzy supra open cover $\{X_i | i \in I\}$ of U, is called finite subcover of U.

3. A family

$$\{K_i | i \in I\} = \left\{\langle u, \mu^+_{K_i}(u), \mu^-_{K_i}(u)\rangle \middle| i \in I\right\}$$

of bipolar fuzzy supra closed sets in U satisfies the finite interseciton property if finite subfamily $\left\{\langle u, \mu^{+}_{K_{i_\alpha}}(u), \mu^{-}_{K_{i_\alpha}}(u)\rangle \,\middle|\, \alpha = 1,2,\dots,n\right\}$

satisfies the condition

$$\bigcap_{\alpha=1}^{n}\left\{\langle u, \mu^{+}_{K_{i_\alpha}}(u), \mu^{-}_{K_{i_\alpha}}(u)\rangle\right\} \neq 0_{BP}.$$

Definition 5.2: Let (U,τ) be a bipolar fuzzy supra topological space. (U,τ) is called bipolar fuzzy supra compact topological spaces if every bipolar fuzzy supra open cover of X has a finite subcover.

Theorem 5.3: (U,τ) is a bipolar fuzzy supra compact topological space if and only if every family of bipolar fuzzy closed sets with finite intersection property has a nonempty intersection.

Proof: $(\Longrightarrow)$ Let (U,τ) be a bipolar fuzzy supra compact topological space and let $\mathcal{A} := \{X_\alpha | \alpha \in \Lambda\}$ be a family of bipolar fuzzy supra closed sets with finite intersection property of U. Let us consider $\bigcap_{\alpha\in\Lambda} X_\alpha = 0_{BP}$. Then

$$1_{BP} = (0_{BP})^c = \left(\bigcap_{\alpha\in\Lambda} X_\alpha\right)^c = \bigcup_{\alpha\in\Lambda}(X_\alpha)^c$$

Therefore, the family $\{(X_\alpha)^c\}_{\alpha\in\Lambda}$ is a bipolar fuzzy supra open cover of U. Since U is bipolar fuzzy supra compact, then this cover has a bipolar fuzzy supra finite subcover, so for $\exists\ \alpha_1, \alpha_2, \dots \alpha_n$, $1_{BP} = \bigcup_{i=1}^{n}\left(\left(X_{\alpha_i}\right)^c\right)$.

Hence

$$0_{BP} = (1_{BP})^c = \left(\bigcup_{i=1}^{n}\left(\left(X_{\alpha_i}\right)^c\right)\right)^c = \bigcap_{i=1}^{n} X_{\alpha_i}$$

This contradicts the fact that $\mathcal{A}$ has the finite intersection property. So $\bigcap_{\alpha\in\Lambda} X_\alpha \neq 0_{BP}$.

$(\Longleftarrow)$ Let us consider U is not compact. Therefore U has a finite open subcover $\mathcal{A} := \{X_\alpha | \alpha \in \Lambda\}$ such that none of any finite subfamily of $\mathcal{A}$ covers U. So, for any $\alpha_1, \alpha_2, \dots \alpha_n \in \Lambda$

$$1_{BP} \neq \bigcup_{i=1}^{n} X_{\alpha_i}.$$

Since $0_{BP} = (1_{BP})^c \neq \left(\bigcup_{i=1}^{n} X_{\alpha_i}\right)^c$ then we have $\bigcap_{i=1}^{n}\left(\left(X_{\alpha_i}\right)^c\right) \neq 0_{BP}$.

Hence the family of $\left\{\left(X_{\alpha_i}\right)^c \middle| \alpha \in \Lambda\right\}$ is a family of bipolar fuzzy supra closed sets with finite intersection property. Since $1_{BP} = \bigcup_{\alpha\in\Lambda} X_\alpha$ then $\bigcap_{\alpha\in\Lambda}((X_\alpha)^c) = \left(\bigcup_{\alpha\in\Lambda} X_\alpha\right)^c = 0_{BP}$.

So intersection of the members of $\left\{\left(X_{\alpha_i}\right)^c \middle| \alpha \in \Lambda\right\}$ is empty. This contradicts the hypothesis, so U is compact.

