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Research Article

Ruled surfaces corresponding to hyper-dual curves

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Abstract

In this paper, we give the definition of the concept of unit hyper-dual sphere. We take a subset of this sphere and show that each curve on this subset represents two ruled surfaces in three dimensional real vector space such that these ruled surfaces have a common base curve and their rulings are perpendicular. Finally, we give some examples to illustrate the applications of our main results.

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1. Introduction

Clifford introduced the algebra of dual numbers $\mathbb{D}$ as an extension of real numbers $\mathbb{R}$ [2]. A dual vector is an ordered triple of dual numbers, and the set of all dual vectors is denoted by $\mathbb{D}^3$. Dual vectors were first applied in mechanism by Study [19] and Kotelnikov [11]. There exists a one-to-one correspondence (known as E. Study mapping) between the directed lines in 3-dimensional real vector space $\mathbb{R}^3$ and the points of unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$ (the set of all unit dual vectors).

The algebra of hyper-dual numbers $\tilde{\mathbb{D}}$ was first defined by Fike to overcome some derivative problems in the complex-step derivative approximation [6, 7]. Afterwards, this number system is used in derivative calculations [6–9]. Cohen and Shoham showed that a hyper-dual number consists of two dual numbers [3]. Futhermore, they interpreted hyper-dual numbers in the sense of Study [19] and Kotelnikov [11], and they used this number system in the motion of multi-body systems [3–5]. Hyper-dual numbers are suitable for software, analysis and design of airspace systems, and robot manipulators [4, 7].

A ruled surface is described as a surface swept out by a straight line moving along a curve [15]. The parametric representation of a ruled surface consists of two curves in $\mathbb{R}^3$ similar to a curve on unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$. Hence, there exists a one-to-one correspondence between the dual curves on $\mathbb{S}^2_{\mathbb{D}}$ and the ruled surfaces in $\mathbb{R}^3$ [20]. Veldkamp gave the

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applications of the dual curves on $\mathbb{S}^2_{\mathbb{D}}$ to theoretical space kinematic [20]. Afterwards, these curves have been used in motion of the robot end-effector [14, 17], in kinematic formulations of the lines trajectories [12, 13] and in kinematic generations of the ruled surfaces [18].

In this paper, we give some basic concepts of hyper-dual numbers. We define unit hyper-dual sphere $\mathbb{S}^2_{\tilde{\mathbb{D}}}$. Using E. Study mapping, we show that there exists a one-to-one correspondence between the points of $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$ (which is a subset of unit hyper-dual sphere $\mathbb{S}^2_{\tilde{\mathbb{D}}}$) and any two intersecting perpendicular directed lines in $\mathbb{R}^3$. We give the definition of hyper-dual curves on $\mathbb{S}^2_{\tilde{\mathbb{D}}}$. By interpreting these curves in the sense of Veldkamp [20], we show that each hyper-dual curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$ represents two ruled surfaces in $\mathbb{R}^3$. It is observed that these ruled surfaces intersect along a common base curve and their rulings are perpendicular. It is also observed that each dual curve on unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$ represents a ruled surface in $\mathbb{R}^3$ while each hyper-dual curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$ represents two ruled surfaces in $\mathbb{R}^3$ such that these two ruled surfaces intersect along a common base curve. Examples of ruled surfaces are given to illustrate the applications of our results.

2. Preliminaries

In this section, definitions and some algebraic properties of the concepts of dual numbers and hyper-dual numbers will be given to provide a background.

2.1. Dual numbers

The set of all dual numbers is defined as

$$\mathbb{D} = \{A = a + \varepsilon a^* : a,\, a^* \in \mathbb{R}\}, \tag{2.1}$$

where ε is the dual unit satisfying

$$\varepsilon \neq 0,\ \varepsilon^2 = 0 \quad \text{and} \quad r\varepsilon = \varepsilon r \quad \text{for all } r \in \mathbb{R}. \tag{2.2}$$

The square root of a dual number $A = a + \varepsilon a^*$ is defined as

$$\sqrt{A} = \sqrt{a} + \varepsilon \frac{a^*}{2\sqrt{a}}, \quad \text{for } a > 0. \tag{2.3}$$

Taylor series expansion of a dual function $f(x+\varepsilon x^*)$ about a point $x+\varepsilon x^* = a+\varepsilon a^* \in \mathbb{D}$ can be given as

$$f(a + \varepsilon a^*) = f(a) + \varepsilon a^* f'(a), \tag{2.4}$$

where the prime represents differentiation with respect to x [20], i.e.

$$f'(x) = \frac{d}{dx} f(x). \tag{2.5}$$

The set of dual vectors is defined by

$$\mathbb{D}^3 = \left\{\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^* : \boldsymbol{a},\, \boldsymbol{a}^* \in \mathbb{R}^3\right\} \tag{2.6}$$

and each element $\hat{A}$ of $\mathbb{D}^3$ is called a dual vector.

The scalar and vector products of any dual vectors $\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ and $\hat{B} = \boldsymbol{b} + \varepsilon \boldsymbol{b}^*$ are defined by

$$\left\langle \hat{A}, \hat{B} \right\rangle_D = \langle \boldsymbol{a}, \boldsymbol{b} \rangle + \varepsilon \left(\langle \boldsymbol{a}, \boldsymbol{b}^* \rangle + \langle \boldsymbol{a}^*, \boldsymbol{b} \rangle \right), \tag{2.7}$$

$$\hat{A} \times_D \hat{B} = \boldsymbol{a} \times \boldsymbol{b} + \varepsilon \left(\boldsymbol{a} \times \boldsymbol{b}^* + \boldsymbol{a}^* \times \boldsymbol{b} \right), \tag{2.8}$$

where "$\langle , \rangle$" and "$\times$" denote, respectively, the usual scalar and vector products in 3-dimensional real vector space $\mathbb{R}^3$.

