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# Existence Results for Hybrid Stochastic Differential Equations Involving $\psi$-Hilfer Fractional Derivative

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Abstract. In this paper, we discuss the existence of solutions for hybrid stochastic differential equations (HSDEs) with the $\psi$-Hilfer fractional derivative. The main tool used in our study is associated with the technique of fixed point theorems due to Dhage.

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## 1. Introduction

The study of fractional calculus is concerned with the generalization of the integer order differentiation and integration to an arbitrary real or complex order. It has played a significant role in various branches of science such as physics, chemistry, chemical physics, electrical networks, control of dynamic systems, science, engineering, biological science, optics and signal processing, see the monographs of Hilfer [15], Kilbas [17] and Podlubny [18]. Especially, numerous works have been devoted to the study of initial value problems, for example, see [2, 7].

In the recent years, some authors have considered Hilfer fractional derivative see [12, 13] and references therein. R. Hilfer [15] proposed a generalized Riemann-Liouville fractional derivative, for short, Hilfer fractional derivative (or composite fractional derivative), which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. This operator appeared in the theoretical simulation of dielectric relaxation in glass forming materials. In [12], Furati et al. considered an initial value problem for a class of nonlinear FDEs involving Hilfer fractional derivative. Recently, Abbas et al. [1] studied the existence and stability results for FDEs with Hilfer fractional derivative. Subsequently, many authors studied the FDEs involving composite fractional derivatives [21–23]. In [24], Vivek et al. investigated dynamics and stability of pantograph equations via Hilfer fractional derivative. Moreover, stochastic perturbation is unavoidable in nature and hence it is important and necessary to consider stochastic effect into the investigation of FDEs, see [5, 6, 8, 11].

One more attractive class of problems involves hybrid stochastic FDEs . For some works on this topic, one can refer to [3, 4, 20, 28].

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The following hybrid differential of first order

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J := [0,T), \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

was studied by Dhage et al. [10], under the assumptions $f \in C(J \times \mathbb{R}, \mathbb{R}|\{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$.
In [28], Zhao et al. investigated the fractional version of the problem (1.1), i.e.,

$$\begin{cases} D^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J,\ \alpha \in (0,1), \\ x(0) = 0, \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}|\{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$, a fixed point theorem in Banach algebras was the main tool used in this work.

From the above works, we develop the theory of HSDEs involving $\psi$-type composite fractional derivative. In this paper, we consider the following HSDEs:

$$D^{\alpha,\beta;\psi}\left(\frac{y(t)}{f(t,y(t))}\right) = b(t,y(t)) + \sigma(t,y(t))\dot{W}(t), \quad t \in [0,T], \tag{1.2}$$

$$I^{1-\gamma;\psi}y(0) = \phi, \tag{1.3}$$

where $D^{\alpha,\beta;\psi}$ is the $\psi$-type composite fractional derivative of type $\beta$ and order $\alpha$, $f : J \times \mathscr{L}_2(\Omega) \to \mathscr{L}_2(\Omega)$ and $b, \sigma : J \times \mathscr{L}_2(\Omega) \to \mathscr{L}_2(\Omega)$, $I^{1-\gamma;\psi}$ is the $\psi$-type Riemann-Liouville fractional integral and $\phi$ be such that $\left(\mathbb{E}(\|\phi\|^2)\right)^{\frac{1}{2}}$ are investigated.

## 2. Prerequisite

In this section, we recall a few known results that are needed in our work. Most of it can be found in [14,16,19,27].
Let $(\Omega, \mathfrak{I}, \mathbb{P})$ be a probability space, where $\Omega$ is sample space, $\mathfrak{I}$ is a $\sigma$-algebra and $\mathbb{P}$ is probability measure. Let $C = C(J, \mathscr{L}_2(\Omega))$ be the space of all second order processes which mean square continuous on $J$, this is a Banach space endowed with the norm $\|y\|_C = \max_t \|y(t)\|_2$, where $\|y(t)\|_2 = \left(\mathbb{E}(u^2(t))\right)^{\frac{1}{2}}$.

