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Research Article

Inference and prediction of progressive Type-II censored data from Unit-Generalized Rayleigh distribution

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Abstract

In this paper, inference and prediction problems are studied under progressively Type-II censored data. When the latent lifetime follows the Unit-Generalized Rayleigh distribution, maximum likelihood estimators of the unknown parameters are established, and corresponding existence and uniqueness are also provided. Besides, the approximate confidence intervals are constructed based on asymptotic approximation theory. For comparison, another alternative generalized point and interval estimates are constructed based on proposed pivotal quantities. Further, point and interval predictions for the censored samples are established by using conventional classical and generalized inferential approaches. Finally, extensive simulation studies are carried out to investigate the performance of different methods, and one real-life example is presented to illustrate the applicability of the obtained results.

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1. Introduction

Nowdays, modern products are always exceedingly durable and feature a long lifetime in their lifecycle. Therefore, sometimes it is difficult to collect complete failure data due to practical constraints like time and cost. Under such situation, the observations appeared frequently as censored data. To be specific, censoring means that there is a part of failure times observed in life testing and other data collection procedures. In practice, Type-I censoring and Type-II censoring are conventional most used censoring schemes (CSs). The Type-I CS allows the test to stop at a predetermined time, whereas the Type-II CS enables that the test stops when the pre-fixed number of failures is obtained. However, both of these CSs do not allow experimenters to withdraw any alive units during the test. In order to give a more general testing way, the progressive censoring scheme is further introduced in practice including progressive Type-I CS and progressive Type-II CS as its special cases, which appears more flexible and allows experimenters to remove survival units at different

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stages of the test. For progressive Type-II CS (PCS-II), it can be described as follows. Suppose a total of n units is put into a life test, and $m(\leqslant n)$ and $(r_1, r_2, \ldots, r_m)$ are the pre-fixed number of failures and CS, respectively. When the first failure occurs, r_1 live units are randomly removed from the remaining $n-1$ units. When the second failure occurs, r_2 live units are randomly withdrawn from the remaining $n-1-(r_1+1)$ units. Following similar procedure, when mth failure occurs, all remaining survival $n-1-\sum_{i=1}^{m-1}(r_i+1)$ units are removed and the test stops. Therefore, a progressively Type-II censored sample of size m can be obtained as $\{T_{1:m:n}, T_{2:m:n}, \ldots, T_{m:m:n}\}$ with CS $(r_1, r_2, \ldots, r_m)$. It is seen clearly that the PCS-II is a more general testing approach than conventional CSs, and commonly used complete and Type-II censoring are its special cases. Progressive censoring has attracted wide attention in both theoretical studies and practical applications and has been discussed by many authors. See, for example, some works of [1,9,13,14]. For more details, one may refer to the monographs by [2,3].

Lifetime models with bounded support are very useful in data analysis. Especially, when the support lies in $(0,1)$, the associated models are called unit lifetime distribution in literature. Besides conventional beta and Kumaraswamy distributions, another type of unit distributions are proposed by using variable transformation (e.g. [4,15,16]). Recently, a Unit-Generalized Rayleigh (UGR) distribution is proposed by [11]. Let random variable X follows the UGR distribution, then its cumulative distribution function (CDF) and probability density function (PDF) are presented as

$$F(x;\theta,\lambda) = 1 - \left[1 - e^{-\lambda(\log x)^2}\right]^{\theta}, 0 < x < 1 \tag{1.1}$$

and

$$f(x;\theta,\lambda) = 2\theta\lambda\left(\frac{1}{x}\right)\left(\log\frac{1}{x}\right)e^{-\lambda(\log x)^2}\left[1 - e^{-\lambda(\log x)^2}\right]^{\theta-1}, 0 < x < 1 \tag{1.2}$$

where $\lambda > 0$ and $\theta > 0$ are scale and shape parameters, respectively. Correspondingly, associated survival function (SF) and hazard function (HF) are given by

$$S(x;\theta,\lambda) = \left[1 - e^{-\lambda(\log x)^2}\right]^{\theta} \text{ and } H(x;\theta,\lambda) = \frac{2\theta\lambda\left(\frac{1}{x}\right)\left(\log\frac{1}{x}\right)}{e^{\lambda(\log x)^2}-1}, 0 < x < 1. \tag{1.3}$$

Hereafter, the UGR distribution with parameters θ and λ is denoted as $UGR(\theta,\lambda)$. This distribution is not only a continuous probability distribution with double-bounded support, but also has a variety of shapes under different choices of parameters θ and λ, which can be used to model various characteristics of practical lifetime data. For illustrations, some plots of the PDF, SF and HF are presented in Figure 1 showing great flexibility. Although the commonly used lifetime distributions feature infinite supports like $(0,\infty)$ and $(-\infty,\infty)$, in various practical areas such as engineering, reliability, survival analysis, medicine, etc, the lifetime of products cannot reach to infinite. Under this situation, it is more reasonable to use a distribution with bounded support for statistical inference, where a bounded model may provide high weights for observations and provide a better fit in data analysis. Besides, the UGR distribution has an invertible closed form CDF, which makes it suitable for computation-intensive activities like the Kumaraswamy distribution. Thus, the UGR distribution discussed in this paper may have its potential theoretical and applicable applicability.

Motivated by such previous reasons and due to the practical applications of the UGR distribution, this paper discusses estimation of the unknown parameters when data are progressively Type-II censored. The maximum likelihood estimates (MLEs) are established, and corresponding approximate confidence intervals (ACIs) are also constructed based on asymptotic approximation theory. For comparison, another general estimation approach based on pivotal quantities is also proposed as a supplement. The numerical

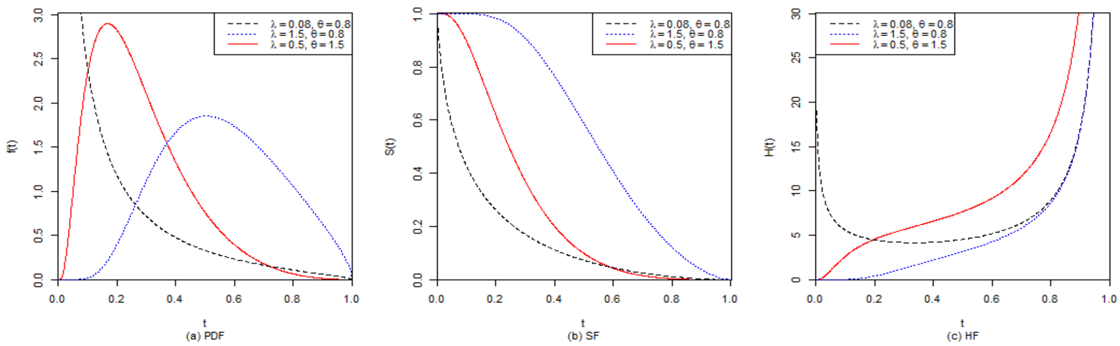

Figure 1. The plots of PDF, SF and HF

results indicate that our proposed general point and interval estimates for both model parameters and prediction issues are superior to traditional results.

The rest of the paper is organized as follows. Conventional likelihood based estimates are discussed in Section 2. Section 3 proposes another pivotal quantity based generalized point and interval estimates. The prediction issues for progressively censored samples are also presented in Section 4 under both likelihood and the proposed general inference approaches, respectively. In Section 5, simulation studies and a real-life analysis are provided for illustration. Finally, some concluding remarks are given in Section 6.

2. Likelihood based estimation

In this section, the MLEs of the unknown parameters are established, and the corresponding ACIs are also constructed based on asymptotic approximation theory.