The modulus of the dual vector $\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ is defined to be

$$\left|\hat{A}\right|_D = \sqrt{\left\langle \hat{A}, \hat{A} \right\rangle_D} = |\boldsymbol{a}| + \varepsilon \frac{\langle \boldsymbol{a}, \boldsymbol{a}^* \rangle}{|\boldsymbol{a}|}, \text{ for } |\boldsymbol{a}| \neq 0. \tag{2.9}$$

If $\left|\hat{A}\right|_D = 1$ (i.e., $|\boldsymbol{a}| = 1$ and $\langle \boldsymbol{a}, \boldsymbol{a}^* \rangle = 0$), then $\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ is called a unit dual vector.

Unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$, consisting of all unit dual vectors, is defined by

$$\mathbb{S}^2_{\mathbb{D}} = \left\{ \hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^* : \left|\hat{A}\right|_D = 1,\ \hat{A} \in \mathbb{D}^3 \right\}. \tag{2.10}$$

Theorem 2.1. [E. Study Mapping] *Each point on unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$ represents a directed line in $\mathbb{R}^3$. In other words, there is a one-to-one correspondence between the points of unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$ and the directed lines in $\mathbb{R}^3$ [19].*

The scalar product of any unit dual vectors $\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ and $\hat{B} = \boldsymbol{b} + \varepsilon \boldsymbol{b}^*$ is

$$\left\langle \hat{A}, \hat{B} \right\rangle_D = \cos\varphi = \cos\theta - \varepsilon\theta^* \sin\theta, \tag{2.11}$$

where $\varphi = \theta + \varepsilon\theta^*$ is a dual angle [19]. If d_1 and d_2 are the directed lines in $\mathbb{R}^3$ corresponding, respectively, to the unit dual vectors $\hat{A}$ and $\hat{B}$, then θ is the angle between the real vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $|\theta^*|$ is the shortest distance between d_1 and d_2, see Fig. 1.

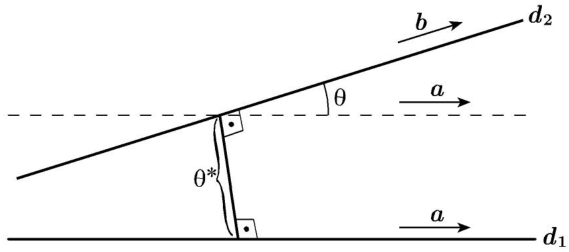

Figure 1. Geometric representation of dual angle $\varphi \in \mathbb{R}^3$

The vector product of any unit dual vectors $\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ and $\hat{B} = \boldsymbol{b} + \varepsilon \boldsymbol{b}^*$ is

$$\hat{A} \times_D \hat{B} = \hat{N} \sin\varphi, \tag{2.12}$$

where the Taylor series expansion of $\sin\varphi$ is $\sin\varphi = \sin\theta + \varepsilon\theta^* \cos\theta$ and where $\hat{N} = \frac{\hat{A} \times_D \hat{B}}{\left|\hat{A} \times_D \hat{B}\right|_D}$ is the common perpendicular direction vector to the dual vectors $\hat{A}$ and $\hat{B}$, directed from $\boldsymbol{a}$ to $\boldsymbol{b}$. For further information about dual numbers, see [1, 2, 5, 20].

2.2. Hyper-dual numbers

The set of all hyper-dual numbers is defined as

$$\tilde{\mathbb{D}} = \{ \mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 : a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R} \}, \tag{2.13}$$

where the dual units ε_1 and ε_2 satisfy

$$\varepsilon_1^2 = \varepsilon_2^2 = (\varepsilon_1\varepsilon_2)^2 = 0 \text{ and } \varepsilon_1 \neq \varepsilon_2,\ \varepsilon_1 \neq 0,\ \varepsilon_2 \neq 0,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 \neq 0. \tag{2.14}$$

The algebra of $\tilde{\mathbb{D}}$ can be embedded in the real exterior algebra $\wedge V$ where V is a real vector space with an orthogonal basis e_1, e_2, e_3, e_4, as follows: let $\varepsilon_1 = e_1 \wedge e_2$ and $\varepsilon_2 = e_3 \wedge e_4$. Then, one can recover the algebra of the $\tilde{\mathbb{D}}$ as this 4-dimensional subalgebra of the exterior algebra $\wedge V$ that is spanned by $\{1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$.

Addition and multiplication of any hyper-dual numbers $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3$ and $\mathbb{B} = b_0 + \varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_1\varepsilon_2 b_3$ are defined, respectively, as

$$\mathbb{A} + \mathbb{B} = (a_0 + b_0) + \varepsilon_1 (a_1 + b_1) + \varepsilon_2 (a_2 + b_2) + \varepsilon_1\varepsilon_2 (a_3 + b_3), \tag{2.15}$$

$$\mathbb{A}\mathbb{B} = (a_0 b_0) + \varepsilon_1 (a_0 b_1 + a_1 b_0) + \varepsilon_2 (a_0 b_2 + a_2 b_0) + \varepsilon_1\varepsilon_2 (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0). \tag{2.16}$$

The multiplicative-inverse of a hyper-dual number $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3$ is

$$\mathbb{A}^{-1} = \frac{1}{\mathbb{A}} = \frac{1}{a_0} - \varepsilon_1 \frac{a_1}{a_0^2} - \varepsilon_2 \frac{a_2}{a_0^2} + \varepsilon_1\varepsilon_2 \left(-\frac{a_3}{a_0^2} + \frac{2a_1 a_2}{a_0^3} \right), \quad \text{if } a_0 \neq 0. \tag{2.17}$$

Thus, a hyper-dual number in the form $\mathbb{A} = 0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 = \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3$ does not have an inverse.