Theorem 5.4: Let (U,τ) and (V,σ) be two bipolar fuzzy supra topological spaces and

$f: (U,\tau) \longrightarrow (V,\sigma)$ be continuous and onto bipolar fuzzy supra mapping. If (U,τ) is a bipolar fuzzy supra compact topological space, then (V,σ) is so.

Proof: Let $\mathcal{A}$ be a bipolar fuzzy supra open cover of V. Then since f is bipolar fuzzy supra continuous, $\mathcal{B} = \{f^{-1}(Y) | Y \in \mathcal{A}\}$ is a bipolar fuzzy supra open cover of U. Since (U,τ) is bipolar fuzzy supra compact, then there exists a finite subcover such that

$$\{f^{-1}(Y_1), f^{-1}(Y_2), \dots, f^{-1}(Y_n)\}$$

of $\mathcal{B}$. Since f is a surjection then $\{Y_1, Y_2, \dots, Y_n\}$ is a bipolar fuzzy supra finite subcover of $\mathcal{A}$.

Definition 5.5: Let (U,τ) be a bipolar fuzzy supra topological space and (X,τ_X) be a bipolar fuzzy supra subspace where $X \in BPF(U)$. A family of $\mathfrak{B}$ subsets of U called the covering of X if the union of its elements contains X.

Theorem 5.6: Let (U,τ) and (V,σ) be two bipolar fuzzy supra topological spaces and $f: (U,\tau) \longrightarrow (V,\sigma)$ be continuous and onto bipolar fuzzy supra mapping. If X is bipolar fuzzy supra compact set in (U,τ) then $f(X)$ is bipolar fuzzy supra compact in (V,σ).

Proof: Straightforward.

Theorem 5.7: Let (U,τ) be a bipolar fuzzy supra compact topological space and X be a bipolar fuzzy supra closed set on U. Then (X,τ_X) is a bipolar fuzzy supra compact topological space.

Proof: Let X be a bipolar fuzzy supra closed subspace of the bipolar fuzzy supra compact space U. Let $\mathcal{A}$ be a bipolar fuzzy supra open covering of X by sets bipolar fuzzy supra open in U. Since X is bipolar fuzzy supra closed, then X^c is bipolar fuzzy supra open. If we add the bipolar fuzzy supra open set X^c to $\mathcal{A}$, we get a bipolar fuzzy supra open cover of U such

$\mathfrak{B} = \mathcal{A} \cup \{X^c\}$. Some finite subcollection of $\mathfrak{B}$ covers A. If subcollection $\mathfrak{B}$ contains $\{X^c\}$ discard $\{X^c\}$, otherwise take the subcollection as it is. The resulting collection is a finite subfamily of $\mathcal{A}$ that covers X.

Definition 5.8: Let (U,τ) and (V,σ) be two bipolar fuzzy supra topological spaces and

$X \in BPF(U), Y \in BPF(V)$. The product of X and Y is a bipolar fuzzy set on $U \times V$ defined by

XxY=$\{\langle (u,v), (\mu_A^+(u) \wedge \mu_B^+(v)), (\mu_A^-(u) \vee \mu_B^-(v))\rangle : (u,v) \in UxV\}$.

We can define a bipolar fuzzy supra topology on $U \times V$. The bipolar fuzzy supra topology on the product set $U \times V$ is given by the coarsest bipolar fuzzy supra topology makes following projections

$p_1: U \times V \longrightarrow V$, $p_1(u,v) = u$ and

$p_2: U \times V \longrightarrow V$, $p_2(u,v) = v$

are bipolar fuzzy supra continuous.

We have

$p_1^{-1}(\mu_G^+)(u,v) = \mu_G^+(p_1(u,v)) = \mu_G^+(u)$ and

$p_1^{-1}(\mu_G^-)(u,v) = \mu_G^-(p_1(u,v)) = \mu_G^-(u)$

for $G \in \tau$. So

$p_1^{-1}(G) = \{\langle (u,v), \mu_G^+(u), \mu_G^-(u)\rangle\} = G \times \mathbf{1}_{BP_V}$.