2.1. Maximum likelihood estimate

Suppose $\{T_{1:m:n}, T_{2:m:n}, \ldots, T_{m:m:n}\}$ is a progressively Type-II censored samples with CS $(r_1, r_2, \ldots, r_m)$ from the UGR distribution with parameters θ and λ, then the log-likelihood function of λ and θ can be expressed as (e.g. [2])

$$\begin{aligned} l(\theta, \lambda) \propto & m(\log 2 + \log\theta + \log\lambda) - \lambda \sum_{i=1}^{m} (\log t_i)^2 \\ & + \sum_{i=1}^{m} \left\{ \log\frac{1}{t_i} + \log\left(\log\frac{1}{t_i}\right) + (\theta r_i + \theta - 1)\log\left[1 - e^{-\lambda(\log t_i)^2}\right] \right\}. \end{aligned} \tag{2.1}$$

where $\{t_1, t_2, \ldots, t_m\}$ is the observation of $\{T_{1:m:n}, T_{2:m:n}, \ldots, T_{m:m:n}\}$.

Since it is complex to directly solve the MLEs of two unknown parameters through the likelihood equation $\frac{\partial l(\theta,\lambda)}{\partial\theta} = \frac{\partial l(\theta,\lambda)}{\partial\lambda} = 0$, the profile likelihood approach is utilized to find the MLEs. Under this procedure, the estimates that maximize the profile likelihood function are equal to the MLEs obtained from the full likelihood function. For more details, one can refer to [5] for a review. In the following, two helpful theorems are proposed to obtain the MLEs.

Theorem 2.1. *Suppose $T_{1:m:n}, T_{2:m:n}, \ldots, T_{m:m:n}$ are order statistics from $UGR(\theta, \lambda)$ under progressive Type-II censoring with CS $(r_1, r_2, \ldots, r_m)$. For given $\lambda > 0$ and $m > 0$, the MLE of θ exists, which can be written as*

$$\tilde{\theta}(\lambda) = \frac{-m}{\sum_{i=1}^{m}(r_i + 1)\log\left[1 - e^{-\lambda(\log t_i)^2}\right]}. \tag{2.2}$$

Proof. See Appendix A. □

Substituting $\tilde{\theta}(\lambda)$ for θ in (2.1), the profile log-likelihood for λ (without the additive constant) can be obtained as

$$l(\lambda) = m\log\lambda - m\log\left\{-\sum_{i=1}^{m}(r_i+1)\log\left[1-e^{-\lambda(\log t_i)^2}\right]\right\} \qquad (2.3)$$
$$-\sum_{i=1}^{m}\left\{\lambda(\log t_i)^2 + \log\left[1-e^{-\lambda(\log t_i)^2}\right]\right\}.$$

Theorem 2.2. *For $m > 0$, the MLE $\hat{\lambda}$ of λ uniquely exists, which can be obtained from equation $p(\lambda) = 0$ with*

$$p(\lambda) = -\frac{m\sum_{i=1}^{m}(r_i+1)\frac{(\log t_i)^2}{e^{\lambda(\log t_i)^2}-1}}{\sum_{i=1}^{m}(r_i+1)\log\left[1-e^{-\lambda(\log t_i)^2}\right]} + \frac{m}{\lambda} - \sum_{i=1}^{m}\left[(\log t_i)^2 + \frac{(\log t_i)^2}{e^{\lambda(\log t_i)^2}-1}\right]. \qquad (2.4)$$

Proof. See Appendix B. □

Generally, the closed form of MLE $\hat{\lambda}$ cannot be obtained from equation (2.4), then one can use some numerical approaches such as the Newton-Raphson method, fixed point iteration to obtain its estimate. Since the solution of (2.4) with respect to λ uniquely exists, it is reasonable to use common bisection method to find the root. Once $\hat{\lambda}$ is obtained, the MLE of θ can be obtained from Theorem 2.1 as

$$\hat{\theta} = \tilde{\theta}(\hat{\lambda}) = \frac{-m}{\sum_{i=1}^{m}(r_i+1)\log\left[1-e^{-\hat{\lambda}(\log t_i)^2}\right]}$$

2.2. Approximate confidence interval estimation

Since the MLEs of parameters θ and λ cannot be obtained in closed form, it is difficult to get their exact distribution as well as corresponding exact confidence intervals. In this subsection, ACIs for θ and λ are constructed based on the Fisher information matrix and asymptotic approximation theory.

The Fisher information matrix of unknown parameters $\eta = (\theta, \lambda)$ is given by

$$I(\eta) = [I_{ij}]_{2\times2} = \begin{pmatrix} -\frac{\partial^2 l(\theta,\lambda)}{\partial\theta^2} & -\frac{\partial^2 l(\theta,\lambda)}{\partial\theta\partial\lambda} \\ -\frac{\partial^2 l(\theta,\lambda)}{\partial\lambda\partial\theta} & -\frac{\partial^2 l(\theta,\lambda)}{\partial\lambda^2} \end{pmatrix}$$

with

$$I_{11} = -\frac{\partial^2 l(\theta,\lambda)}{\partial\theta^2} = \frac{m}{\theta^2},$$
$$I_{22} = -\frac{\partial^2 l(\theta,\lambda)}{\partial\lambda^2} = \frac{m}{\lambda^2} + \sum_{i=1}^{m}(\theta r_i + \theta - 1)\frac{(\log t_i)^4 e^{\lambda(\log t_i)^2}}{\left[e^{\lambda(\log t_i)^2}-1\right]^2},$$
$$I_{12} = I_{21} = -\frac{\partial^2 l(\theta,\lambda)}{\partial\lambda\partial\theta} = -\sum_{i=1}^{m}(r_i+1)\frac{(\log t_i)^2}{e^{\lambda(\log t_i)^2}-1}.$$

Under some mild regularity conditions (see [17]), the asymptotic distribution of the MLE $\hat{\eta}$ is given by

$$\hat{\eta} - \eta \xrightarrow{d} N(0, I^{-1}(\hat{\eta}))$$

where '$\xrightarrow{d}$' denotes convergence in distribution and $I^{-1}(\hat{\eta})$ is the inverse of the observed Fisher information matrix $I(\eta)$ with

$$I^{-1}(\hat{\eta}) = \begin{pmatrix} Var(\hat{\theta}) & Cov(\hat{\theta},\hat{\lambda}) \\ Cov(\hat{\lambda},\hat{\theta}) & Var(\hat{\lambda}) \end{pmatrix}.$$

Hence, for arbitrary $0 < \gamma < 1$, the $100(1-\gamma)\%$ ACIs of the parameters θ and λ can be constructed as

$$\left(\hat{\theta} - z_{\gamma/2}\sqrt{Var(\hat{\theta})}, \hat{\theta} + z_{\gamma/2}\sqrt{Var(\hat{\theta})}\right)$$

and

$$\left(\hat{\lambda} - z_{\gamma/2}\sqrt{Var(\hat{\lambda})}, \hat{\lambda} + z_{\gamma/2}\sqrt{Var(\hat{\lambda})}\right)$$

respectively, where z_γ is the upper γ-th quantile of the standard normal distribution.

Remark 2.3. Denote $R_x = S(x;\theta,\lambda)$. For a unit whose lifetime follows $UGR(\theta,\lambda)$, once the MLEs of θ and λ are obtained, the corresponding MLE for R_x can be expressed as

$$\hat{R}_x = \left[1 - e^{-\hat{\lambda}(\log x)^2}\right]^{\hat{\theta}}, 0 < x < 1.$$

Then the reliability of this unit can be calculated at a given time x. Furthermore, the delta method (see, e.g. [22]) can be utilized to find the asymptotic normal distribution of R_x as $\frac{R_x - \hat{R}_x}{\sqrt{\widehat{Var}(\hat{R}_x)}} \xrightarrow{d} N(0,1)$ where $\widehat{Var}(\hat{R}_x) = (\nabla\hat{R}_x)^T I^{-1}(\hat{\eta})(\nabla\hat{R}_x)$ and $(\nabla\hat{R}_x)^T = \left(\frac{\partial\hat{R}_x}{\partial\theta}, \frac{\partial\hat{R}_x}{\partial\lambda}\right)^T\Big|_{\eta=\hat{\eta}}$. Therefore, $100(1-\gamma)\%$ ACI for R_x can be established as

$$\left(\hat{R}_x - z_{\gamma/2}\sqrt{\widehat{Var}(\hat{R}_x)}, \hat{R}_x + z_{\gamma/2}\sqrt{\widehat{Var}(\hat{R}_x)}\right)$$

3. Pivotal quantities based estimation

In practical data analysis, small and median sample size sometimes may yield poor performance for MLEs. Under such situation, another pivotal quantity based generalized estimators are constructed for supplement and comparison.