We define $y_r(t) = (\psi(t))^r y(t)$, $r \geq 0$. Let $C_{\gamma,\psi}(J, \mathscr{L}_2(\Omega))$ be the space of all continuous processes $y$ such that $y_r \in C(J, \mathscr{L}_2(\Omega))$ which is indeed a Banach space endowed with the norm

$$\|y\|_C = \max_t \left\{(\psi(t))^r \|y(t)\|_2\right\}.$$

**Remark 2.1.** For $u \in \mathscr{L}_2$, there holds the following *Itô* isometry property:

$$\mathbb{E}\left\|\int_0^t u(s)dW(s)\right\|^2 = \int_0^t \mathbb{E}\|u(s)\|^2 ds,$$

where $\{W(t)\}_{t\geq 0}$ is a Wiener (Brownian motion) process.

**Definition 2.2** ( [22]). The Riemann-Liouville fractional integral of order $\alpha$ for a continuous function $f$ is defined as

$$\left(I^{\alpha;\psi}f\right)(t) = \frac{1}{\Gamma(\psi)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} f(s)ds, \quad \alpha > 0.$$

**Definition 2.3** ( [22]). Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$. The $\psi$-composite fractional derivative of type $\beta$ and order $\alpha$ of the function $f$ is given by

$$(D^{\alpha,\beta;\psi}f)(t) = \left(I^{\beta(1-\alpha);\psi}\frac{d}{dt}I^{(1-\alpha)(1-\beta);\psi}f\right)(t); \quad \text{for a.e.} \quad t \in [0,1]. \tag{2.1}$$

**Properties 2.4** ( [25,27]). Let $\alpha \in (0,1)$, $\beta \in [0,1]$ and $\gamma = \alpha + \beta - \alpha\beta$.

(1) The operator $(D^{\alpha,\beta}f)(t)$ can be written as

$$\begin{aligned}(D^{\alpha,\beta;\psi}f)(t) &= \left(I^{\beta(1-\alpha);\psi}\frac{d}{dt}I^{(1-\alpha);\psi}f\right)(t) \\ &= \left(I^{\beta(1-\alpha);\psi}D^{\gamma;\psi}f\right)(t) \quad \text{for a.e.,} \quad t \in J.\end{aligned}$$

Moreover, the parameter $\gamma$ satisfies $\gamma \in (0,1]$, $\alpha \leq \gamma$, $\gamma > \beta$, $1-\gamma < 1-\beta(1-\alpha)$.

(2) For $\beta = 0$, the generalization (2.1) coincides with the fractional derivative of Riemann-Liouville type $D^{\alpha,0;\psi} = D^{\alpha;\psi}$, where for $\beta = 1$, the generalization (2.1) coincides with the fractional derivative of Caputo type $D^{\alpha,1;\psi} = D^{\alpha;\psi}$.

(3) If $D^{\beta(1-\alpha);\psi} f$ exists, then

$$\left(D^{\alpha,\beta;\psi} I^{\alpha;\psi} f\right)(t) = \left(I^{\beta(1-\alpha);\psi} D^{\beta(1-\alpha);\psi} f\right)(t); \quad \text{for a.e.,} \quad t \in J.$$

Furthermore, if $f \in C_{\gamma,\psi}(J, \mathscr{L}_2(\Omega))$, then

$$\left(D^{\alpha,\beta;\psi} I^{\alpha;\psi} f\right)(t) = f(t); \quad \text{for a.e.,} \quad t \in [0, T].$$

(4) If $D^{\gamma;\psi} f$ exists and in $C_{\gamma,\psi}(J, \mathscr{L}_2(\Omega))$, then

$$\begin{aligned}\left(I^{\alpha,\psi} D^{\alpha,\beta;\psi} f\right)(t) &= \left(I^{\gamma;\psi} D^{\gamma;\psi} f\right)(t) \\ &= f(t) - \frac{I^{\gamma-1;\psi}(0^+)}{\Gamma(\gamma)}(\psi(t))^{\gamma-1}; \quad \text{for a.e.,} \quad t \in J.\end{aligned}$$

