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UNBOUNDED PERTURBATION TO EVOLUTION PROBLEMS WITH TIME-DEPENDENT SUBDIFFERENTIAL OPERATORS

SARRA BOUDADA* AND MUSTAPHA FATEH YAROU**
* LMPA LABORATORY, JIJEL UNIVERSITY, ALGERIA, ORCID ID: 0000-0002-9193-6933
**DEPARTMENT OF MATHEMATICS, LMPA LABORATORY, JIJEL UNIVERSITY, ALGERIA, ORCID ID: 0000-0003-4083-1813

Abstract. In this paper, we consider a nonlinear evolution inclusion governed by the subdifferential of a proper convex lower semicontinuous function in a separable Hilbert space. The right-hand side contains a set-valued perturbation with nonempty closed convex and not necessary bounded values. The existence of absolutely continuous solution is stated under different assumptions on the perturbation.

1. Introduction

Nonlinear evolution equations with subdifferential operators plays an important role in the theory of differential inclusions and have been widely investigated by many authors (see [1], [3], [5], [11], [15], [14], [17], [18], [19], [20], [21]), [22], [25]. Such problems appear often in problems of optimal control theory, mechanics and differential games, see for instance [9], [10], [12], [23]. In this work, we prove some existence results for evolution problems governed by subdifferential operator of the form

$$(\mathcal{P}) \begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + G(t, x(t)) \text{ a.e. } t \in [0, T]; \\ x(0) = x_0 \in dom\, \varphi(0, \cdot), \end{cases}$$

in a separable Hilbert space, where φ is a proper convex lower semicontinuous function, $\partial\varphi(\cdot, \cdot)$ is the subdifferntial of φ and $G(\cdot, \cdot)$ is a set-valued mapping with convex closed nonempty values playing the role of a perturbation to the problem. For the unperturbed problem, that is when $G \equiv 0$, the existence and uniqueness of solution have been obtained under various assumptions by many authors, see for instance ([10], [11], [16], [23]). In [16], the author introduced an assumption expressed in terms of the conjugate function $\varphi^*(t, \cdot)$ of the convex function $\varphi(t, \cdot)$, namely, there exists a Lipschitz function $k : H \to \mathbf{R}_+$ and an absolutely continuous function $a : [0, T] \to \mathbf{R}$ with $\dot{a} \in L^2_{\mathbf{R}}([0, T])$ such that, for all $x \in H$ and $s, t \in [0, T]$,

$$\varphi^*(t, x) \leq \varphi^*(s, x) + k(x) \mid a(t) - a(s) \mid .$$

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Some extensions, dealing with set-valued or single-valued perturbations, have been obtained under in general a compactness assumption on the subdifferential ([13], [17]) or on the perturbation [19]. The authors in [19] proved the existence of an absolutely continuous solution with set-valued perturbation satisfying the linear growth condition

$$G(t,x) \subset \beta(t)(1+ \| x \|)K \text{ for all } t \in [T_0,T] \text{ and } x \in H,$$

for some compact subset K and some non-negative function $\beta(\cdot) \in L^2_{\mathbf{R}}([T_0,T])$, $(T_0 \geq 0)$. In the particular case of the so-called sweeping process, i.e., for $\varphi(t,x)$ taken as the indicator function of a closed moving set $C(t,x)$, [13] established the existence of solution with prox-regular sets $C(t,x)$ and $G(\cdot,\cdot)$ with unnecessary bounded closed convex values. For other results, we refer to [6], [24] and the references therein. The main purpose in this paper is to study, in the setting of infinite dimensional Hilbert space H, the perturbed problem $(\mathcal{P})$, and to show how the approach from [13] can be adapted to yield the existence of solutions for $(\mathcal{P})$ with unbounded perturbation, under various assumptions. The paper is organized as follows. In section 2, we give some preliminaries and we recall some results which will be used in the paper. In section 3, we establish the existence theorem for the considered problem $(\mathcal{P})$ for a globally upper hemicontinuous perturbation, then we extend the result obtained in $[0,T]$ to the whole interval $\mathbf{R}_+$. Finally, we weaken the result by taking the perturbation G measurable in the time t and upper semicontinuous in the state x.

2. Preliminaries

Throughout the paper, H is a separable Hilbert space whose inner product is denoted by $\langle \cdot,\cdot \rangle$ and the associated norm by $\| \cdot \|$ and $[0,T]$ is an interval of $\mathbf{R}$. We will denote by $\mathbf{B}$ the closed unit ball of H, $\mathcal{P}_c(H)$ the family of all nonempty closed sets of H and $\mathcal{P}_{cc}(H)$ (resp. $\mathcal{P}_{ck}(H)$) the set of nonempty closed (resp. compact) convex subsets of H.

Let $\varphi : H \to \mathbf{R} \cup \{+\infty\}$ be an extended real-valued lower semicontinuous function, which is proper in the sense that its effective domain $dom\, \varphi$ defined by $dom\ \varphi := \{x \in H \ : \ \varphi(x) < +\infty\}$ is nonempty and, as usual, its Fenchel conjugate is defined by $\varphi^*(v) := \sup\limits_{x \in H} [\langle v, x\rangle - \varphi(x)]$. The subdifferential $\partial\varphi(x)$ of φ at $x \in dom\ \varphi$ is

$$\partial\varphi(x) = \{v \in H : \ \langle v, y - x\rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in dom\ \varphi\}$$

and its effective domain is $dom\ \partial\varphi = \{x \in \ H : \ \partial\varphi(x) \neq \emptyset\}$. It is well known that if φ is a proper lower semicontinuous convex function, then its subdifferential operator $\partial\varphi$ is a maximal monotone operator and then satisfies the closure property. The function φ is said to be inf-ball compact if for every $r > 0$, the set $\{x \in H : \ \varphi(x) \leq r\}$ is ball-compact, i.e., its intersection with any closed ball in H is compact.