Taylor series expansion of a hyper-dual function $f(x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3)$ about a point $x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3 = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 \in \tilde{\mathbb{D}}$ can be given as

$$f(a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3) = f(a_0) + \varepsilon_1 a_1 f'(a_0) + \varepsilon_2 a_2 f'(a_0) + \varepsilon_1\varepsilon_2 (a_3 f'(a_0) + a_1 a_2 f''(a_0)), \tag{2.18}$$

where the prime represents differentiation with respect to x_0, i.e.

$$f'(x_0) = \frac{d}{dx_0} f(x_0), \tag{2.19}$$

see [6–9].

A hyper-dual number $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3$ can be given in terms of two dual numbers as

$$\mathbb{A} = A + \varepsilon^* A^*, \tag{2.20}$$

where $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^*$ and $A = a_0 + \varepsilon a_1$, $A^* = a_2 + \varepsilon a_3 \in \mathbb{D}$.

The addition and multiplication rules of two hyper-dual numbers $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 = A + \varepsilon^* A^*$ and $\mathbb{B} = b_0 + \varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_1\varepsilon_2 b_3 = B + \varepsilon^* B^*$ given, respectively, by Eqs. (2.15) and (2.16) can be expressed differently as

$$\mathbb{A} + \mathbb{B} = (A + B) + \varepsilon^* \left(A^* + B^* \right), \tag{2.21}$$

$$\mathbb{A}\mathbb{B} = AB + \varepsilon^* \left(AB^* + A^* B \right). \tag{2.22}$$

An alternative representation of the multiplicative-inverse of a hyper-dual number $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 = A + \varepsilon^* A^*$ given by Eq. (2.17) can be given as

$$\mathbb{A}^{-1} = \frac{1}{A} - \varepsilon^* \frac{A^*}{A^2}, \quad \text{for } a_0 \neq 0. \tag{2.23}$$

This means that a hyper-dual number $\mathbb{A} = A + \varepsilon^* A^*$ providing $A = 0 + \varepsilon a_1 = \varepsilon a_1$ does not have an inverse.

If we extend the real vectors $\boldsymbol{a}$ and $\boldsymbol{p} \times \boldsymbol{a}$ in a dual vector $\hat{A} = \boldsymbol{a} + \varepsilon (\boldsymbol{p} \times \boldsymbol{a})$, respectively, to the dual vectors $\hat{A}$ and $\hat{P} \times_D \hat{A}$ then we obtain the hyper-dual vector

$$\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \left(\hat{P} \times_D \hat{A} \right). \tag{2.24}$$

Scalar and vector products of any hyper-dual vectors $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \left(\hat{P} \times_D \hat{A} \right)$ and $\widetilde{\mathbb{B}} = \hat{B} + \varepsilon^* \left(\hat{K} \times_D \hat{B} \right)$ can be given, respectively, as

$$\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = \left| \hat{A} \right|_D \left| \hat{B} \right|_D \cos \tilde{\varphi} \tag{2.25}$$

$$\widetilde{\mathbb{A}} \times_{HD} \widetilde{\mathbb{B}} = \left| \hat{A} \right|_D \left| \hat{B} \right|_D \boldsymbol{n} \sin \tilde{\varphi}, \tag{2.26}$$

where $\tilde{\varphi}$ is a hyper-dual angle and $\boldsymbol{n}$ is the common perpendicular direction vector to the hyper-dual vectors $\widetilde{\mathbb{A}}$ and $\widetilde{\mathbb{B}}$, directed from $\hat{A}$ to $\hat{B}$. For further information about hyper-dual numbers, see [3–5].

3. Hyper-dual numbers and ruled surfaces

In this section, we express some basic concepts of hyper-dual numbers. Using these expressions, we define a subset $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$ of unit hyper-dual sphere $\mathbb{S}^2_{\tilde{\mathbb{D}}}$ such that each element of $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$ represents two intersecting and perpendicular directed lines in $\mathbb{R}^3$. Moreover, we show that each hyper-dual curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$ represents two ruled surfaces in $\mathbb{R}^3$. These ruled surfaces have a common base curve and their rulings are perpendicular.

3.1. Some basic concepts of hyper-dual numbers

The square root of a hyper-dual number $\mathbb{A} = A + \varepsilon^* A^*$ can be defined by

$$\sqrt{\mathbb{A}} = \sqrt{A} + \varepsilon^* \frac{A^*}{2\sqrt{A}}, \quad \text{for } a_0 > 0 \tag{3.1}$$

or

$$\sqrt{\mathbb{A}} = \sqrt{a_0} + \varepsilon \frac{a_1}{2\sqrt{a_0}} + \varepsilon^* \frac{a_2}{2\sqrt{a_0}} + \varepsilon\varepsilon^* \left(\frac{a_3}{2\sqrt{a_0}} - \frac{a_1 a_2}{4a_0\sqrt{a_0}} \right), \text{ for } a_0 > 0. \tag{3.2}$$

The set of all hyper-dual vectors is defined to be

$$\begin{aligned} \tilde{\mathbb{D}}^3 &= \left\{ \widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* : \hat{A},\ \hat{A}^* \in \mathbb{D}^3 \right\} && (3.3)\\ &= \left\{ \widetilde{\mathbb{A}} = \boldsymbol{a}_0 + \varepsilon \boldsymbol{a}_1 + \varepsilon^* \boldsymbol{a}_2 + \varepsilon\varepsilon^* \boldsymbol{a}_3 : \boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in \mathbb{R}^3 \right\}, && (3.4) \end{aligned}$$

and each element $\widetilde{\mathbb{A}}$ of $\tilde{\mathbb{D}}^3$ is called a hyper-dual vector.