First of all, a very helpful result is provided as follows.

Theorem 3.1. *Suppose $T_{1:m:n}, T_{2,m,n}, \ldots, T_{m:m:n}$ are order statistics from $UGR(\theta,\lambda)$ under progressive Type-II censoring with CS $(r_1, r_2, \ldots, r_m)$. Then pivotal quantities*

$$S_1(\lambda) = 2\sum_{i=1}^{m-1}\log\left\{1 + \frac{-\left[n - \sum_{j=1}^{i}(r_j+1)\right] + \sum_{j=i+1}^{m}(r_j+1)\frac{\log\left[1-e^{-\lambda(\log(T_{j:m:n}))^2}\right]}{\log\left[1-e^{-\lambda(\log(T_{i:m:n}))^2}\right]}}{\left[n - \sum_{j=1}^{i}(r_j+1)\right] + \sum_{j=1}^{i}(r_j+1)\frac{\log\left[1-e^{-\lambda(\log(T_{j:m:n})^2)}\right]}{\log\left[1-e^{-\lambda(\log(T_{i:m:n})^2)}\right]}}\right\} \tag{3.1}$$

and

$$S_2(\theta,\lambda) = -2\theta\sum_{j=1}^{m}(r_j+1)\log\left[1 - e^{-\lambda(\log(T_{j:m:n}))^2}\right] \tag{3.2}$$

follow chi-square distributions with $2m-2$ and $2m$ degrees of freedom, respectively. Furthermore, $S_1(\lambda)$ and $S_2(\theta,\lambda)$ are statistically independent.

Proof. See Appendix C. □

In order to construct generalized estimators for the unknown parameters, another useful result is given in the following lemma.

Lemma 3.2. *Let $G(\lambda) = \frac{\log\left[1-e^{-\lambda(\log a)^2}\right]}{\log\left[1-e^{-\lambda(\log b)^2}\right]}, 0 < b < a < 1$. Then function $G(\lambda)$ increases in λ with $\lim_{\lambda\to0^+} G(\lambda) = 1$ and $\lim_{\lambda\to+\infty} G(\lambda) = +\infty$.*

Proof. See Appendix D. □

Based on the lemma 3.2, one has the following result by direct computation.

Corollary 3.3. *Pivotal quantity $S_1(\lambda)$ increases in λ with range $(0,+\infty)$.*

From Theorem 3.1 and Corollary 3.3, for a given $S_1 \sim \chi^2(2m-2)$, equation $S_1(\lambda) = S_1$ has an unique solution with respect to λ, and the solution is denoted as $h(S_1; T)$ which could be regarded as the generalized estimate of parameter λ from the perspective of inverse moment estimation, where $T = \{T_{1:m:n}, T_{2,m,n}, \ldots, T_{m:m:n}\}$ being the progressively Type-II censored samples. Furthermore, it is noted from Theorem 3.1 that $S_2(\theta, \lambda)$ decreases in λ with range $(0, +\infty)$, then one has

$$\theta = \frac{S_2}{-2\sum_{j=1}^{m}(r_j+1)\log\left[1-e^{-\lambda(\log(T_{j:m:n}))^2}\right]} \text{ with } S_2 \sim \chi^2(2m). \tag{3.3}$$

According to the substitution method of [21], a generalized pivotal quantity Q for θ can be established by substituting $h(S_1; T)$ for λ in (3.3) as follows

$$Q = \frac{S_2}{H\left[h(S_1;t)\right]} \text{ with } H(\lambda) = -2\sum_{j=1}^{m}(r_j+1)\log\left[1-e^{-\lambda(\log(T_{j:m:n}))^2}\right]$$

where t denotes the observation of T. Therefore, an algorithm is proposed to construct the generalized estimates of λ and θ as follows.

Algorithm 1. Generalized estimation for λ and θ.

Step 1 Generate N realizations from $\chi^2(2m-2)$ as $S_{11}, S_{12}, \ldots, S_{1N}$.

Step 2 Compute $\lambda_l = h(S_{1l}; t), l = 1, 2, \ldots, N$, then the generalized point estimate (GPE) for λ can be expressed as $\acute{\lambda} = \frac{1}{N}\sum_{l=1}^{N}\lambda_l$.

Step 3 Generate N samples from $\chi^2(2m)$ as $S_{21}, S_{22}, \ldots, S_{2N}$.

Step 4 Compute

$$\theta_l = \frac{S_{2l}}{-2\sum_{j=1}^{m}(r_j+1)\log\left[1-e^{-\lambda_l[\log(t_j)]^2}\right]}, l = 1, 2, \ldots, N,$$

then the associated GPE for θ can be calculated by $\acute{\theta} = \frac{1}{N}\sum_{l=1}^{N}\theta_l$.

Step 5 Arrange all estimates of θ and λ in an ascending order as $\theta_{[1]}, \theta_{[2]}, \ldots, \theta_{[N]}$ and $\lambda_{[1]}, \lambda_{[2]}, \ldots, \lambda_{[N]}$, respectively. For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)\%$ confidence intervals of θ and λ can be established as

$$\left(\theta_{[j]}, \theta_{[j+N-[N\gamma+1]]}\right), j = 1, 2, \ldots, [N\gamma]$$

and

$$\left(\lambda_{[j]}, \lambda_{[j+N-[N\gamma+1]]}\right), j = 1, 2, \ldots, [N\gamma],$$

where $[t]$ denotes the greatest integer less than or equal to t. Therefore, the $100(1-\gamma)\%$ generalized confidence intervals (GCIs) of θ and λ can be constructed as the $j^\star$th one satisfying

$$\theta_{[j^\star+N-[N\gamma+1]]} - \theta_{[j^\star]} = \min_{j=1}^{[N_\gamma]}\left(\theta_{[j+N-[N\gamma+1]]} - \theta_{[j]}\right)$$

and

$$\lambda_{[j^\star+N-[N\gamma+1]]} - \lambda_{[j^\star]} = \min_{j=1}^{[N_\gamma]}\left(\lambda_{[j+N-[N\gamma+1]]} - \lambda_{[j]}\right)$$

respectively.

Remark 3.4. For a unit whose lifetime follows $UGR(\theta, \lambda)$, the estimation of its reliability can be also obtained based on this generalized method. For a given time x, substituting θ_l and $\lambda_l, l = 1, \ldots, N$ for θ and λ in SF specifically presented in expression (1.3), all estimates for reliability R_x can be calculated as $R_{x1}, \ldots, R_{xN}$. Therefore, the GPE for R_x can be computed by $\acute{R}_x = \frac{1}{N}\sum_{l=1}^{N}R_{xl}$. Arrange all estimates for R_x in an ascending order as $R_{x[1]}, \ldots, R_{x[N]}$, then its $100(1-\gamma)\%$ GCI can be constructed as

$$\left(R_{x[j^\star]}, R_{x[j^\star+N-[N\gamma+1]]}\right)$$

when $j^\star$th one satisfying $R_{x[j^\star+N-[N\gamma+1]]} - R_{x[j^\star]} = \min_{j=1}^{[N_\gamma]}\left(R_{x[j+N-[N\gamma+1]]} - R_{x[j]}\right)$.

4. Prediction

Besides parameter estimation, lifetime prediction is also widely concerned in both theoretical studies and practical applications. Many authors have studied the problem of prediction. See, for instance, [7, 8, 10, 12, 18]. It is observed that $\sum_{i=1}^{m} r_i$ units are randomly removed in progressive Type-II censoring, and the failure time of these units cannot be observed. In this section, We interest in predicting the lifetimes of the random removal units.