For any subset C of H, $\overline{co}\, C$ stands for the closed convex hull of C and $\sigma(\cdot, C)$ represents the support function of C, that is, for all $\xi \in H$, $\sigma(\xi, C) = \sup\limits_{x \in C} \langle \xi, x\rangle$. We denote by $Proj(\cdot, C)$ the metric projection mapping onto the closed set C, defined by $Proj(x, C) := \{v \in C : d(x, C) = \|v - x\|\}$. A set-valued mapping $G : E \to \mathcal{P}_c(H)$ from a Hausdorff topological space E into subsets of H is said to be upper semicontinuous if, for any open subset $V \subset H$, the set $\{x \in E : G(x) \ \subset V\}$ is open in E. G is said to be scalarly upper semicontinuous or upper hemicontinuous if, for any $y \in \ H$, the real-valued function $x \mapsto \ \sigma(y, G(x))$ is upper semicontinuous. For more details concerning the properties of maximal monotone operators in Hilbert space, we refer to [2] and [4]. Basic facts of convex analysis and set-valued mappings can be found in [8]. Let us recall the following result due to [19].

Proposition 2.1. *Let $\varphi : [T_0,T]\times H \to \mathbf{R}_+ \cup \{+\infty\}$ be such that:*

(H_1) *for each $t \in [T_0,T]$, $\varphi(t,\cdot)$ is proper convex lower semicontinuous;*

(H_2) *there exist a ρ-Lipschitzean function $k : H \to \mathbf{R}_+$ and an absolutely continuous function $a : [T_0,T] \to \mathbf{R}$, with a non-negative derivative $\dot a \in L^2_{\mathbf{R}}([T_0,T])$, such that*

$$\varphi^*(t,x) \le \varphi^*(s,x) + k(x) \mid a(t) - a(s) \mid$$

for every $(t,s,x) \in [T_0,T] \times [T_0,T] \times H$.

If $h \in L^2_H([T_0,T])$ and $x_0 \in dom\ \varphi(T_0,\cdot)$, then the problem

$$(\mathcal{P}_h) \begin{cases} -\dot x(t) \in \partial\varphi(t,x(t)) + h(t) \text{ a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in dom\ \varphi(T_0,\cdot) \end{cases}$$

admits a unique absolutely continuous solution $x(\cdot)$ that satisfies

$$\int_{T_0}^{T} \| \dot x(t) \|^2 \, dt \le 2c_0 \int_{T_0}^{T} \dot a^2(t)dt + \sigma \int_{T_0}^{T} \|h(t)\|^2 dt + c_1$$

with $c_0 = \frac{1}{2}\big(k^2(0) + 3(\rho+1)^2\big)$, $\sigma = k^2(0) + 3(\rho+1)^2 + 4$, and

$$c_1 = 2\big(T - T_0 + \varphi(T_0,x(T_0)) - \varphi(T,x(T))\big)$$

and for $T_0 \le t_1 \le t_2 \le T$

$$|\varphi(t_2,x(t_2)) - \varphi(t_1,x(t_1))| \le$$

$$\int_{t_1}^{t_2} \big(k(0) + (\rho+1) \| \dot x(t) + h(t) \| \big)\big(\dot a(t) + |h|(t)\big)dt + \int_{t_1}^{t_2} \| \dot x(t) + h(t) \|^2 \, dt.$$

We close this section with a set-valued version of Scorza-Dragoni theorem due to [7], Corollary 2.2.

Corollary 2.2. *Let $I = [T_0,T]$ and λ the Lebesgue measure on I, with σ-algebra $\mathcal{L}(I)$. Let X be a Polish space and Y be a compact convex metrizable subset of a Hausdorff locally convex space. Let $G : I \times X \to \mathcal{P}_{ck}(Y)$ be a multifunction that satisfies the following hypotheses:*

(i) *$\forall t \in I$, $Graph\, G_t$ is closed in $X \times Y$;*

(ii) *$\forall x \in X$, the multifunction $t \mapsto G(t,x)$ admits a measurable selection.*

Then, there exists a measurable multifunction $G_0 : I \times X \to \mathcal{P}_{ck}(Y) \cup \{\emptyset\}$, which has the following properties:

(1) *there is a λ-null set N such that $G_0(t,x) \subset G(t,x)$, $\forall t \notin N$, $\forall x \in X$;*

(2) *if $u : I \to X$ and $v : I \to Y$ are $\mathcal{L}(I)$-measurable functions with $v(t) \in G(t,u(t))$ a.e., then $v(t) \in G_0(t,u(t))$ a.e.;*

(3) *for every $\varepsilon > 0$, there is a compact subset $J_\varepsilon \subset I$ such that $\lambda(I \setminus J_\varepsilon) < \varepsilon$, the graph of the restriction $G_0/J_\varepsilon \times X$ is closed and $\emptyset \ne G_0(t,x) \subset G(t,x)$, $\forall (t,x) \in J_\varepsilon \times X$.*

3. The Main Results

Now we are able to proved our first result for the problem $(\mathcal{P})$ with unbounded perturbation. In the development, we will use some ideas from [13] and [19].