The scalar and vector products of any hyper-dual vectors $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* = \boldsymbol{a}_0 + \varepsilon \boldsymbol{a}_1 + \varepsilon^* \boldsymbol{a}_2 + \varepsilon\varepsilon^* \boldsymbol{a}_3$ and $\widetilde{\mathbb{B}} = \hat{B} + \varepsilon^* \hat{B}^* = \boldsymbol{b}_0 + \varepsilon \boldsymbol{b}_1 + \varepsilon^* \boldsymbol{b}_2 + \varepsilon\varepsilon^* \boldsymbol{b}_3$ are defined, respectively, by

$$\begin{aligned} \left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} &= \left\langle \hat{A}, \hat{B} \right\rangle_D + \varepsilon^* \left(\left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D \right) && (3.5)\\ &= \langle \boldsymbol{a}_0, \boldsymbol{b}_0 \rangle + \varepsilon(\langle \boldsymbol{a}_0, \boldsymbol{b}_1 \rangle + \langle \boldsymbol{a}_1, \boldsymbol{b}_0 \rangle) + \varepsilon^*(\langle \boldsymbol{a}_0, \boldsymbol{b}_2 \rangle + \langle \boldsymbol{a}_2, \boldsymbol{b}_0 \rangle)\\ &\quad + \varepsilon\varepsilon^*(\langle \boldsymbol{a}_0, \boldsymbol{b}_3 \rangle + \langle \boldsymbol{a}_1, \boldsymbol{b}_2 \rangle + \langle \boldsymbol{a}_2, \boldsymbol{b}_1 \rangle + \langle \boldsymbol{a}_3, \boldsymbol{b}_0 \rangle), && (3.6)\\ \widetilde{\mathbb{A}} \times_{HD} \widetilde{\mathbb{B}} &= \hat{A} \times_D \hat{B} + \varepsilon^* \left(\hat{A} \times_D \hat{B}^* + \hat{A}^* \times_D \hat{B} \right) && (3.7)\\ &= \boldsymbol{a}_0 \times \boldsymbol{b}_0 + \varepsilon(\boldsymbol{a}_0 \times \boldsymbol{b}_1 + \boldsymbol{a}_1 \times \boldsymbol{b}_0) + \varepsilon^*(\boldsymbol{a}_0 \times \boldsymbol{b}_2 + \boldsymbol{a}_2 \times \boldsymbol{b}_0)\\ &\quad + \varepsilon\varepsilon^*(\boldsymbol{a}_0 \times \boldsymbol{b}_3 + \boldsymbol{a}_1 \times \boldsymbol{b}_2 + \boldsymbol{a}_2 \times \boldsymbol{b}_1 + \boldsymbol{a}_3 \times \boldsymbol{b}_0). && (3.8) \end{aligned}$$

It is obvious that $\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD}$ and $\widetilde{\mathbb{A}} \times_{HD} \widetilde{\mathbb{B}}$ are, respectively, a hyper-dual number and a hyper-dual vector.

The norm of a hyper-dual vector $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* = \boldsymbol{a}_0 + \varepsilon \boldsymbol{a}_1 + \varepsilon^* \boldsymbol{a}_2 + \varepsilon\varepsilon^* \boldsymbol{a}_3$ is defined to be

$$\begin{aligned} N_{\widetilde{\mathbb{A}}} &= \left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{A}} \right\rangle_{HD} = \left| \hat{A} \right|_D^2 + 2\varepsilon^* \left\langle \hat{A}, \hat{A}^* \right\rangle_D && (3.9)\\ &= |\boldsymbol{a}_0|^2 + 2\left(\varepsilon \langle \boldsymbol{a}_0, \boldsymbol{a}_1 \rangle + \varepsilon^* \langle \boldsymbol{a}_0, \boldsymbol{a}_2 \rangle + \varepsilon\varepsilon^*(\langle \boldsymbol{a}_0, \boldsymbol{a}_3 \rangle + \langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle) \right). && (3.10) \end{aligned}$$

The modulus (i.e., square root of the norm) of the hyper-dual vector $\widetilde{\mathbb{A}}$ is also defined to be

$$\begin{aligned} \left| \widetilde{\mathbb{A}} \right|_{HD} &= \sqrt{\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{A}} \right\rangle_{HD}} = \left| \hat{A} \right|_D + \varepsilon^* \frac{\left\langle \hat{A}, \hat{A}^* \right\rangle_D}{\left| \hat{A} \right|_D} && (3.11)\\ &= |\boldsymbol{a}_0| + \varepsilon \frac{\langle \boldsymbol{a}_0, \boldsymbol{a}_1 \rangle}{|\boldsymbol{a}_0|} + \varepsilon^* \frac{\langle \boldsymbol{a}_0, \boldsymbol{a}_2 \rangle}{|\boldsymbol{a}_0|}\\ &\quad + \varepsilon\varepsilon^* \left(\frac{\langle \boldsymbol{a}_0, \boldsymbol{a}_3 \rangle}{|\boldsymbol{a}_0|} + \frac{\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle}{|\boldsymbol{a}_0|} - \frac{\langle \boldsymbol{a}_0, \boldsymbol{a}_1 \rangle \langle \boldsymbol{a}_0, \boldsymbol{a}_2 \rangle}{|\boldsymbol{a}_0|^3} \right), && (3.12) \end{aligned}$$

where $|\boldsymbol{a}_0| \neq 0$.