Let $Y_{ij}, i = 1, \ldots, m, j = 1, \ldots, r_i$ be the jth order statistic of r_i units removed at ith failure, the prediction for $Y_{ij}, i = 1, \ldots, m, j = 1, \ldots, r_i$ will be established by likelihood and the pivotal quantities based generalized approaches. Recall that $T_{1:m:n} < T_{2:m:n} < \cdots < T_{m:m:n}$ are the order statistics from $UGR(\theta, \lambda)$ under progressive Type-II censoring test with CS $(r_1, r_2, \ldots, r_m)$. Due to the Markovian property of progressively Type-II censored order statistics, from [6], then the conditional density function of the jth order statistic out of r_i removed units can be written as

$$f(y_{ij}|t_i) = j\begin{pmatrix} r_i \\ j \end{pmatrix} f(y_{ij})\left[F(y_{ij}) - F(t_i)\right]^{j-1}\left[1 - F(y_{ij})\right]^{r_i - j}\left[1 - F(t_i)\right]^{-r_i}, y_{ij} > t_i. \tag{4.1}$$

From (1.1) and (1.2), the predicition distribution can be expressed as

$$\begin{aligned} f(y_{ij}|t_i) =& 2\theta\lambda j\begin{pmatrix} r_i \\ j \end{pmatrix}\left(\frac{1}{y_{ij}}\right)\log\left(\frac{1}{y_{ij}}\right)e^{-\lambda(\log y_{ij})^2}\left[1 - e^{-\lambda(\log y_{ij})^2}\right]^{\theta-1} \\ &\times\left[1 - e^{-\lambda(\log t_i)^2}\right]^{-\theta r_i}\left\{\left[1 - e^{-\lambda(\log t_i)^2}\right]^{\theta} - \left[1 - e^{-\lambda(\log y_{ij})^2}\right]^{\theta}\right\}^{j-1} \\ &\times\left[1 - e^{-\lambda(\log y_{ij})^2}\right]^{\theta(r_i - j)}, y_{ij} > t_i. \end{aligned} \tag{4.2}$$

Before proceeding, a very helpful result is proposed to establish the likelihood and pivotal quantities based prediction.

Theorem 4.1. *Let $T_{i:m:n}, i = 1, \ldots, m$ be the order statistics from $UGR(\theta, \lambda)$ under progressive Type-II censoring with CS $(r_1, r_2, \ldots, r_m)$ and Y_{ij} be the jth order statistic of r_i removed units at ith failure. Then one has that variable*

$$Z_{ij} = \frac{\left[1 - e^{-\lambda(\log(Y_{ij}))^2}\right]^{\theta}}{\left[1 - e^{-\lambda(\log(T_{i:m:n}))^2}\right]^{\theta}}, i = 1, \ldots, m, j = 1, \ldots, r_i \tag{4.3}$$

follows $Beta(r_i - j + 1, j)$ distribution.

Proof. See Appendix E. □

- likelihood based prediction

  Denote b_α is the upper α-th quantile of $Beta(r_i - j + 1, j)$ with $0 < \alpha < 1$, let $Z_{ij} = b_{0.5}$ and substituting MLEs $\hat{\lambda}$ and $\hat{\theta}$ for unknown parameters λ and θ respectively, then the likelihood based point prediction estimation (LPPE) $\hat{y}_{ij}$ of Y_{ij} can be expressed as

$$\hat{y}_{ij} = \exp\left\{-\sqrt{-\frac{1}{\hat{\lambda}}\log\left[1 - b_{0.5}^{1/\hat{\theta}}\left[1 - e^{-\hat{\lambda}(\log t_i)^2}\right]\right]}\right\}. \tag{4.4}$$

  Further, since Z_{ij} follows Beta distribution $Beta(r_i - j + 1, j)$, then one has

$$p\left(b_{1-\alpha/2} < Z_{ij} < b_{\alpha/2}\right) = 1 - \alpha.$$

By direct transformation and replacing λ and θ by their MLEs $\hat{\lambda}$ and $\hat{\theta}$ respectively, then the $100(1-\gamma)\%$ likelihood based prediction interval (LPI) of Y_{ij} can be also constructed as

$$\left(\exp\left\{-\sqrt{-\frac{1}{\hat{\lambda}}\log\left[1-b_{\gamma/2}^{1/\hat{\theta}}\left[1-e^{-\hat{\lambda}(\log t_i)^2}\right]\right]}\right\}, \exp\left\{-\sqrt{-\frac{1}{\hat{\lambda}}\log\left[1-b_{1-\gamma/2}^{1/\hat{\theta}}\left[1-e^{-\hat{\lambda}(\log t_i)^2}\right]\right]}\right\} \right)$$

- pivotal quantities based prediction

  According to Theorem 4.1 and using the substitution method of [21], the point and interval prediction can be constructed for $Y_{ij}, i = 1,\ldots,m, j = 1,\ldots,r_i$ in following Algorithm 2 based on previous proposed pivotal quantities.

  Algorithm 2. Generalized predictions for Y_{ij}.

  Step 1 Using Steps 1-4 of algorithm 1, generate N parameters θ and λ as $\theta_1,\theta_2,\ldots,\theta_N$ and $\lambda_1,\lambda_2,\ldots,\lambda_N$.

  Step 2 Compute the predicitons for $Y_{ij}, i = 1,\ldots,m, j = 1,\ldots,r_i$ by

$$y_{ij}^{(l)} = e^{-\sqrt{-\frac{1}{\lambda_l}\log\left[1-(b_{0.5})^{1/\theta_l}\left[1-e^{-\lambda_l(\log t_i)^2}\right]\right]}}, l = 1,2,\ldots,N.$$

  Step 3 The generalized point prediction estimation (GPPE) of Y_{ij} can be computed by $\acute{y}_{ij} = \frac{1}{N}\sum_{l=1}^{N} y_{ij}^{(l)}$.

  Step 4 Arrange all predictors of Y_{ij} in an ascending order as $y_{ij}^{[1]}, y_{ij}^{[2]},\ldots,y_{ij}^{[N]}$. For arbitrary $0 < \gamma < 1$, a $100(1-\gamma)\%$ prediction interval for Y_{ij} can be obtained as

$$\left(y_{ij}^{[k]}, y_{ij}^{[k+N-[N\gamma+1]]}\right), k = 1,2,\ldots,[N\gamma],$$

  where $[t]$ denotes the greatest integer less than or equal to t. Therefore, the $100(1-\gamma)\%$ generalized prediction interval (GPI) for Y_{ij} can be constructed as k^*th prediction interval satisfying

$$y_{ij}^{[k^*+N-[N\gamma+1]]} - y_{ij}^{[k^*]} = \min_{k=1}^{[N\gamma]}\left(y_{ij}^{[k+N-[N\gamma+1]]} - y_{ij}^{[k]}\right).$$

5. Numerical illustration

5.1. Simulation studies

In this section, the performance of proposed estimates is investigated through extensive Monte-Carlo simulations. For comparison, average bias (ABs) and mean square errors (MSEs) are used to evaluate the results of point estimates. Besides, the performance of interval estimates is evaluated by average lengths (ALs) and converage probabilities (CPs). In this simulation, the algorithm suggested by [3] is used to generate progressively Type-II censored data. For different sample size, three kinds of CSs are adopted as follows.