Theorem 3.1. *Assume that* $\varphi : [0,T] \times H \to \mathbf{R}_+ \cup \{+\infty\}$ *satisfies* $(H_1), (H_2)$ *and*

(H_3) *φ is inf-ball compact for every $t \in [0,T]$.*

Let $G : [0,T] \times H \to \mathcal{P}_{cc}(H)$ be such that

(H_4) *G is upper hemicontinuous with respect to both variables;*

(H_5) *for any $(t,x) \in [0,T] \times H$, the mapping $Proj(0, G(t,x))$ is measurable on $[0,T]$ and there exist some real $\alpha > 0$ such that for all $(t,x) \in [0,T] \times H$,*

$$\|Proj(0, G(t,x))\| = d(0, G(t,x)) \leq \alpha.$$

Then, for any $x_0 \in dom\,\varphi(0,\cdot)$ the problem $(\mathcal{P})$ has at least one absolutely continuous solution, satisfying $\int_0^T \|\dot{x}(t)\|^2 dt \leq c$, where $c = 2c_0 \int_0^T \dot{a}^2(t)dt + \sigma\alpha^2 T + 2\big(T + \varphi(0,x_0)\big)$ and $c_0 = \frac{1}{2}\big(k^2(0) + 3(\rho+1)^2\big)$.

Proof. For each $(t,x) \in [0,T] \times H$, denote by $g(t,x)$ the element of minimal norm of the closed convex set $G(t,x)$ of H, that is, $g(t,x) = Proj(0, G(t,x))$. First, we shall construct a sequence of absolutely continuous mappings $(x_n(\cdot))_n$. Define, for every $n \geq 1$, the classical partition of $[0,T]$: for each $0 \leq k \leq n$, $t_k^n = k\dfrac{T}{n}$. Put $x(t_0^n) = x_0$, and choose y_0^n the element of minimal norm of $G(t_0^n, x_0)$, by (H_5) one has

$$\| y_0^n \| \leq \alpha, \tag{3.1}$$

and consider the following differential inclusion on the interval $[t_0^n, t_1^n]$:

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + y_0^n \text{ for a.e. } t \in [t_0^n, t_1^n], \\ x(t_0^n) = x_0 \in dom\ \varphi(t_0^n, \cdot), \end{cases}$$

by (3.1) observe that the map $t \mapsto y_0^n$ is in $L^2_H([t_0^n, t_1^n])$, then, by Proposition 2.1 the last problem has a unique absolutely continuous solution that we denote by $x_0^n : [t_0^n, t_1^n] \to H$.

Likewise, for each $k \in \{0, ..., n-1\}$ we can construct a finite sequence of absolutely continuous mappings $x_k^n(\cdot) : [t_k^n, t_{k+1}^n] \to H$ such that

$$\begin{cases} -\dot{x}_k^n(t) \in \partial\varphi(t, x_k^n(t)) + y_k^n \text{ a.e. } t \in [t_k^n, t_{k+1}^n], \\ x_k^n(t_k^n) = x_{k-1}^n(t_k^n) \in dom\varphi(t_k^n, \cdot). \end{cases} \tag{3.2}$$

where $y_k^n = Proj(0, G(t_k^n, x_{k-1}^n(t_k^n)))$. Recall that, in view of Proposition 2.1, the following inequality holds true in each subinterval $[t_k^n, t_{k+1}^n]$ for any $k \in \{0, ..., n-1\}$

$$\begin{aligned} \int_{t_k{}^n}^{t_{k+1}^n} \| \dot{x}_k^n(t) \|^2\, dt &\leq 2c_0 \int_{t_k{}^n}^{t_{k+1}^n} \dot{a}^2(t)dt + \sigma \int_{t_k{}^n}^{t_{k+1}^n} \| y_k^n \|^2\, dt + c_k \\ &\leq 2c_0 \int_{t_k{}^n}^{t_{k+1}^n} \dot{a}^2(t)dt + \sigma \int_{t_k{}^n}^{t_{k+1}^n} \alpha^2 dt + c_k, \end{aligned} \tag{3.3}$$

with $c_0 = \frac{1}{2}(k^2(0) + 3(\rho+1)^2)$, $\sigma = k^2(0) + 3(\rho+1)^2 + 4$ and

$$c_k = 2[(t_{k+1}^n - t_k^n) + \varphi(t_k^n, x_k^n(t_k^n)) - \varphi(t_{k+1}^n, x_{k+1}^n(t_{k+1}^n))].$$

Now, define x_n and g_n from $[0,T]$ to H by

$$x_n(t) = x_k^n(t) \text{ if } t \in [t_k^n, t_{k+1}^n[;\ x_n(T) = x_{n-1}^n(T),$$

$$g_n(t) = y_k^n \text{ if } t \in [t_k^n, t_{k+1}^n[;\ g_n(T) = y_{n-1}^n$$

for any $k \in \{0, ..., n-1\}$. Clearly, $x_n(\cdot)$ is absolutely continuous on $[0,T]$. Consider the mapping $\delta_n : [0,T] \to [0,T]$ such that for any $k \in \{0, ..., n-1\}$

$$\delta_n(t) = t_k^n \text{ if } t \in [t_k^n, t_{k+1}^n[;\ \delta_n(T) = T$$

then, observe that for each $t \in [0,T]$, $|\delta_n(t) - t| \le |t_{k+1}^n - t_k^n| = \dfrac{T}{n}$, so $\delta_n(t) \to t$. Thus, for each $n \ge 1$, we have the following:

(i) $g_n(t) \in G(\delta_n(t), x_n(\delta_n(t)))$, $\forall t \in [0,T]$, $\forall x \in H$;

(ii) $\forall t \in [0,T]:$ $\|g_n(t)\| \le \alpha$;

(iii) $-\dot{x}_n(t) \in \partial\varphi(t, x_n(t)) + g_n(\delta_n(t))$ a.e. $t \in [0,T]$, $x_n(0) = x_0$.