**Lemma 2.5** ( [22]). *Let $f : J \times R \to R$ be a function such that $f(\cdot, y(\cdot)) \in C_{\gamma;\psi}[J, R]$ for any $y \in C_{\gamma;\psi}[J, R]$. A function $y \in C_{\gamma;\psi}[J, R]$ is a solution of fractional initial value problem:*

$$\begin{cases} D_{0^+}^{\alpha,\beta;\psi} y(t) = f(t, y(t)),\ 0 < \alpha < 1,\ 0 \le \beta \le 1, \\ I_{0^+}^{1-\gamma;\psi} y(0) = y_0, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases}$$

*if and only if y satisfies the following Volterra integral equation:*

$$y(t) = \frac{y_0(\psi(t))^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-1} f(s, y(s))ds.$$

Further details can be found in [1]. From Lemma 2.5 we have the following result.

**Lemma 2.6.** *Given $y \in C_{\gamma,\psi}(J, \mathscr{L}_2(\Omega))$, the integral solution of the initial value problem*

$$\begin{aligned} D^{\alpha,\beta;\psi}\left(\frac{y(t)}{f(t, y(t))}\right) &= z(t), \quad t \in J, \\ I^{1-\gamma;\psi} y(t) &= \phi, \end{aligned}$$

*is given by*

$$y(t) = f(t, y(t))\left(\frac{\phi}{\Gamma(\gamma)}(\psi(t))^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-1} z(s)ds\right), \quad t \in J.$$

The following lemma (fixed point theorem due to [3, 9]) is the fundamental in the proof of our main result.

**Theorem 2.7.** *Let $S$ be a nonempty, closed convex and bounded subset of the Banach algebra $E$ and let $A : E \to E$ and $B : S \to E$ be two operators satisfying:*

(1) *$\mathscr{A}$ is Lipschitzian with lipschitz constant $k$.*
(2) *$\mathscr{B}$ is completely continuous.*
(3) *$u = \mathscr{A}u\mathscr{B}v \Rightarrow u \in S$ for all $v \in S$.*
(4) *$Mk < 1$, where $M = \|\mathscr{B}(s)\| = sup\,\{\|\mathscr{B}(u) : u \in S\|\}$.*

*Then, the operator equation $u = \mathscr{A}u\mathscr{B}x$ has a solution in $S$.*

## 3. Main Results

We are now in a position to state our main results. We adopt some ideas from [25, 26]. Assume that the function $f : J \times \mathscr{L}_2(\Omega) \to \mathscr{L}_2(\Omega)$ satisfies the following hypotheses:

(H1) The function $f$ is bounded continuous and there exists a positive bounded function $q$ with bound $\|q\|$ such that

$$E\left(\|f(t, u(t)) - f(t, v(t))\|\right) \le q(t)\,\|u(t) - v(t)\|, \quad t \in J, \quad u, v \in \mathscr{L}_2(\Omega).$$

(H2) There exists a function $\delta \in C(J, \mathbb{R}^+)$ and continuous non-decreasing function $\zeta : [0, \infty) \to [0, \infty)$ such that for all measurable and continuous functions $b, \sigma$ the following conditions are satisfied:

$$\mathbb{E}\left(\|b(t, u(t))\|\right) \leq \delta(t)\zeta(\|u\|), \quad t \in J, \quad u \in \mathscr{L}_2(t),$$
$$\mathbb{E}\left(\|\sigma(t, u(t))\|\right) \leq \delta(t)\zeta(\|u\|), \quad t \in J, \quad u \in \mathscr{L}_2(t).$$

(H3) There exists a number $\rho > 0$ such that

$$\rho \geq \mathscr{K}\left[\frac{\|\phi\|}{\Gamma(\gamma)} + \frac{(\psi(T))}{\Gamma(\alpha+1)\|\delta\|}\zeta(\rho) + \frac{\mathscr{M}}{\Gamma(\alpha)}\|\delta\|\,\zeta(\rho)\right], \tag{3.1}$$

where $\|f(t, u(t))\| \leq \mathscr{K}$, for all $(t, x) inJ \times \mathscr{L}_2(\Omega)$ and $\mathscr{M} = T(\psi(t))^{1-\gamma}\Gamma(2\alpha - 1)$.