By the same procedure

$p_2^{-1}(H) = \{\langle (u,v), \mu_H^+(u), \mu_H^-(v)\rangle\} = \mathbf{1}_{BP_U} \times H$ for $H \in \sigma$.

Hence $\{G \times \mathbf{1}_{BP_V} | G \in \tau\} \cup \{\mathbf{1}_{BP_U} \times H | H \in \sigma\}$ is a base (A family $\mathscr{B} \subset BPF(U)$ is called a base for a bipolar fuzzy supra topology if $\mathscr{B} \subset \tau$ and every member of τ is a union of a number of members of $\mathscr{B}$.) for the product bipolar fuzzy supra topology δ on $U \times V$. Since τ and σ are bipolar fuzzy supra topologies, the families $\{G \times \mathbf{1}_{BP_V} | G \in \tau\}$ and $\{\mathbf{1}_{BP_U} \times H | H \in \sigma\}$ are closed under arbitrary unions and so we have

$\delta = \{(G \times \mathbf{1}_{BP_V}) \cup (\mathbf{1}_{BP_U} \times H) | G \in \tau, H \in \sigma\}$.

Theorem 5.9: Let (U,τ) and (V,σ) be two bipolar fuzzy supra topological spaces and $(U \times V, \delta)$ be their product. Then $(U \times V, \delta)$ is a compact bipolar fuzzy supra topological space if and only if (U,τ) and (V,σ) are bipolar fuzzy supra compact.

Proof: (⇒:) Since projections p_1 and p_2 are bipolar fuzzy supra continuous, we have the result.

(⇐:) Let U and V be two bipolar fuzzy supra compact spaces.

$\mathcal{A} = \{(G_i \times \mathbf{1}_{BP_V}) \cup (\mathbf{1}_{BP_U} \times H_i) : i \in J\}$ is a bipolar fuzzy supra open cover of $U \times V$, where $G_i = \{\langle u, \mu_{G_i}^+(u), \mu_{G_i}^-(u)\rangle : u \in U\} \in \tau$ and

$H_i = \{\langle v, \mu_{H_i}^+(v), \mu_{H_i}^-(v)\rangle : v \in V\} \in \sigma$.

We suggest that $\{G_i : i \in J\}$ is cover of U and $\{H_i : i \in J\}$ is a cover of V. We need to show that

$\bigvee_{i \in J} \mu_{G_i}^+(u) = 1, \bigwedge_{j \in J} \mu_{G_i}^-(u) = -1$

and

$\bigvee_{i \in J} \mu_{H_i}^+(v) = 1, \bigwedge_{j \in J} \mu_{H_i}^-(v) = -1$.

We have $(G_i \times \mathbf{1}_{BP_V}) \cup (\mathbf{1}_{BP_U} \times H_i) = \{\langle (u,v), \mu_{G_i}^+(u) \vee \mu_{H_i}^+(v), \mu_{G_i}^-(u) \wedge \mu_{H_i}^-(v)\rangle\}$. Hence,

$\bigvee\{\mu_{G_i}^+(u) : i \in J\} \vee \bigvee\{\mu_{H_i}^+(v) : i \in J\} = \bigvee_{i \in J}\{\mu_{G_i}^+(u) \vee \mu_{H_i}^+(v)\} = 1$

and

$\bigwedge\{\mu^-_{G_i}(u) : i \in J\} \wedge \bigwedge\{\mu^-_{H_i}(v)\ i \in J\} = \bigwedge_{i\in J}\{\mu^-_{G_i}(u) \wedge \mu^-_{H_i}(v)\} = -1.$

Then we have a finite subset J^* of J for which

$\bigvee_{i\in J^*} \mu^+_{G_i}(u) = 1, \bigwedge_{j\in J^*} \mu^-_{G_i}(u) = -1$

and

$\bigvee_{i\in J^*} \mu^+_{H_i}(v) = 1, \bigwedge_{j\in J^*} \mu^-_{H_i}(v) = -1.$

Hereby, $\{(G_i \times \mathbf{1}_{\boldsymbol{BP}_V}) \cup (\mathbf{1}_{\boldsymbol{BP}_U} \times H_i) : i \in J^*\}$ is a finite subcover of $\mathcal{A}$. Then $(U \times V, \delta)$ is bipolar fuzzy supra compact topological space.