Further, from (3.3) we have:

$$\sum_{k=0}^{n-1} \int_{t_k{}^n}^{t_{k+1}^n} \|\dot{x}_n(t)\|^2 dt \le 2c_0 \sum_{k=0}^{n-1} \int_{t_k{}^n}^{t_{k+1}^n} \dot{a}^2(t)dt + \sigma\alpha^2 \sum_{k=0}^{n-1} \int_{t_k{}^n}^{t_{k+1}^n} dt + \sum_{k=0}^{n-1} c_k,$$

equivalently

$$\int_0^T \|\dot{x}_n(t)\|^2 dt \le 2c_0 \int_0^T \dot{a}^2(t)dt + \sigma\alpha^2 \int_0^T dt + c_n \le 2c_0 \int_0^T \dot{a}^2(t)dt + \sigma\alpha^2 T + c_n,$$

with $c_n = 2(T + \varphi(0,x_0) - \varphi(T, x_n(T)))$, because φ is non-negative, putting $c' = 2(T + \varphi(0,x_0))$, we may write

$$\int_0^T \|\dot{x}_n(t)\|^2 dt \le 2c_0 \int_0^T \dot{a}^2(t)dt + \sigma\alpha^2 T + c',$$

then $\int_0^T \|\dot{x}_n(t)\|^2 dt \le c$, where $c = 2c_0 \int_0^T \dot{a}^2(t)dt + \sigma\alpha^2 T + c'$, so

$$\sup_{n \in \mathbf{N}} \int_0^T \|\dot{x}_n(t)\|^2 dt \le c \tag{3.4}$$

and thus $L = \sup_{n\in\mathbf{N}} \|\dot{x}_n(t)\|_{L^2_H([0,T])} < +\infty$.

Now, let us prove the uniform convergence of some subsequence of $x_n(\cdot)$ to some absolutely continuous mapping $x(\cdot)$. Using the Cauchy-Schwarz inequality and (3.4) for all $s \in [0,T]$ we obtain

$$\| x_n(s) - x_n(0) \|^2 = \| x_n(s) - x_0 \|^2 \le s \int_0^s \| \dot{x}_n(t) \|^2 \, dt \le Tc$$

and hence

$$\| x_n(s) \|^2 \le 2 \| x_0 \|^2 + 2 \| x_n(s) - x_0 \|^2 \le 2 \| x_0 \|^2 + 2Tc.$$

Consequently, for each n, we get $\| x_n(\cdot) \|_\infty^2 \le 2 \| x_0 \|^2 + 2Tc$. Then

$$\| x_n(\cdot) \|_\infty \le M, \tag{3.5}$$

where $M = (2 \| x_0 \|^2 + 2Tc)^{\frac{1}{2}}$. Therefore

$$\| x_n(t) - x_n(s) \| = \| \int_s^t \dot{x}_n(\tau)d\tau \| \le (t-s)^{\frac{1}{2}} \Big(\int_s^t \| \dot{x}_n(\tau) \|^2 \, d\tau \Big)^{\frac{1}{2}} \le (t-s)^{\frac{1}{2}} L,$$

so along with (3.5), the set $\{(x_n(\cdot))_n\}$ is bounded and equicontinuous in $C_H([0,T])$, recall that, in view of Proposition 2.1, for any fixed $t \in [0,T]$ and any n, one has

$$|\varphi(t, x_n(t)) - \varphi(0, x(0))| \le \sup_{n\in\mathbf{N}} \int_0^t (k(0) + (\rho+1) \| \dot{x}_n(t) + \alpha \|)(\dot{a}(t) + \alpha)dt$$

$$+\sup_{n\in\mathbf{N}}\int_0^t \| \dot{x}_n(t)+\alpha \|^2 \, dt < +\infty,$$

since φ is inf-ball compact by assumption, the set $\{x_n(t);\ n \in \ \mathbf{N}\}$ is relatively compact in H, so by Ascoli's theorem, we can extract a subsequence of $(x_n(\cdot))_n$ that converges uniformly on $[0,T]$ to some map $x(\cdot) \in C_H([0,T])$. From (3.4), $(\dot{x}_n)_n$ is bounded in $L^2_H([0,T])$, we may then extract a subsequence from the latter subsequence converging weakly in $L^2_H([0,T])$ to some map $v(\cdot)$. The equality $x_n(t) = x_n(0) + \int_0^t \dot{x}_n(s)ds$ for all $t \in [0,T]$ then yields $x(t) = x(0) + \int_0^t v(s)ds$ for all $t \in [0,T]$ and hence the map $x(\cdot)$ is absolutely continuous on $[0,T]$ with $\dot{x}(\cdot) = v(\cdot)$ a.e.
Finally, we show now that $x(\cdot)$ is a solution of $(\mathcal{P})$ on $[0,T]$. Define the step mapping $z_n(t) = g_n(\delta_n(t))$ for all $t \in [0,T]$, one has for almost all $t \in [0,T]$, $-\dot{x}_n(t) \in \partial\,\varphi(t, x_n(t)) + z_n(t)$ with