CS-1: $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n-m$;

CS-2: $r_1 = n-m$ and $r_2 = r_3 = \cdots = r_m = 0$;

CS-3: $\begin{cases} r_1 = r_2 = \cdots = r_{n-m} = 1, r_{n-m+1} = \cdots = r_m = 0, & \text{for } n \le 2m \\ r_1 = r_2 = \cdots = r_{m-1} = 1, r_m = n-2m+1, & \text{for } n > 2m \end{cases}$

In order to calculate the MLE of λ, the bisection method is utilized to find the estimates, where the existence and uniqueness is guaranteed by Theorem 2.2. For quantities based generalized estimation, it is seen from Theorem 3.1 and Corollary 3.3 that $S_1(\lambda)$ increases in λ and $S_2(\theta,\lambda)$ decreases with respect to θ in the full range $(0,1)$, then pivotal quantities based estimates can be also obtained by using bisection method. Based on different choices of parameters and sample sizes, the simulation is conducted based on 10000 times replications, and the evaluation criterion quantities are reported in Tables 1-4, where the confidence level is 0.95.

Table 1. ABs and MSEs (within bracket) for parameters with $(\theta, \lambda) = (1.5, 1)$

n	m	CS	$\hat{\theta}$	$\hat{\lambda}$	$\acute{\theta}$	$\acute{\lambda}$
20	10	CS-1	1.3116[4.7772]	0.3451[0.2146]	1.2639[4.4195]	0.2912[0.1464]
		CS-2	0.7894[1.6935]	0.2873[0.1448]	0.7515[1.5875]	0.2694[0.1353]
		CS-3	1.1007[3.3891]	0.3197[0.1865]	1.0296[3.1545]	0.2691[0.1251]
	15	CS-1	0.6735[0.9247]	0.2701[0.1379]	0.6153[0.7645]	0.2536[0.1074]
		CS-2	0.5427[0.5676]	0.2534[0.1100]	0.5100[0.5308]	0.2422[0.0969]
		CS-3	0.5634[0.6399]	0.2482[0.1113]	0.5173[0.5548]	0.2303[0.0916]
30	15	CS-1	0.7243[1.0252]	0.2461[0.1030]	0.7133[0.9740]	0.2243[0.0797]
		CS-2	0.5216[0.5275]	0.2301[0.0881]	0.5105[0.5265]	0.2204[0.0833]
		CS-3	0.6575[0.8319]	0.2329[0.0932]	0.6284[0.7378]	0.2135[0.0747]
	20	CS-1	0.5533[0.6234]	0.2195[0.0849]	0.5472[0.5897]	0.2132[0.0750]
		CS-2	0.4709[0.4751]	0.2124[0.0785]	0.4660[0.4470]	0.2047[0.0722]
		CS-3	0.5129[0.5309]	0.2124[0.0788]	0.4922[0.4863]	0.2028[0.0716]
50	20	CS-1	0.6728[0.8795]	0.2049[0.0675]	0.6616[0.8445]	0.1873[0.0563]
		CS-2	0.4702[0.4471]	0.1965[0.0656]	0.4367[0.3880]	0.1825[0.0560]
		CS-3	0.6235[0.7699]	0.1978[0.0653]	0.5583[0.6091]	0.1738[0.0485]
	30	CS-1	0.4756[0.4753]	0.1817[0.0567]	0.4557[0.4050]	0.1682[0.0442]
		CS-2	0.3672[0.2803]	0.1697[0.0516]	0.3659[0.2598]	0.1581[0.0414]
		CS-3	0.4165[0.3550]	0.1699[0.0472]	0.3735[0.2765]	0.1595[0.0421]

Table 2. ALs and CPs (within bracket) for parameters with $(\theta, \lambda) = (1.5, 1)$

n	m	CS	$\hat{\theta}$	$\hat{\lambda}$	$\acute{\theta}$	$\acute{\lambda}$
20	10	CS-1	7.3037[0.9630]	1.6127[0.9730]	5.9958[0.9520]	1.4918[0.9590]
		CS-2	3.7869[0.9640]	1.3708[0.9720]	3.4165[0.9610]	1.3180[0.9580]
		CS-3	5.5947[0.9650]	1.4485[0.9610]	4.7050[0.9350]	1.3537[0.9450]
	15	CS-1	3.4360[0.9720]	1.3090[0.9640]	2.9821[0.9610]	1.2229[0.9530]
		CS-2	2.7160[0.9730]	1.2199[0.9650]	2.4510[0.9670]	1.1587[0.9420]
		CS-3	2.7858[0.9690]	1.1933[0.9610]	2.5078[0.9620]	1.1349[0.9460]
30	15	CS-1	4.1261[0.9580]	1.2474[0.9740]	3.5609[0.9510]	1.1599[0.9370]
		CS-2	2.6011[0.9650]	1.1056[0.9650]	2.4224[0.9570]	1.0689[0.9580]
		CS-3	3.3554[0.9570]	1.1208[0.9640]	3.0275[0.9460]	1.0691[0.9440]
	20	CS-1	2.8817[0.9620]	1.0889[0.9650]	2.6196[0.9570]	1.0392[0.9560]
		CS-2	2.2307[0.9580]	1.0121[0.9610]	2.1059[0.9510]	0.9859[0.9450]
		CS-3	2.4017[0.9610]	0.9755[0.9580]	2.2033[0.9560]	0.9404[0.9360]
50	20	CS-1	3.7160[0.9550]	1.0450[0.9740]	3.2780[0.9430]	0.9872[0.9530]
		CS-2	2.1750[0.9670]	0.9277[0.9550]	2.0105[0.9450]	0.8803[0.9460]
		CS-3	3.2722[0.9610]	0.9823[0.9730]	2.8800[0.9630]	0.9342[0.9570]
	30	CS-1	2.3387[0.9640]	0.8658[0.9620]	2.1405[0.9460]	0.8277[0.9540]
		CS-2	1.6940[0.9630]	0.8153[0.9540]	1.6439[0.9410]	0.7808[0.9480]
		CS-3	1.8756[0.9520]	0.7682[0.9510]	1.7383[0.9420]	0.7435[0.9390]

From Tables 1 and 3, it can be observed for point estimates that

(1) For fixed n and CS, MSEs and ABs of both likelihood and pivotal quantities based estimates decrease with the increase of m.

(2) Under the fixed combinations of m, n, the performance of estimates from CS-2 and CS-3 is better than the results from CS-1 in general.

(3) Under fixed n, m and CS, the MSEs and ABs of the GPEs are smaller than those of MLEs.

In addition, it can be noted from Tables 2 and 4 for interval estimates that

(1) For fixed n and CS, the ALs of both ACIs and GCIs decrease with the increase of m.

Table 3. ABs and MSEs (within bracket) for parameters with $(\theta, \lambda) = (0.8, 1)$

n	m	CS	$\hat{\theta}$	$\hat{\lambda}$	$\acute{\theta}$	$\acute{\lambda}$
20	10	CS-1	0.6826[1.7701]	0.4344[0.3861]	0.6344[1.5217]	0.3689[0.2794]
		CS-2	0.4159[0.5793]	0.3595[0.2689]	0.3833[0.4869]	0.3506[0.2555]
		CS-3	0.5797[1.1327]	0.4012[0.3166]	0.5127[1.0443]	0.3281[0.2214]
	15	CS-1	0.3311[0.3164]	0.3423[0.2281]	0.3154[0.3373]	0.3094[0.1879]
		CS-2	0.2828[0.2219]	0.3178[0.2116]	0.2700[0.1765]	0.3059[0.1694]
		CS-3	0.2760[0.1845]	0.2955[0.1655]	0.2528[0.1528]	0.2743[0.1499]
30	15	CS-1	0.4486[0.7316]	0.3252[0.2125]	0.4457[0.7289]	0.2859[0.1576]
		CS-2	0.2912[0.2132]	0.2754[0.1519]	0.2644[0.1675]	0.2610[0.1227]
		CS-3	0.3569[0.3508]	0.2899[0.1617]	0.3446[0.3454]	0.2634[0.1296]
	20	CS-1	0.2817[0.2091]	0.2720[0.1398]	0.2721[0.1747]	0.2531[0.1115]
		CS-2	0.2352[0.1140]	0.2637[0.1348]	0.2039[0.0822]	0.2514[0.1109]
		CS-3	0.2392[0.1176]	0.2456[0.1077]	0.2153[0.0965]	0.2345[0.0968]
50	20	CS-1	0.3972[0.5156]	0.2585[0.1223]	0.3694[0.4114]	0.2373[0.1009]
		CS-2	0.2183[0.1079]	0.2303[0.0979]	0.2027[0.0825]	0.2271[0.0913]
		CS-3	0.3366[0.3147]	0.2406[0.1074]	0.3242[0.3272]	0.2258[0.0920]
	30	CS-1	0.2123[0.0947]	0.2080[0.0768]	0.2094[0.0953]	0.1981[0.0660]
		CS-2	0.1647[0.0519]	0.2005[0.0705]	0.1581[0.0457]	0.1919[0.0656]
		CS-3	0.1977[0.0828]	0.1904[0.0671]	0.1836[0.0743]	0.1903[0.0630]