If $\left|\widetilde{\mathbb{A}}\right|_{HD} = 1$ (i.e., $\left|\hat{A}\right|_D = 1$ and $\left\langle \hat{A}, \hat{A}^* \right\rangle_D = 0$), then $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ is called a unit hyper-dual vector.

Definition 3.1. [Unit hyper-dual sphere] Unit hyper-dual sphere $\mathbb{S}^2_{\tilde{\mathbb{D}}}$, consisting of all unit hyper-dual vectors, is defined as

$$\mathbb{S}^2_{\tilde{\mathbb{D}}} = \left\{ \widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* : \left|\widetilde{\mathbb{A}}\right|_{HD} = 1; \quad \hat{A},\, \hat{A}^* \in \mathbb{D}^3 \right\}. \tag{3.13}$$

Theorem 3.2. *Let us take a subset of unit hyper-dual sphere $\mathbb{S}^2_{\tilde{\mathbb{D}}}$ as*

$$\mathbb{S}^2_{\tilde{\mathbb{D}}_1} = \left\{ \widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* : \left|\hat{A}^*\right|_D = 1,\ \widetilde{\mathbb{A}} \in \mathbb{S}^2_{\tilde{\mathbb{D}}} \right\} \subset \mathbb{S}^2_{\tilde{\mathbb{D}}}. \tag{3.14}$$

Then, there exists a one-to-one correspondence between the points of $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$ and any two intersecting perpendicular directed lines in $\mathbb{R}^3$.

Proof. Since $\widetilde{\mathbb{A}} \in \mathbb{S}^2_{\tilde{\mathbb{D}}_1}$, $\hat{A}$ and $\hat{A}^*$ are unit dual vectors and $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ is a unit hyper-dual vector satisfying $\left|\hat{A}\right|_D = 1$ and $\left\langle \hat{A}, \hat{A}^* \right\rangle_D = 0$. According to Theorem 2.1, the unit dual vectors $\hat{A}$ and $\hat{A}^*$ represent the directed lines d_1 and d_2 in $\mathbb{R}^3$, respectively. Using Eq. (2.11), the dual angle $\varphi = \theta + \varepsilon\theta^*$ between $\hat{A}$ and $\hat{A}^*$ can be given as

$$\left\langle \hat{A}, \hat{A}^* \right\rangle_D = \cos\theta - \varepsilon\theta^* \sin\theta = \cos\varphi. \tag{3.15}$$

From $\left\langle \hat{A}, \hat{A}^* \right\rangle_D = 0$, we get $\theta = \dfrac{\pi}{2}$ and $\theta^* = 0$. Thus, the lines d_1 and d_2 are perpendicular and intersecting in $\mathbb{R}^3$. □

3.2. Ruled surfaces constructed by hyper-dual curves on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$

A ruled surface in $\mathbb{R}^3$ is a surface swept out by a straight line moving along a curve. The various positions of the generating line are called the rulings of the surface. Such a surface can be given by the parametrization

$$\Phi(t, v) = \beta(t) + v\gamma(t), \qquad t \in I = (a, b) \subset \mathbb{R}, \quad v \in \mathbb{R}. \tag{3.16}$$

Here; $\beta(t)$ is the base curve of $\Phi(t, v)$ and the unit vector $\gamma(t)$ is the director curve of $\Phi(t, v)$ [15].

A dual curve in $\mathbb{D}^3$ can be defined as

$$\begin{aligned} \hat{\Gamma} :\ & I \subset \mathbb{R} \longrightarrow \mathbb{D}^3 \\ & t \longrightarrow \hat{\Gamma}(t) = (a_1(t) + \varepsilon a_1^*(t), a_2(t) + \varepsilon a_2^*(t), a_3(t) + \varepsilon a_3^*(t)) \\ & \qquad\qquad = \boldsymbol{a}(t) + \varepsilon \boldsymbol{a}^*(t), \end{aligned} \tag{3.17}$$

where I is an open interval in $\mathbb{R}$ and $\boldsymbol{a}(t) = (a_1(t), a_2(t), a_3(t))$, $\boldsymbol{a}^*(t) = (a_1^*(t), a_2^*(t), a_3^*(t)) \in \mathbb{R}^3$. If every real valued functions $a_i(t)$ and $a_i^*(t)$ are differentiable for $i = 1, 2, 3$, then the dual space curve $\hat{\Gamma}(t)$ is differentiable. And if $\left|\hat{\Gamma}(t)\right|_D = 1$, then the dual curve $\hat{\Gamma}(t)$ is on unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$ [16].

Let $\hat{\Gamma}(t) = \boldsymbol{a}(t) + \varepsilon \boldsymbol{a}^*(t)$ be a dual curve on the unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$. Then, the ruled surface corresponding to the dual curve $\hat{\Gamma}(t)$ can be given in $\mathbb{R}^3$ as

$$\Phi(t, u) = \boldsymbol{a}(t) \times \boldsymbol{a}^*(t) + u\boldsymbol{a}(t), \qquad t \in I \subset \mathbb{R}, \quad u \in \mathbb{R} \tag{3.18}$$

where $\alpha(t) = \boldsymbol{a}(t) \times \boldsymbol{a}^*(t)$ is the base curve and $\boldsymbol{a}(t)$ is the director curve of $\Phi(t, u)$ [10,20].