$$z_n(t) \in \ G(\delta_n(t), x_n(\delta_n(t))). \tag{3.6}$$

Since $\| \ g_n(\delta_n(t)) \ \| \leq \alpha$ for all $n \in \mathbf{N}$ and $t \in [0,T]$, we may suppose that the sequence $(z_n(\cdot))_n$ converges weakly in $L^1_H([0,T])$ to a mapping $z(\cdot) \in L^1_H([0,T])$ with $\| \ z(t) \ \| \leq \alpha$ a.e. $t \in [0,T]$. By Mazur's Theorem, there exists

$$\xi_n \in \overline{co}\{z_q, \ \ q \geq n\} \tag{3.7}$$

such that $(\xi_n(\cdot))_n$ converges strongly in $L^1_H([0,T])$ to $z(\cdot)$. Extracting a subsequence if necessary, we may suppose that $(\xi_n(\cdot))_n$ converges a.e. to $z(\cdot)$, then there is a Lebesgue negligible set $S \subset [0,T]$ such that for every $t \in [0,T] \setminus S$, on one hand $\xi_n(t) \to z(t)$ strongly in H, and on the other hand the inclusion (3.6) holds true for every integer $n \geq 1$ as well as the inclusion

$$z(t) \in \bigcap_n \overline{co}\{z_q(t), \ \ q \geq n\}.$$

From the inclusion (3.6), for any $n \in \mathbf{N},\ t \in [0,T] \setminus S$ and any $y \in H$:

$$\langle y, z_n(t)\rangle \leq \sigma(y, G(\delta_n(t), x_n(\delta_n(t)))), \tag{3.8}$$

further, for each $n \in \mathbf{N}$ and any $t \in [0,T] \setminus S$, from (3.7) we have

$$\langle y, \xi_k(t)\rangle \leq \sup_{q\geq n}\langle y, z_q(t)\rangle \quad \forall\, k \geq n, \tag{3.9}$$

taking the limit in (3.9) as $k \to +\infty$ and by (3.8) one obtains

$$\langle y, z(t)\rangle \leq \sup_{q\geq n}\langle y, z_q(t)\rangle \leq \sup_{q\geq n}\sigma(y, G(\delta_q(t), x_q(\delta_q(t)))),$$

which ensures that $\langle y, z(t)\rangle \leq \limsup_{n\to+\infty}\sigma(y, G(\delta_n(t), x_n(\delta_n(t))))$. Since $\sigma(y, G(\cdot,\cdot))$ is upper semicontinuous on $[0,T] \times H$, then for every $t \in \ [0,T] \setminus S$ and every $y \in H$, $\langle y, z(t)\rangle \leq \sigma(y, G(t, x(t))$, then $z(t) \in G(t, x(t))$ a.e. Further, since $(\dot{x}_n(\cdot) + z_n(\cdot))_n$ converges weakly in $L^1_H([0,T])$ to $\dot{x}(\cdot) + z(\cdot)$ and $(x_n(\cdot))_n$ converges strongly in $L^1_H([0,T])$ to $x(\cdot)$ and since the operator $\partial\varphi(t,\cdot)$ satisfies the closure property as the subdifferential of a proper lower semicontinuous function one obtains $\dot{x}(t) + z(t) \in -\partial\varphi(t, x(t))$ a.e., with $z(t) \in G(t, x(t))$ a.e.
Taking the limit in inequality (3.4) and using the preceding convergence, we get $\int_0^T \|\dot{x}(t)\|^2 dt \leq c$. $\square$

Note that, obviously, Theorem 3.1 yields for any finite interval of the form $[T_k, T_{k+1}]$ for all $k \in \mathbf{N}$. So, the next Corollary proves on the whole interval $\mathbf{R}_+ := [0, +\infty[$ the existence of solution to the above evolution problem.

Corollary 3.2. *Let $\varphi : \mathbf{R}_+ \times H \to \mathbf{R}_+ \cup \{+\infty\}$ and $G : \mathbf{R}_+ \times H \to \mathcal{P}_{cc}(H)$ be such that the following assumptions hold:*

- (H_1') *the function $x \mapsto \varphi(t, x)$ is proper convex lower semicontinuous, for each $t \in \mathbf{R}_+$.*
- (H_2') *there exist a ρ-Lipschitzean function $k : H \to \mathbf{R}_+$ and an absolutely continuous function $a : \mathbf{R}_+ \to \mathbf{R}$, with a non-negative derivative $\dot{a} \in L^2_{\mathbf{R}}(\mathbf{R}_+)$, such that*

$$\varphi^*(t, x) \leq \varphi^*(s, x) + k(x) \mid a(t) - a(s) \mid$$

  for every $(t, s, x) \in \mathbf{R}_+ \times \mathbf{R}_+ \times H$,
- (H_3') *φ is inf-ball compact for every $t \in \mathbf{R}_+$,*
- (H_4') *G is upper hemicontinuous with respect to both variables,*
- (H_5') *there exists a non-negative function $\alpha(\cdot) \in L^\infty_{loc}(\mathbf{R}_+)$ such that $d(0, G(t, x)) \leq \alpha(t)$ for all $t \in \mathbf{R}_+$ and $x \in H$.*