Table 4. ALs and CPs (within bracket) for parameters with $(\theta, \lambda) = (0.8, 1)$

n	m	CS	$\hat{\theta}$	$\hat{\lambda}$	$\acute{\theta}$	$\acute{\lambda}$
20	10	CS-1	3.3261[0.9650]	1.8580[0.9530]	2.7861[0.9590]	1.7204[0.9460]
		CS-2	1.8200[0.9680]	1.6282[0.9570]	1.6327[0.9470]	1.5726[0.9430]
		CS-3	2.5863[0.9620]	1.6924[0.9570]	2.1711[0.9540]	1.5444[0.9460]
	15	CS-1	1.5453[0.9750]	1.5209[0.9590]	1.4032[0.9470]	1.4203[0.9300]
		CS-2	1.3014[0.9730]	1.4741[0.9590]	1.1922[0.9450]	1.4066[0.9500]
		CS-3	1.2839[0.9710]	1.4141[0.9570]	1.1827[0.9480]	1.3544[0.9550]
30	15	CS-1	2.1060[0.9720]	1.4411[0.9540]	1.9090[0.9660]	1.3420[0.9450]
		CS-2	1.2817[0.9690]	1.3005[0.9540]	1.1697[0.9470]	1.2633[0.9530]
		CS-3	1.6455[0.9780]	1.3123[0.9570]	1.5019[0.9500]	1.2481[0.9540]
	20	CS-1	1.3132[0.9560]	1.2412[0.9440]	1.1950[0.9330]	1.1834[0.9370]
		CS-2	1.0485[0.9630]	1.2091[0.9500]	0.9624[0.9550]	1.1632[0.9440]
		CS-3	1.0976[0.9680]	1.1557[0.9580]	1.0029[0.9480]	1.1056[0.9410]
50	20	CS-1	1.8349[0.9660]	1.1673[0.9500]	1.6266[0.9570]	1.1095[0.9480]
		CS-2	1.0122[0.9680]	1.0723[0.9440]	0.9455[0.9470]	1.0390[0.9360]
		CS-3	1.5487[0.9750]	1.1032[0.9480]	1.4134[0.9510]	1.0478[0.9440]
	30	CS-1	1.0087[0.9680]	0.9809[0.9610]	0.9549[0.9440]	0.9295[0.9360]
		CS-2	0.7930[0.9690]	0.9521[0.9470]	0.7482[0.9450]	0.9260[0.9430]
		CS-3	0.8936[0.9690]	0.8935[0.9460]	0.8348[0.9590]	0.8761[0.9430]

(2) In the most of cases, the ALs of interval estimates obtained from CS-2 and CS-3 perform better than those from CS-1 under the fixed combinations of m and n.

(3) Under fixed n, m and CS, the ALs of GCIs are smaller than those of ACIs in most cases. Besides, compared to the ACIs, the CPs of the GCIs appear slightly smaller, one possible explanation for this phenomenon is that comparing with ACIs the GCI of different parameters provides a balance between criteria AL and CP. But in general, the CPs of different intevals are all close to the nominal significance level.

5.2. Illustrated example

In this illustration, a set of real-life data from [23] about the lifetime of 21 light bulbs from a constant-stress test is used. By dividing its limitation observation 130.47, the transformed data are given as follows.

0.0267 0.0371 0.0661 0.0683 0.0715 0.1469 0.1505 0.1564 0.2084 0.2164 0.3115

0.3216 0.3770 0.3948 0.4273 0.5487 0.5752 0.7065 0.7843 0.7898 0.9225

It is necessary to check whether the UGR distribution can fit the data. Based on the 21 observations above, the MLEs of θ and λ are estimated as $\hat{\theta} = 0.5596$ and $\hat{\lambda} = 0.2101$, and the Kolmogorov-Smirnov distance is 0.1009 with the associated p-value 0.9685, which implies that the UGR distribution can fit these data properly. Besides, in order to more intuitively show the fitting effect of the UGR distribution on above data, probability-probability (P-P) and Quantile-Quantile (Q-Q) plots in Figure 2 are presented. It is also seen in visual that the UGR distribution could be used as a proper model.

Based on original lifetime of the bulb data, a group of progressively Type-II censored sample with $n = 21$, $m = 16$, $r_1 = 5$ and $r_i = 0, i = 2, \dots, 16$ are generated as follows.

0.0267 0.0371 0.0661 0.0715 0.1469 0.1564 0.2164 0.3115

0.3216 0.3770 0.3948 0.4273 0.5487 0.7065 0.7843 0.9225

where in this dataset, the removed observations at first failure are 0.0683, 0.1505, 0.2084, 0.5752 and 0.7898 successively. The point and interval estimates of the unknown parameters and R_x ($x = 0.1, 0.2, 0.3, 0.4, 0.5$) are presented in Tables 5 and 6, respectively. It is noted that the MLEs and GPEs are very close to each other, and that the lengths of GCIs are all smaller than those of ACIs. Moreover, the profile log-likelihood function $l(\lambda)$ of the parameter λ is presented in Figure 2, which also show in visual that the MLE $\hat{\lambda}$ uniquely exists.

Further, the results of predictions for $Y_{1j}, j = 1, 2, \dots, r_1$ are shown in Table 7. it is seen that both LPPE and GPPE are similar and close to the removed observations. Meanwhile, for prediction intervals, it is noted that GPIs perform better than LPIs in terms of the interval lengths, and that both kinds of prediction intervals cover the associated true values of removed data.

Table 5. Estimation for θ and λ

parameter	MLE	GPE	ACI[Interval length]	GCI[Interval length]
θ	0.5552	0.5279	(0.2331,0.8773)[0.6442]	(0.1990,0.8271)[0.6281]
λ	0.2102	0.1950	(0.0629,0.3575)[0.2946]	(0.0714,0.3485)[0.2771]

Table 6. Estimation for $R_x, x = 0.1, 0.2, 0.3, 0.4, 0.5$

R_x	MLE	GPE	ACI[Interval length]	GCI[Interval length]
$R_{0.1}$	0.8020	0.7778	(0.4536,1.1503)[0.6967]	(0.6317,0.9067)[0.2750]
$R_{0.2}$	0.6177	0.6065	(0.2584,0.9770)[0.7186]	(0.4389,0.7565)[0.3176]
$R_{0.3}$	0.4761	0.4763	(0.1695,0.7827)[0.6132]	(0.3077,0.6456)[0.3379]
$R_{0.4}$	0.3638	0.3733	(0.1188,0.6087)[0.4899]	(0.2098,0.5481)[0.3383]
$R_{0.5}$	0.2723	0.2891	(0.0848,0.4598)[0.3750]	(0.1402,0.4679)[0.3277]

6. Concluding remarks

In this paper, inference for progressively Type-II censored data from Unit-Generalized Rayleigh distribution is studied, and different statistical inferential approaches are proposed for parameter estimation as well as the prediction problems. The maximum likelihood estimators and related approximate confidence intervals are constructed, where the