Definition 3.3. [Hyper-dual curve] A hyper-dual curve in $\tilde{\mathbb{D}}^3$ can be defined as

$$\begin{aligned} \tilde{\Gamma} :\ & I \subset \mathbb{R} \longrightarrow \tilde{\mathbb{D}}^3 \\ & t \longrightarrow \tilde{\Gamma}(t) = \hat{A}(t) + \varepsilon^* \hat{A}^*(t) \end{aligned} \tag{3.19}$$

where I is an open interval in $\mathbb{R}$. If $\hat{A}(t)$ and $\hat{A}^*(t)$ are differentiable dual curves in $\mathbb{D}^3$, then the hyper-dual curve $\tilde{\Gamma}(t)$ in $\tilde{\mathbb{D}}^3$ is differentiable. And if $\left|\tilde{\Gamma}(t)\right|_{HD} = 1$, then $\tilde{\Gamma}(t)$ is a hyper-dual curve on unit hyper-dual sphere $\mathbb{S}^2_{\tilde{\mathbb{D}}}$. Moreover if $\tilde{\Gamma}(t)$ is a hyper-dual curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}}$ and $\left|\hat{A}^*(t)\right|_D = 1$, then $\tilde{\Gamma}(t)$ is a hyper-dual curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$.

Theorem 3.4. *Let $\tilde{\Gamma}(t) = \hat{A}(t) + \varepsilon^* \hat{A}^*(t)$ be a hyper-dual curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$. Then, each hyper-dual curve $\tilde{\Gamma}(t)$ represents two ruled surfaces in $\mathbb{R}^3$ such that these surfaces have a common base curve and the position vectors of their director curves are perpendicular.*

Proof. Since $\tilde{\Gamma}(t) = \hat{A}(t) + \varepsilon^* \hat{A}^*(t)$ is a hyper-dual curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$, $\hat{A}(t)$ and $\hat{A}^*(t)$ are dual curves on unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$. These curves $\hat{A}(t)$ and $\hat{A}^*(t)$ can be expressed as

$$\hat{A}(t) = \boldsymbol{a}_0(t) + \varepsilon \boldsymbol{a}_1(t) \text{ and } \hat{A}^*(t) = \boldsymbol{a}_2(t) + \varepsilon \boldsymbol{a}_3(t), \tag{3.20}$$

where $\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t) \in \mathbb{R}^3$. The scalar product of $\hat{A}(t) = \boldsymbol{a}_0(t) + \varepsilon \boldsymbol{a}_1(t)$ and $\hat{A}^*(t) = \boldsymbol{a}_2(t) + \varepsilon \boldsymbol{a}_3(t)$ is

$$\left\langle \hat{A}(t), \hat{A}^*(t) \right\rangle_D = \langle \boldsymbol{a}_0(t), \boldsymbol{a}_2(t) \rangle + \varepsilon(\langle \boldsymbol{a}_0(t), \boldsymbol{a}_3(t) \rangle + \langle \boldsymbol{a}_1(t), \boldsymbol{a}_2(t) \rangle). \tag{3.21}$$

Since $\tilde{\Gamma}(t)$ is a hyper-dual curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$, it is also a hyper-dual curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}}$, and thus $\left\langle \hat{A}(t), \hat{A}^*(t) \right\rangle_D = 0$. This means that

$$\langle \boldsymbol{a}_0(t), \boldsymbol{a}_2(t) \rangle = 0 \text{ and } \langle \boldsymbol{a}_0(t), \boldsymbol{a}_3(t) \rangle + \langle \boldsymbol{a}_1(t), \boldsymbol{a}_2(t) \rangle = 0. \tag{3.22}$$

Using Eq. (3.18), the ruled surfaces corresponding to $\hat{A}(t) = \boldsymbol{a}_0(t) + \varepsilon \boldsymbol{a}_1(t)$ and $\hat{A}^*(t) = \boldsymbol{a}_2(t) + \varepsilon \boldsymbol{a}_3(t)$ can be given, respectively, as

$$\Phi_1(t, u_1) = \boldsymbol{a}_0(t) \times \boldsymbol{a}_1(t) + u_1 \boldsymbol{a}_0(t), \quad u_1 \in \mathbb{R}, \tag{3.23}$$

$$\Phi_2(t, u_2) = \boldsymbol{a}_2(t) \times \boldsymbol{a}_3(t) + u_2 \boldsymbol{a}_2(t), \quad u_2 \in \mathbb{R}, \tag{3.24}$$

where $\alpha_1(t) = \boldsymbol{a}_0(t) \times \boldsymbol{a}_1(t)$ and $\alpha_2(t) = \boldsymbol{a}_2(t) \times \boldsymbol{a}_3(t)$ are the base curves of $\Phi_1(t, u_1)$ and $\Phi_2(t, u_2)$, respectively. Also, $\boldsymbol{a}_0(t)$ and $\boldsymbol{a}_2(t)$ are the director curves of $\Phi_1(t, u_1)$ and $\Phi_2(t, u_2)$, recpectively.

For $t = t_0$, let us denote $\Phi_1(t_0, u_1)$ by the line $m_{t_0}(u_1)$ and $\Phi_2(t_0, u_2)$ by the line $n_{t_0}(u_2)$. It is obvious that $m_t(u_1)$ and $n_t(u_2)$ are, recpectively, the rulings of the surfaces $\Phi_1(t, u_1)$ and $\Phi_2(t, u_2)$, for all $t \in I$. Moreover, $m_{t_0}(u_1)$ is a line corresponding to the unit dual vector $\hat{A}(t_0) = \boldsymbol{a}_0(t_0) + \varepsilon \boldsymbol{a}_1(t_0)$ and $n_{t_0}(u_2)$ is a line corresponding to the unit dual vector $\hat{A}^*(t_0) = \boldsymbol{a}_2(t_0) + \varepsilon \boldsymbol{a}_3(t_0)$, where $\boldsymbol{a}_0(t_0)$ and $\boldsymbol{a}_2(t_0)$ are the direction vectors of $m_{t_0}(u_1)$ and $n_{t_0}(u_2)$, respectively.