(H4) $\|q(t)\|\left[\frac{\|\phi\|}{\Gamma(\gamma)} + \frac{(\psi(T))}{\Gamma(\alpha+1)\|\delta\|}\zeta(\rho) + \frac{\mathscr{M}}{\Gamma(\alpha)}\|\delta\|\,\zeta(\rho)\right] < 1.$

**Theorem 3.1.** *Assume that the hypotheses (H1)-(H4) are satisfied. Then, the initial value problem (1.2)-(1.3) has at least one solution on $J$.*

*Proof.* According to [3], we split the proof into a sequence of steps:
Define a subset $S$ of $C$ as

$$S = \{y \in C : \|y\|_C \leq \rho\},$$

where $\rho$ satisfies inequality (3.1). Clearly, $S$ is closed, convex and bounded subset of the Banach space $C$. By Lemma 2.6, the initial value problem (1.2)-(1.3) is equivalent to the integral equation

$$y(t) = f(t, y(t))\left[\frac{\phi}{\Gamma(\gamma)}(\psi(t))^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} b(s, y(s))ds \right.$$
$$\left. + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\sigma(s, y(s))dW(s)\right],$$

for $t \in J$.
Define two operators $\mathscr{A} : C \to C$ by

$$\mathscr{A}y(t) = f(t, y(t)), \quad t \in J,$$

and $\mathscr{B} : S \to C$ by

$$\mathscr{B}y(t) = \frac{\phi}{\Gamma(\gamma)}(\psi(t))^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} b(s, y(s))ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\sigma(s, y(s))dW(s), \quad t \in J.$$

Then, $y = \mathscr{A}y\mathscr{B}y$. We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Lemma 2.6.
**Claim 1.** We first show that $\mathscr{A}$ is Lipschitz on $C$.
Let $x, y \in C$. Then, by (H1) we have

$$\begin{aligned}
&\left\|(\psi(t))^{1-\gamma}\mathscr{A}x(t) - (\psi(t))^{1-\gamma}\mathscr{A}y(t)\right\| \\
&= (\psi(t))^{1-\gamma}\|f(t, x(t)) - f(t, y(t))\| \\
&\leq q(t)(\psi(t))^{1-\gamma}\|x(t) - y(t)\| \\
&\leq \|q\|\,\|x - y\|_C,
\end{aligned}$$

for all $t \in J$. Taking the minimum over the interval $J$, we obtain

$$\|\mathscr{A}x - \mathscr{A}y\|_C \leq \|q\|\,\|x - y\|,$$

for all $x, y \in C$. So $\mathscr{A}$ is Lipschitz on $C$ with Lipschitz constant $\|q\|$.
**Claim 2.** The operator $\mathscr{B}$ is a completely continuous on $S$.
We show that $\mathscr{B}$ is continuous on $S$. Then, after some simple calculations, we get

$$\lim_{h\to\infty}(\psi(t))^{1-\gamma}\mathscr{B}y_n(t) = (\psi(t))^{1-\gamma}\mathscr{B}y(t),$$

for all $t \in J$. This shows that $\mathscr{B}$ is continuous on $S$. It sufficient to show that $B(s)$ is uniformly bounded and equi-continuous set in $C$. First, we have

$$\begin{aligned}(\psi(t))^{1-\gamma}\|By(t)\| &= \bigg\|\frac{\phi}{\Gamma(\gamma)}(\psi(t))^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}b(s,y(s))ds \\ &\qquad + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\sigma(s,y(s))dW(s)\bigg\| \\ &\leq \frac{\|\phi\|}{\Gamma(\gamma)} + \Delta_1 + \Delta_2,\end{aligned}$$

where

$$\begin{aligned}\Delta_1 &= \frac{\|\delta\|\,\zeta(\rho)(\psi(T))}{\Gamma(\gamma+1)}\\ \Delta_2 &= \frac{\mathscr{M}\,\|\delta\|\,\zeta(\rho)}{\Gamma(\alpha)},\end{aligned}$$

for all $t \in J$. Taking the maximum over the interval $J$, the above inequality becomes

$$\|\mathscr{B}y\|_C \leq \frac{\|\phi\|}{\Gamma(\gamma)} + \Delta_1 + \Delta_2, \quad \forall \quad y \in S.$$

This shows that $\mathscr{B}$ is uniformly bounded on $S$.