Table 7. Prediction for $Y_{1j}, j = 1, 2, 3, 4, 5, 6$

y_{11}	LPPE	GPPE	LPI[Interval length]	GPI[Interval length]
y_{12}	0.0825	0.0851	(0.0290,0.3121)[0.2831]	(0.0528,0.1240)[0.0712]
y_{13}	0.1735	0.1829	(0.0497,0.4986)[0.4489]	(0.1122,0.2764)[0.1642]
y_{14}	0.2951	0.3204	(0.0901,0.6856)[0.5955]	(0.1856,0.4821)[0.2965]
y_{15}	0.4647	0.5001	(0.1569,0.8612)[0.7043]	(0.3161,0.7348)[0.4187]
y_{16}	0.7140	0.7398	(0.2788,0.9821)[0.7033]	(0.5460,0.9737)[0.4277]

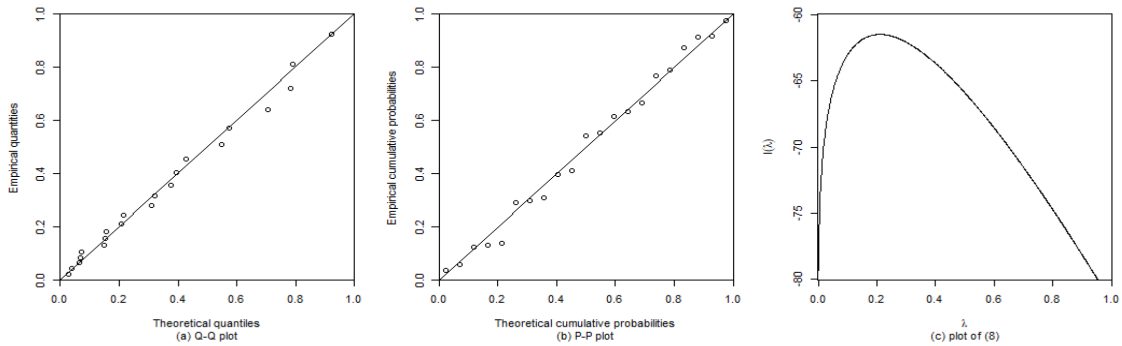

Figure 2. The plots of bulbs data

existence and uniqueness are also established for MLEs. Furthermore, another alternative pivotal quantities based generalized point and interval estimates are proposed for comparison. In addition, to investigate the lifetimes of the randomly censored units, associated prediction issues are also constructed based on the likelihood and the proposed generalized approaches. Extensive simulation studies and one real-life example are conducted to evaluate the performance of our methods, and the results indicate that both likelihood and generalized estimation work satisfactory, and that the proposed generalized results perform better than likelihood estimates for both parameter and prediction estimation problems. Although the studies in this paper focus on statistical inference for UGR distribution with progressively Type-II censored data, the results could be extended to other kinds of failure data with proper modifications, such as records data, complete and progressive first-failure censored data, among others.

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Appendix A. Proof of Theorem 2.1

Taking derivative of (2.1) with respect to θ and equating it to zero, one could find the expression (2.2) directly. For fixed $\lambda > 0$, according to the inequality $\log \frac{\theta}{\tilde{\theta}} \leqslant \frac{\theta}{\tilde{\theta}} - 1$, it's seen that

$$m \log \theta = m \log \frac{\theta}{\tilde{\theta}} + m \log \tilde{\theta} \leqslant m \frac{\theta}{\tilde{\theta}} - m + m \log \tilde{\theta}.$$

Using above equality and the expression

$$m = -\tilde{\theta}\left(\lambda\right) \sum_{i=1}^{m} (r_i + 1) \log \left[1 - e^{-\lambda (\log t_i)^2}\right],$$

it can be easily found that

$$\begin{aligned} l(\lambda, \theta) \leqslant\ & m\left(\log 2 + \log \lambda\right) + m \log m - m \log \left\{-\sum_{i=1}^{m} (r_i + 1) \log \left[1 - e^{-\lambda (\log t_i)^2}\right]\right\} \\ & - m - \lambda \sum_{i=1}^{m} (\log t_i)^2 + \sum_{i=1}^{m} \left\{\log\left(\frac{1}{t_i}\right) + \log\left(\log \frac{1}{t_i}\right) - \log\left[1 - e^{-\lambda (\log t_i)^2}\right]\right\} \\ & = l\left(\lambda, \tilde{\theta}\right). \end{aligned}$$

Equality holds if and only if $\theta = \tilde{\theta}$. Therefore, this proves the assertion.

Appendix B. Proof of Theorem 2.2

Clearly, the MLE of λ can be obtained from equation $\frac{dl(\lambda)}{d\lambda} = p(\lambda) = 0$. In order to show the uniqueness and existence of MLE $\hat{\lambda}$, it is equivalent to prove that $p(\lambda)$ is a monotone function with λ and changes from positive to negative in range $\lambda \in (0, +\infty)$.

For brevity, let $u_i = (\log t_i)^2$, it's seen that $u_i \in (0, +\infty)$ and $u_1 > u_2 > \cdots > u_m$, then the expression (2.4) can be rewritten as

$$p(\lambda) = -m \frac{\sum_{i=1}^{m} (r_i + 1) \frac{u_i}{e^{\lambda u_i} - 1}}{\sum_{i=1}^{m} (r_i + 1) \log(1 - e^{-\lambda u_i})} + \frac{m}{\lambda} - \sum_{i=1}^{m} \left(u_i + \frac{u_i}{e^{\lambda u_i} - 1}\right).$$

In order to obtain the limits of function $p(\lambda)$ as $\lambda \to 0^+$ and $\lambda \to +\infty$ respectively, some necessary preparations are shown as follows, which can be proved directly.

(1) $\log(1 + x) < x$ in range $x \in (0, +\infty)$.
(2) $\frac{e^x - 1}{x} \to 1$ as $x \to 0$.
(3) $\frac{e^x - 1 - x}{x^2/2} \to 1$ as $x \to 0$.
(4) $\frac{log(1+x)}{x} \to 1$ as $x \to 0$.
(5) For $i = 1, 2, \ldots, m - 1$ and $k = 1, 2, \ldots, m$, let $S_1 = \sum_{k>i} (r_k + 1) e^{\lambda\left(u_i^2 - u_k^2\right)}$, $S_2 = \sum_{k<i} (r_k + 1) e^{\lambda\left(u_i^2 - u_k^2\right)}$ and $S_3 = \sum_{k=i} (r_k + 1) e^{\lambda\left(u_i^2 - u_k^2\right)}$, then $\lim_{\lambda \to +\infty} S_1 = \infty$, $\lim_{\lambda \to +\infty} S_2 = 0$, and $\lim_{\lambda \to +\infty} S_3 = \sum_{k=i} (r_k + 1)$.

Based on above results (2)-(5), it is observed that

$$\lim_{\lambda \to 0^+} p(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \to +\infty} p(\lambda) = -\sum_{i=1}^{m-1} (u_m - u_i) \leq 0.$$

Therefore, function $p(\lambda)$ changes from positive to negative when $\lambda \in (0, \infty)$ implying that the MLE $\hat{\lambda}$ of λ exists.