Since $\tilde{\Gamma}(t_0) = \hat{A}(t_0) + \varepsilon^* \hat{A}^*(t_0) \in \mathbb{S}^2_{\tilde{\mathbb{D}}_1}$, $\tilde{\Gamma}(t_0)$ represents two intersecting perpendicular lines (which are $m_{t_0}(u_1)$ and $n_{t_0}(u_2)$) in $\mathbb{R}^3$. Let us denote the intersection point of the lines $m_t(u_1)$ and $n_t(u_2)$ by $k(t)$, for all $t \in I$. Then, according to E. Study mapping the moments of the vectors $\boldsymbol{a}_0(t)$ and $\boldsymbol{a}_2(t)$ with respect to the origin O can be given as

$$\boldsymbol{a}_1(t) = k(t) \times \boldsymbol{a}_0(t), \tag{3.25}$$

$$\boldsymbol{a}_3(t) = k(t) \times \boldsymbol{a}_2(t), \tag{3.26}$$

respectively. Inserting Eq. (3.25) in Eq. (3.23), we get

$$\begin{aligned}
\Phi_1(t, u_1) &= \boldsymbol{a}_0(t) \times \boldsymbol{a}_1(t) + u_1 \boldsymbol{a}_0(t) \\
&= \boldsymbol{a}_0(t) \times (k(t) \times \boldsymbol{a}_0(t)) + u_1 \boldsymbol{a}_0(t) \\
&= \langle \boldsymbol{a}_0(t), \boldsymbol{a}_0(t) \rangle k(t) - \langle \boldsymbol{a}_0(t), k(t) \rangle \boldsymbol{a}_0(t) + u_1 \boldsymbol{a}_0(t) \\
&= k(t) - \langle \boldsymbol{a}_0(t), k(t) \rangle \boldsymbol{a}_0(t) + u_1 \boldsymbol{a}_0(t) \\
&= k(t) + (u_1 - \langle \boldsymbol{a}_0(t), k(t) \rangle) \boldsymbol{a}_0(t),
\end{aligned} \tag{3.27}$$

where $\langle \boldsymbol{a}_0(t), \boldsymbol{a}_0(t)\rangle = 1$. And inserting $v_1 = u_1 - \langle \boldsymbol{a}_0(t), k(t)\rangle$ in Eq. (3.27), we also get Eq. (3.23) as

$$\Phi_1(t, v_1) = k(t) + v_1\boldsymbol{a}_0(t), \quad v_1 \in \mathbb{R}. \tag{3.28}$$

Similarly, we can obtain Eq. (3.24) as

$$\Phi_2(t, v_2) = k(t) + v_2\boldsymbol{a}_2(t), \quad v_2 \in \mathbb{R}. \tag{3.29}$$

From Eqs. (3.28) and (3.29), it can be seen that ruled surfaces $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ possess a common base curve that is $k(t)$. And from Eq. (3.22), it can be seen that the position vectors of the director curves $\boldsymbol{a}_0(t)$ and $\boldsymbol{a}_2(t)$ of the surfaces $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ are perpendicular, see Fig. 2.

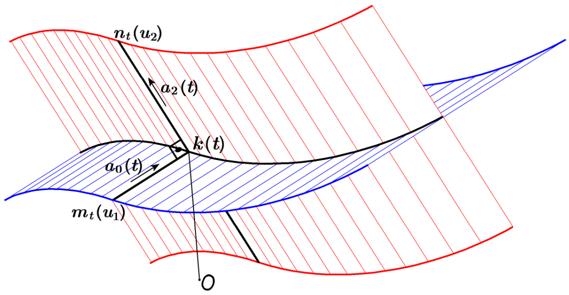

Figure 2. Geometric representation of two ruled surfaces in $\mathbb{R}^3$ corresponding to the hyper-dual curve $\tilde{\Gamma}(t)$ on $\mathbb{S}^2_{\mathbb{D}_1}$.

□

Theorem 3.5. *Let $\Phi_1(t, v_1) = k(t) + v_1\boldsymbol{a}_0(t)$ and $\Phi_2(t, v_2) = k(t) + v_2\boldsymbol{a}_2(t)$ be the ruled surfaces corresponding to the hyper-dual curve $\tilde{\Gamma}(t) = \hat{A}(t)+\varepsilon^*\hat{A}^*(t)$ on $\mathbb{S}^2_{\mathbb{D}_1}$, where $\hat{A}(t) = \boldsymbol{a}_0(t)+\varepsilon\boldsymbol{a}_1(t)$ and $\hat{A}^*(t) = \boldsymbol{a}_2(t)+\varepsilon\boldsymbol{a}_3(t)$. Then, the normal vectors of the surfaces $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ are perpendicular along the common base curve $k(t)$ if and only if the velocity vector $\frac{d}{dt}k(t) = k'(t)$ is perpendicular to $\boldsymbol{a}_0(t)$ or $\boldsymbol{a}_2(t)$.*

Proof. The normal vectors of $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ can be obtained, respectively, as

$$\boldsymbol{n}_1(t, v_1) = \boldsymbol{a}_0(t) \times \left(k'(t) + v_1\boldsymbol{a}_0'(t)\right), \tag{3.30}$$

$$\boldsymbol{n}_2(t, v_2) = \boldsymbol{a}_2(t) \times \left(k'(t) + v_2\boldsymbol{a}_2'(t)\right). \tag{3.31}$$