Next, we show that $B$ is an equi-continuous in $C$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $y \in C$. Then, we have

$$\left\|(\psi(t_2))^{1-\gamma}(\mathscr{B}y)(t_2) - (\psi(t_1))^{1-\gamma}(\mathscr{B}y)(t_1)\right\| \leq \frac{\|\delta\|\,\zeta(\rho)}{\Gamma(\alpha)}\,\|X_1 + X_2 - X_3 - X_4\|,$$

where

$$\begin{aligned}X_1 &= \int_0^{t_2} \psi'(s)(\psi(t_2))^{1-\gamma}(\psi(t_2)-\psi(s))^{\alpha-1}\,ds,\\ X_2 &= \int_0^{t_2} \psi'(s)(\psi(t_2))^{1-\gamma}(\psi(t_2)-\psi(s))^{\alpha-1}\,dW(s),\\ X_3 &= \int_0^{t_1} \psi'(s)(\psi(t_1))^{1-\gamma}(\psi(t_1)-\psi(s))^{\alpha-1}\,ds,\\ X_4 &= \int_0^{t_1} \psi'(s)(\psi(t_1))^{1-\gamma}(\psi(t_2)-\psi(s))^{\alpha-1}\,dW(s),\end{aligned}$$

and after some computations, we will see the right-hand side of the above inequality tends to zero, independently of $y \in S$ as $t_2 \to t_1$. Therefore, it follows from Arzela-Ascoli theorem that $\mathscr{B}$ is a completely continuous operator on $S$.

**Claim 3.** Let $y \in C$ and $z \in S$ be arbitrary elements such that $y = \mathscr{A}y\mathscr{B}z$. Then, we have

$$\begin{aligned}(\psi(t))^{1-\gamma}\|y(t)\| &= (\psi(t))^{1-\gamma}\|\mathscr{A}y(t)\|\,\|\mathscr{B}z(t)\|\\ &= \|f(t,y(t))\|\,\|\theta\|,\end{aligned}$$

where

$$\begin{aligned}\theta = \frac{\phi}{\Gamma(\gamma)}(\psi(t))^{\gamma-1} &+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}b(s,y(s))ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\sigma(s,y(s))dW(s).\end{aligned}$$

Therefore,

$$(\psi(t))^{1-\gamma}\|y(t)\| \leq \mathscr{K}\left[\frac{\|\phi\|}{\Gamma(\gamma)} + \Delta_1 + \Delta_2\right].$$

Taking the maimum over $J$, we obtain

$$\|y(t)\| \leq \mathscr{K}\left[\frac{\|\phi\|}{\Gamma(\gamma)} + \Delta_1 + \Delta_2\right] \leq \rho, \quad y \in S.$$

**Claim 4.** According to Theorem 2.7, now we show that $Mk < 1$. This obvious by (H4), since we have

$$\begin{aligned} M &= \|\mathscr{B}(s)\| \\ &= \sup\{\|\mathscr{B}(z) : z \in S\,\|\} \\ &\leq \frac{\|\phi\|}{\Gamma(\gamma)} + \Delta_1 + \Delta_2, \quad k = \|\delta\|\,. \end{aligned}$$

Thus, all the condition of Theorem 2.7 are satisfied and hence the operator equation $y = \mathscr{A}y\mathscr{B}y$ has a solution in $S$. In consequence, the problem (1.2)-(1.3) has a solution on $J$. □

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All authors have equal contribution in this manuscript.

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