Then, for any $x_0 \in dom\ \varphi(0, \cdot)$, there exists a mapping $x : \mathbf{R}_+ \to H$ which is locally absolutely continuous on $\mathbf{R}_+$ and satisfies

$$(\mathcal{P}_1) \begin{cases} -\dot{x}(t) \in \partial \varphi(t, x(t)) + G(t, x(t)) \;\; a.e. \;\; t \in \mathbf{R}_+, \\ \qquad x(0) = x_0 \in dom\ \varphi(0, \cdot). \end{cases}$$

Proof. We follow the idea of the proof of Theorem 4 in [13]. We consider the partition of $\mathbf{R}_+$ by the points $T_n = n$ for all $n \in \mathbf{N}$. It will suffice to apply Theorem 3.1 in an appropriate way on each interval $[T_n, T_{n+1}]$. By Theorem 3.1, there exists a absolutely continuous solution $x_0 : [T_0, T_1] \to H$ of the differential inclusion

$$-\dot{x}_0(t) \in \partial \varphi(t, x_0(t)) + G(t, x_0(t)),\ t \in [T_0, T_1];\ x_0(T_0) = x_0 \in dom\varphi(T_0, \cdot).$$

Likewise, for each $i \in \{0, \cdots, n-1\}$ we construct an absolutely continuous mapping $x_i : [T_i, T_{i+1}] \to H$ such that

$$\begin{cases} -\dot{x}_i(t) \in \partial \varphi(t, x_i(t)) + G(t, x_i(t)) \;\; \text{a.e.} \;\; t \in [T_i, T_{i+1}], \\ \qquad x_i(T_i) = x_{i-1}(T_i) \in dom\varphi(T_i, \cdot). \end{cases} \tag{3.10}$$

Taking $x : \mathbf{R}_+ \to H$ defined by $x(t) := x_n(t)$ for all $t \in [T_n, T_{n+1}[$ and $n \in \mathbf{N}$, it is readily seen that x is locally absolutely continuous solution of $(\mathcal{P}_1)$ on $\mathbf{R}_+$. □

In the next theorem, we weaken the hypothesis on G by taking G having a measurable selection with respect to the first variable and upper hemicontinuous on H.

Theorem 3.3. *Under the assumptions of Theorem 3.1 on φ, let $G : [0, T] \times H \to \mathcal{P}_{cc}(H)$ be such that:*

- (a) *for all $t \in [0, T]$, $G(t, \cdot)$ is upper hemicontinuous on H,*
- (b) *for any $x \in H$, $G(\cdot, x)$ has a λ-measurable selection,*
- (c) *for some compact convex subset $K \subset \mathbf{B}$ and some real number $\gamma > 0$, for all $(t, x) \in [0, T] \times H$, one has $G(t, x) \subset \gamma(1+ \parallel x \parallel)K$.*

Then, for any $x_0 \in dom\varphi(0,\cdot)$ *the Cauchy problem* $(\mathcal{P})$ *admits at least one absolutely continuous solution, more precisely, there exist an absolutely continuous mapping* $x(\cdot) : [0,T] \to H$ *and an integrable mapping* $g : [0,T] \to H$ *such that* $x(0) = x_0$, $x(t) \in dom\varphi(t, x(t))$, *for all* $t \in [0,T]$, *and for almost every* $t \in [0,T]$, $g(t) \in G(t, x(t))$ *and* $-\dot{x}(t) - g(t) \in \partial\varphi(t, x(t))$.

Proof. Choose some positive numbers α, R such that $\alpha = \gamma(1 + R)$ and $R = \sqrt{2}(\|x_0\|^2 + Tc)^{\frac{1}{2}}$, where c is as in Theorem 3.1, and fix a continuous function $\psi : \mathbf{R}_+ \to [0,1]$ such that

$$\psi(\tau) = \begin{cases} 1 & \text{if } \tau \leq R, \\ 0 & \text{if } \tau \geq R+1. \end{cases} \tag{3.11}$$

Let us consider the compact convex metric space $Y := \gamma(1+R)K$, which is a Borel subset of H, and let us define a set-valued mapping $\widehat{G} : [0,T] \times H \to \mathcal{P}_{ck}(Y)$ by

$$\widehat{G}(t,x) := \psi(\|x\|)G(t,x),$$

obviously, $\widehat{G}(\cdot,x)$ has a measurable selection for all $x \in H$ and for each $t \in [0,T]$, the graph of $\widehat{G}(t,\cdot)$ is closed in $H \times Y$, therefore, in view of Corollary 2.2, there exists a measurable set-valued mapping $G_0 : [0,T] \times H \to \mathcal{P}_{ck}(Y) \cup \{\emptyset\}$ such that:

(i) there is a λ-negligible set $N \subset [0,T]$, such that

$$G_0(t,x) \subset \widehat{G}(t,x) \text{ for all } t \notin N \text{ and for all } x \in H; \tag{3.12}$$