6. CONCLUSION

We have introduced bipolar fuzzy supra topological spaces and investigated some of their important properties. In future, we plan to study on neighborhood structures of bipolar fuzzy point [11] and other topological structures such as separation axioms and connectedness in bipolar fuzzy supra topological space.

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Authors' Contribution

The authors contributed equally to the study.

The Declaration of Ethics Committee Approval

This study does not require ethics committee permission or any special permission.

The Declaration of Research and Publication Ethics

The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

REFERENCES

[1] L. Zadeh, "Fuzzy sets," Information and Control, vol. 8, pp. 338–353, 1965.

[2] L. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I," Inform. Sci., vol. 8, pp. 199–249, 1975.

[3] K. T. Atanassov, "Intuitionistic fuzzy sets," Fuzzy sets and Systems, vol. 20, pp. 87-96, 1986.

[4] P.K Maji., R. Biswas and A. R. Roy, "Fuzzy soft sets," J. Fuzzy Math., vol. 9, no. 3, pp. 589–602, 2001.

[5] W-R. Zhang, "Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis," NAFIPS/IFIS/NASA'94, pp. 305-309, 1994.

[6] K. M. Lee, "Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolarvalued fuzzy sets," J. Fuzzy Logic Intelligent Systems, vol. 14, pp. 125-129, 2004.

[7] M. S. Anitha, K. L. Muruganantha and K. Arjunan, "Notes on bipolar valued fuzzy subgroups of a group," The Bulletin of Society for Mathematical Services and Standards, vol. 7, pp. 40-45, 2013.

[8] S. P. Subbian and M. Kamaraj, "Bipolar-valued fuzzy ideals of ring and bipolar-valued fuzzy ideal extensions in subrings," International Journal of Mathematics Trends and Technology, vol. 61, no. 3, pp. 155-163, 2018.

[9] B. Pazar Varol, An approach to bipolar fuzzy submodules, TWMS J. App. and Eng. Math., 11 (1), 168-175, 2021.

[10] M. Azhagappan and M. Kamaraj, "Notes on bipolar valued fuzzy RW-closaed and bipolar valued fuzzy RW-open sets in bipolar valued fuzzy topological spaces," International Journal of Mathematical Archive, vol. 7, no. 3, pp. 30-36, 2016.

[11] J. Kim, S. K. Samanta, P. K. Lim, J. G. Lee, K. Hur, "Bipolar fuzzy topological spaces," Annals of Fuzzy Mathematics and Informatics, vol. 17, no. 3, pp. 205-229, 2019.

[12] A. S. Mashhour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, "On supratopological spaces," Indian J. Pure Appl. Math., vol. 14, no 4, pp. 502-510, 1983.

[13] M. E. Abd El-Monsef and A. E. Ramadan, "On Fuzzy Supra Topological Spaces," Indian J. Pure Appl. Math., vol. 18, no. 4, pp. 322-329, 1987.

[14] M. E. Abd El-Latif, "Some properties of fuzzy supra soft topological spaces," Europen Journal of Pure and Applied Mathematics, vol. 12, no. 3, pp. 999-1017, 2019.

[15] T. M. Al-Shami and M. E. Shafei, "On supra topological ordered spaces," Arab Journal of Basic and Applied Science, vol. 26, no. 1, pp. 433-445, 2019.

[16] T. M. Al-Shami and M. E. Shafei, "Two types of separation axioms on supra soft topological spaces," Demonstratio Mathematica, vol. 52, pp. 147-165, 2019.

[17] T. M. Al-Shami, "Paracompactness on supra topological spaces," Journal of Linear and Topological Algebra, vol. 9, no. 2, pp. 121-127, 2020.

[18] T. M. Al-Shami, M. Al Shumrani and C. Özel, "Another form of supra ordered separation axioms," Journla of Mathematical Extension, vol. 15, no 1, pp. 105-125, 2021.

[19] N. Turanlı, "An overview of intuitionistic fuzzy supratopological spaces," Hacettepe Journal of Mathematics and Statistics, vol. 32, pp. 17-26, 2003.