Next, Taking derivative of $p(\lambda)$, one has

$$p'(\lambda) = m\frac{\left[\sum_{i=1}^{m}(r_i+1)\frac{u_i^2 e^{\lambda u_i}}{(e^{\lambda u_i}-1)^2}\right]\left[\sum_{i=1}^{m}(r_i+1)\log(1-e^{-\lambda u_i})\right]+\left[\sum_{i=1}^{m}(r_i+1)\frac{u_i}{e^{\lambda u_i}-1}\right]^2}{\left[\sum_{i=1}^{m}(r_i+1)\log(1-e^{-\lambda u_i})\right]^2} + \sum_{i=1}^{m}\frac{u_i^2 e^{\lambda u_i}}{(e^{\lambda u_i}-1)^2} - \frac{m}{\lambda^2}.$$

For the first item of the derivative $p'(\lambda)$, using Cauchy-Schwarz inequality and $\log(1+x) < x, x > 0$, it is seen that

$$m\frac{\left[\sum_{i=1}^{m}(r_i+1)\frac{u_i^2 e^{\lambda u_i}}{(e^{\lambda u_i}-1)^2}\right]\left[\sum_{i=1}^{m}(r_i+1)\log(1-e^{-\lambda u_i})\right]+\left[\sum_{i=1}^{m}(r_i+1)\frac{u_i}{e^{\lambda u_i}-1}\right]^2}{\left[\sum_{i=1}^{m}(r_i+1)\log(1-e^{-\lambda u_i})\right]^2}$$
$$< m\frac{\left[\sum_{i=1}^{m}(r_i+1)\frac{u_i}{e^{\lambda u_i}-1}\right]^2 - \left[\sum_{i=1}^{m}(r_i+1)\frac{u_i^2 e^{\lambda u_i}}{(e^{\lambda u_i}-1)^2}\right]\left[\sum_{i=1}^{m}(r_i+1)e^{-\lambda u_i}\right]}{\left[\sum_{i=1}^{m}(r_i+1)\log(1-e^{-\lambda u_i})\right]^2} \leqslant 0.$$

Moreover, for the last two items of the derivative $p'(\lambda)$, using the inequality $z^k e^{-z} < (1-e^{-z})^k, k = 1, 2, z > 0$, it is found that

$$\sum_{i=1}^{m}\frac{u_i^2 e^{\lambda u_i}}{(e^{\lambda u_i}-1)^2} - \frac{m}{\lambda^2} = \sum_{i=1}^{m}\frac{u_i^2 e^{-\lambda u_i}}{(1-e^{-\lambda u_i})^2} - \frac{m}{\lambda^2} < \sum_{i=1}^{m}\frac{u_i^2 e^{-\lambda u_i}}{(\lambda u_i)^2 e^{-\lambda u_i}} - \frac{m}{\lambda^2} = 0.$$

Therefore, one has that $p'(\lambda) < 0$ implying that function $p(\lambda)$ decreases in λ and the MLE of λ is unique.

Appendix C. Proof of Theorem 3.1

Since $T_{i:m:n}, i = 1, 2, \ldots, m$ are the first m progressively Type-II censored order statistics of size n from $UGR(\theta, \lambda)$, then $V_{i:m:n} = -\theta\log[1 - e^{-\lambda(\log(T_{i:m:n}))^2}], i = 1, 2, \ldots, m$ are corresponding progressively Type-II censored samples from standard exponential distribution with mean 1. According to Viveros and Balakrishnan [20], it can be seen that

$$\begin{aligned}
W_1 &= nV_{1:m:n}\\
W_2 &= [n-(r_1+1)](V_{2:m:n} - V_{1:m:n})\\
W_3 &= \left[n - \sum_{i=1}^{2}(r_i+1)\right](V_{3:m:n} - V_{2:m:n})\\
&\ldots\\
W_m &= \left[n - \sum_{i=1}^{m-1}(r_i+1)\right](V_{m:m:n} - V_{(m-1):m:n})
\end{aligned}$$

are independent and identically distributed from standard exponential distribution. Let $D_i = \sum_{j=1}^{i} W_j, i = 1, 2, \ldots, m$, and $U_{(i)} = D_i/D_m, i = 1, 2, \ldots, m-1$, it can be obtained from [19] that $U_{(1)}, U_{(2)}, \ldots, U_{(m-1)}$ are order statistics from the uniform (0,1) distribution with sample size $m-1$. Furthermore, $U_{(1)} < U_{(2)} < \cdots < U_{(m-1)}$ are also independent with

$$\begin{aligned}
D_i &= \left[n - \sum_{j=1}^{i}(r_j+1)\right]V_{i:m:n} + \sum_{j=1}^{i}(r_j+1)V_{j:m:n}\\
&= -\theta\left\{\left[n - \sum_{j=1}^{i}(r_j+1)\right]\log\left[1 - e^{\lambda(\log(T_{i:m:n}))^2}\right] + \sum_{j=1}^{i}(r_j+1)\log\left[1 - e^{\lambda(\log(T_{j:m:n}))^2}\right]\right\}.
\end{aligned}$$

By the theory of sampling distribution, it is obtained that

$$S_1(\lambda) = \sum_{i=1}^{m-1} \left[-2\log U_{(i)}\right] \quad \text{and} \quad S_2(\theta,\lambda) = 2\sum_{j=1}^{m} W_j$$

are statistically independent and have chi-square distributions with $2m-2$ and $2m$ degrees of freedom respectively.

Appendix D. Proof of Lemma 3.2

To prove that $G(\lambda)$ increases in λ is equivalent to showing $G'(\lambda)$ is always greater than 0. By taking derivative $G(\lambda)$ with respect to λ, it is seen that

$$G'(\lambda) = \frac{\log\left[1-e^{-\lambda(\log a)^2}\right]}{\log\left[1-e^{-\lambda(\log b)^2}\right]}\left\{\frac{(\log a)^2}{\left[e^{\lambda(\log a)^2}-1\right]\log\left[1-e^{-\lambda(\log a)^2}\right]} - \frac{(\log b)^2}{\left[e^{\lambda(\log b)^2}-1\right]\log\left[1-e^{-\lambda(\log b)^2}\right]}\right\}.$$

Let $h(t) = (\log t)^2$ and $g(t) = \left[e^{\lambda(\log t)^2} - 1\right]\log\left[1 - e^{-\lambda(\log t)^2}\right]$, $t \in (0,1)$, $G'(\lambda)$ can be written as

$$G'(\lambda) = -G(\lambda)\left[\frac{h(b)}{g(b)} - \frac{h(a)}{g(a)}\right].$$

Firstly, it's easy to find that $G(\lambda) > 0$. Moreover, according to Cauchy mean value theorem, there exists $\xi \in (a,b) \subset (0,1)$ such that $\frac{h(b)}{g(b)} - \frac{h(a)}{g(a)} = \frac{h'(\xi)}{g'(\xi)}$. Due to

$$\frac{h'(\xi)}{g'(\xi)} = \frac{1}{\lambda\left[e^{\lambda(\log\xi)^2}\log\left[1-e^{-\lambda(\log\xi)^2}\right]+1\right]}$$

and

$$\left[e^{\lambda(\log\xi)^2}\log\left[1 - e^{-\lambda(\log\xi)^2}\right] + 1\right] < \left[e^{\lambda(\log\xi)^2}\cdot\left[-e^{-\lambda(\log\xi)^2}\right] + 1\right] < 0,$$

it is noted that

$$\frac{h(b)}{g(b)} - \frac{h(a)}{g(a)} = \frac{h'(\xi)}{g'(\xi)} < 0.$$

Therefore

$$G'(\lambda) = -G(\lambda)\left[\frac{h(b)}{g(b)} - \frac{h(b)}{g(b)}\right] > 0,$$

then one has that $G'(\lambda) > 0$ implying that the function $G(\lambda)$ increases in λ. In addition, the limitations of $G(\lambda)$ at $\lambda = 0$ and ∞ could be obtained by direct computation which is omitted here for concision. Therefore, the assertion is completed.

Appendix E. Proof of Theorem 4.1

Let $z_{ij} = \frac{\left[1-e^{-\lambda(\log y_{ij})^2}\right]^\theta}{\left[1-e^{-\lambda(\log t_i)^2}\right]^\theta}$, for given $T_{i:m:n} = t_i$ and using distributional theory, it can be seen that the PDF of $Y_{i:r_s}$ can be written as

$$f_{Z_{ij}}(z_{ij}|t_i) = j\begin{pmatrix} r_i \\ j \end{pmatrix} z_{ij}^{r_i-j}(1 - z_{ij})^{j-1}, 0 < z_{ij} < 1.$$

Therefore, Theorem 4.1 is proved.