(ii) for every $n \geq 1$, there is a compact subset $J_n \subset [0,T]$ such that $\lambda([0,T] \setminus J_n) < \frac{1}{n}$, the graph of the restriction $G_0/J_n \times H$ is closed and $\emptyset \neq G_0(t,x) \subset \widehat{G}(t,x)$, $\forall (t,x) \in J_n \times H$;

further, (ii) implies that there exists an increasing sequence $(J_n)_{n\geq 1}$ of compact subsets of $[0,T]$ such that, for each $n \geq 1$, $G_0/J_n \times H$ is upper semicontinuous with convex compact values. So, by the set-valued version of Dugundji's extension theorem, for each $n \geq 1$, there exists some upper semicontinuous extension $\overline{G}_n$ of $G_0/J_n \times H$ to $[0,T] \times H$ satisfying

$$\overline{G}_n(t,x) \subset \gamma(1+ \| x \|)K, \text{ for all } (t,x) \in [0,T] \times H$$

and $\overline{G}_n(t,x) = G_0(t,x)$ on $J_n \times H$. Further, $d(0, \overline{G}_n(t,x)) \leq \alpha$, $\forall (t,x) \in J_n \times H$. Due to Theorem 3.1 , for each $n \geq 1$, there exists an absolutely continuous map $x_n(\cdot) : [0,T] \to H$ and an integrable map $g_n : [0,T] \to H$ such that $x_n(0) = x_0$ and for almost all $t \in [0,T]$, $-\dot{x}_n(t) \in \partial\varphi(t, x_n(t)) + g_n(t)$, and

$$g_n(t) \in \overline{G}_n(t, x_n(t)), \tag{3.13}$$

with

$$\| g_n(t) \| \leq \alpha \tag{3.14}$$

and

$$\sup_{n\in\mathbf{N}} \int_0^T \| \dot{x}_n(t) \|^2 \, dt \leq c, \tag{3.15}$$

and thus $L = \sup\limits_{n\in\mathbf{N}} \| \dot{x}_n \|_{L^2_H([0,T]} < +\infty$. As in the proof of Theorem 3.1, using the Cauchy-Schwarz inequality and by (3.15) we obtain, for each n,

$$\| x_n(\cdot) \|_\infty \leq \alpha. \tag{3.16}$$

Therefore, for all $s, t \in [0, T]$ one has

$$\| x_n(t) - x_n(s) \| \leq (t-s)^{\frac{1}{2}} (\int_0^T \| \dot{x}_n(\tau) \|^2 d\tau)^{\frac{1}{2}} \leq (t-s)^{\frac{1}{2}} L,$$

so along with (3.16), the set $\{x_n(\cdot), n \in \mathbf{N}\}$ is bounded and equicontinuous in $\mathrm{C}_H([0,T])$. Recall that, in view of Proposition 2.1, for any fixed $t \in [0,T]$ and any n, one has

$$|\varphi(t, x_n(t)) - \varphi(0, x(0))| < +\infty.$$

So, since φ is inf-ball compact, the set $\{x_n(t), n \in \mathbf{N}\}$ is relatively compact in H. By Ascoli's Theorem, we can extract a subsequence of $(x_n(\cdot))_n$ that converges uniformly on $[0,T]$ to some continuous map $x(\cdot) \in \mathrm{C}_H([0,T])$, that is

$$x_n(\cdot) \to x(\cdot) \quad \text{strongly in} \quad L^2_H([0,T]). \tag{3.17}$$

By (3.15), the sequence $(\dot{x}_n)_n$ is bounded in $L^2_H([0,T])$, we may then extract a subsequence converging weakly in $L^2_H([0,T])$ to some map $v(\cdot)$. The equality

$$x_n(t) = x_n(0) + \int_0^t \dot{x}_n(s) ds, \quad \text{for all } t \in [0,T],$$

then yields

$$x(t) = x(0) + \int_0^t v(s) ds \quad \text{for all } t \in [0,T]$$

and hence the map $x(\cdot)$ is absolutely continuous on $[0,T]$ with $\dot{x}(\cdot) = v(\cdot)$ for almost all $t \in [0,T]$ and

$$\dot{x}_n(\cdot) \to \dot{x}(\cdot) \quad \text{weakly in} \quad L^2_H([0,T]). \tag{3.18}$$

Due to (3.14), we may also suppose that, for some map $g(\cdot) \in L^2_H([0,T])$, one has

$$g_n(\cdot) \to g(\cdot) \quad \text{weakly in} \quad L^2_H([0,T]). \tag{3.19}$$

Taking (3.17), (3.18) and (3.19) into account, as in the proof of Theorem 3.1 we have, via the closure property of the subdifferential operator $\partial\varphi(t,\cdot)$ for almost all $t \in [0,T]$ the required inclusion, that is,

$$\dot{x}(t) + g(t) \in -\partial\varphi(t, x(t)) \quad \text{a.e. } t \in [0,T]. \tag{3.20}$$

It remains to prove that $g(t) \in G(t, x(t))$ for almost every $t \in [0,T]$.
Due to (3.19), by Mazur theorem, there exists a sequence $(\xi_n(\cdot))_n$ in $L^1_H([0,T])$ such that

$$\xi_n(\cdot) \in co\{g_q(\cdot), q \geq n\} \quad \text{for all} \quad n \geq 1, \tag{3.21}$$

which converges strongly in $L^1_H([0,T])$ to $g(\cdot)$. Thus, extracting a subsequence if necessary we may suppose that $\xi_n(t) \to g(t)$ for almost every $t \in [0,T]$. So, this along with (3.21), implies that, for some negligible subset $N_1 \subset [0,T]$,

$$g(t) \in \bigcap_n \overline{co}\{g_q(t), q \geq n\} \quad \text{for all} \ \ t \in [0,T] \setminus N_1. \tag{3.22}$$

Taking (3.13) into account, we may also suppose that, for all $n \geq 1$ and for all $t \in [0,T] \setminus N_1$,

$$g_n(t) \in \overline{G}_n(t, x_n(t)). \tag{3.23}$$

Consider the λ-negligible subset $N_2 = ([0,T] \setminus \cup_n J_n) \cup N \cup N_1$, we are going to prove that $g(t) \in G(t, x(t))$ for all $t \in [0,T] \setminus N_2$. Fix any $\tau \in [0,T] \setminus N_2$, from (3.22) and (3.23), it follows that, for any $y \in H$,

$$\langle y, g(\tau)\rangle \leq \limsup_n \sigma(y, \overline{G}_n(\tau, x_n(\tau))). \tag{3.24}$$

On the other hand, by definition of N_2, there exists an integer $p(\tau)$ such that $\tau \in J_{p(\tau)} \setminus N$ and $(J_n)_n$ being increasing, one has $\tau \in J_n$ for all $n \geq p(\tau)$. Consequently, for all $n \geq p(\tau)$,

$$\overline{G}_n(\tau, x_n(\tau)) = G_0(\tau, x_n(\tau)) \subset \widehat{G}(\tau, x_n(\tau)). \tag{3.25}$$

The inclusion coming from (3.12). Note that, by (3.16) one has, for all $n \geq 1$ and for almost all $t \in [0,T]$,

$$\| x_n(t) \| \leq R,$$

and hence by (3.11), for all $n \geq 1$,

$$\widehat{G}(\tau, x_n(\tau)) = G(\tau, x_n(\tau)). \tag{3.26}$$

Therefore, due to (3.24), (3.25) and (3.26) and the fact that $G(\tau, \cdot)$ is scalarly upper semicontinuous, we have

$$\langle y, g(\tau)\rangle \leq \sigma(y, G(\tau, x(\tau))),$$

this being true for any $y \in H$, and $G(\tau, x(\tau))$ being closed and convex, it results that $g(t) \in G(t, x(t))$. Since the latter is satisfied for any $\tau \in [0,T] \setminus N_2$, one has $g(t) \in G(t, x(t))$ a.e. $t \in [0,T]$. This, along with (3.20), proves that $x(\cdot)$ is a solution of $(\mathcal{P})$. □

4. Application

Let φ be the indicator function of a nonempty closed convex moving set $C(t)$, that is, $\varphi(t,x) = I_{C(t)}(x) = 0$ if $x \in C(t)$ and $+\infty$ otherwise. It is well-known that $\partial I_{C(t)}(x) = N_{C(t)}(x)$ the normal cone to C(t) at x. Then problem $(\mathcal{P})$ becomes

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t) + G(t, x(t)) \;\text{ a.e. } t \in [0,T], \\ x(0) = x_0 \in C(0). \end{cases}$$

Problems of this form are known as "sweeping process" and arise in elastoplasticity, contact dynamics, friction dynamics, and granular material (see Moreau [15]). The sweeping process model is also of great interest in nonsmooth mechanics, convex optimization, mathematical economics and more recently in the modeling and simulation of switched electrical circuits as well as the modeling of crowd motion. As an example, let consider dynamics that correspond to an electrical circuit containing nonsmooth devices like diodes. A diode is a device that constitutes a rectifier which permits the easy flow of charges in one direction and restrains the flow in the opposite direction. The ideal model diode is a simple switch.The problem is the following:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{L} + \frac{1}{L}u \\ y = -x_2 \;\text{ and }\; y_L \in \partial\Phi(y); \end{cases}$$

where $R > 0$ is a resistor, $L > 0$ an inductor, $C > 0$ a capacitor, u is the voltage supply, $x_1(t)$ is the time integral of the current across the capacitance, $x_2(t) = i(t)$ is the current across the circuit, y_L is the voltage of the diode and Φ is the electrical superpotential of the diode. Setting $\varphi = \Phi \circ C$, we get $\partial\varphi(x) = B\partial\Phi(B^t x)$. Therefore, the dynamic of the system is of the form $-\dot{x}(t) \in Ax(t) + \partial\varphi(x(t))$, where

$$A = \begin{pmatrix} 0 & 0 \\ \frac{-1}{LC} & \frac{-R}{L} \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

If we suppose that the diode is ideal, then its superpotential and subdifferential are respectively given by $\Phi(x) = I_{\mathbf{R}_+}(x)$ and $\partial\Phi(x) = N_{\mathbf{R}_+}(x) = \begin{cases} \emptyset \text{ if } x < 0 \\ ]-\infty, 0] \text{ if } x = 0 \\ 0 \text{ if } x > 0. \end{cases}$

5. Conclusion

We have established existence results for nonlinear evolution inclusions which are driven by time dependent subdifferential operators, by using a specific and adapted discretization, with technical nuances, in both convex analysis and non-smooth analysis. We generalize the results when the perturbation, that is, the external forces applied on the system, is with convex but not necessary bounded values. In a forthcoming work, we deal with a nonconvex perturbation by the relaxation (convexification) approach.

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Sarra Boudada,
LMPA Laboratory, Jijel University, PB 98, Algeria, Phone: +213 790386705, Orcid Id: 0000-0002-9193-6933
Email address: `sarraboudada@gmail.com`

Mustapha Fateh Yarou,
LMPA Laboratory, Department of Mathematics, Jijel University, PB 98, Algeria, Phone: +213 662419610, Orcid Id: 0000-0003-4083-1813
Email address: `mfyarou@yahoo.com`