Since the surfaces $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ intersect along the common base curve $k(t)$ if $v_1 = v_2 = 0$, we get the normal vectors $\boldsymbol{n}_1(t, v_1)$ and $\boldsymbol{n}_2(t, v_2)$ along the base curve $k(t)$ as

$$\boldsymbol{n}_1(t, 0) = \boldsymbol{a}_0(t) \times k'(t), \tag{3.32}$$

$$\boldsymbol{n}_2(t, 0) = \boldsymbol{a}_2(t) \times k'(t), \tag{3.33}$$

for all $t \in I$. Then, we obtain the scalar product of these vectors as

$$\langle \boldsymbol{n}_1(t, 0), \boldsymbol{n}_2(t, 0)\rangle = -\left\langle \boldsymbol{a}_0(t), k'(t)\right\rangle \left\langle k'(t), \boldsymbol{a}_2(t)\right\rangle. \tag{3.34}$$

This means that $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are perpendicular along $k(t)$ if and only if $\langle \boldsymbol{a}_0(t), k'(t)\rangle = 0$ or $\langle k'(t), \boldsymbol{a}_2(t)\rangle = 0$. □

Proposition 3.6. *Consider two ruled surfaces $\Phi_1(t, v_1) = k(t) + v_1\boldsymbol{a}_0(t)$ and $\Phi_2(t, v_2) = k(t) + v_2\boldsymbol{a}_2(t)$ corresponding to the hyper-dual curve $\tilde{\Gamma}(t) \in \mathbb{S}^2_{\mathbb{D}_1}$ such that their normal vectors are perpendicular along their common base curve $k(t)$. If $k(t)$ is the principal curve of $\Phi_1(t, v_1)$ (resp., $\Phi_2(t, v_2)$), then $k(t)$ is also the principal curve of $\Phi_2(t, v_2)$ (resp., $\Phi_1(t, v_2)$).*

Proof. Let $k(t)$ be a curve both on the surfaces $\Phi_1(t,v_1)=k(t)+v_1\boldsymbol{a}_0(t)$ and $\Phi_2(t,v_2)=k(t)+v_2\boldsymbol{a}_2(t)$. And assume that the Darboux frames (see [15]) along the curve $k(t)$ on $\Phi_1(t,v_1)$ and $\Phi_2(t,v_2)$ are, respectively, $\{\boldsymbol{t}_1(t),\boldsymbol{y}_1(t),\boldsymbol{n}_1(t)\}$ and $\{\boldsymbol{t}_2(t),\boldsymbol{y}_2(t),\boldsymbol{n}_2(t)\}$, that means

$$\boldsymbol{t}_1(t,0)=\boldsymbol{t}_2(t)=\frac{d}{dt}k(t)=k'(t)=\boldsymbol{t}(t), \tag{3.35}$$

$$\boldsymbol{n}_1(t,0)=\boldsymbol{a}_0(t)\times k'(t)=\boldsymbol{a}_0(t)\times\boldsymbol{t}(t), \tag{3.36}$$

$$\boldsymbol{n}_2(t,0)=\boldsymbol{a}_2(t)\times k'(t)=\boldsymbol{a}_2(t)\times\boldsymbol{t}(t), \tag{3.37}$$

$$\boldsymbol{y}_1(t,0)=\boldsymbol{n}_1(t,0)\times\boldsymbol{t}_1(t)=\boldsymbol{n}_1(t,0)\times\boldsymbol{t}(t), \tag{3.38}$$

$$\boldsymbol{y}_2(t,0)=\boldsymbol{n}_2(t,0)\times\boldsymbol{t}_2(t)=\boldsymbol{n}_2(t,0)\times\boldsymbol{t}(t). \tag{3.39}$$

Moreover, we have

$$\frac{d}{dt}\boldsymbol{n}_1(t,0)=-k_{n_1}\boldsymbol{t}(t)-t_{g_1}\boldsymbol{y}_1(t,0), \tag{3.40}$$

$$\frac{d}{dt}\boldsymbol{n}_2(t,0)=-k_{n_2}\boldsymbol{t}(t)-t_{g_2}\boldsymbol{y}_2(t,0), \tag{3.41}$$

where k_{n_1}, k_{n_2} are the normal curvatures and t_{g_1}, t_{g_2} are the geodesic torsions. If $t_{g_1}=0$ or $t_{g_2}=0$, then $k(t)$ is a principal curve. Since the normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are perpendicular,

$$\langle\boldsymbol{n}_1(t,0),\boldsymbol{n}_2(t,0)\rangle=0. \tag{3.42}$$

By taking the derivative of this equation, we get

$$\frac{d}{dt}\langle\boldsymbol{n}_1(t,0),\boldsymbol{n}_2(t,0)\rangle=\left\langle\frac{d}{dt}\boldsymbol{n}_1(t,0),\boldsymbol{n}_2(t,0)\right\rangle+\left\langle\boldsymbol{n}_1(t,0),\frac{d}{dt}\boldsymbol{n}_2(t,0)\right\rangle.$$

Using Eqs. (3.40-42), we obtain

$$\langle -k_{n_1}\boldsymbol{t}(t)-t_{g_1}\boldsymbol{y}_1(t,0),\boldsymbol{n}_2(t,0)\rangle+\langle\boldsymbol{n}_1(t,0),-k_{n_2}\boldsymbol{t}(t)-t_{g_2}\boldsymbol{y}_2(t,0)\rangle=0. \tag{3.43}$$

And since $\langle\boldsymbol{n}_1(t,0),\boldsymbol{t}(t)\rangle=\langle\boldsymbol{t}(t),\boldsymbol{n}_2(t,0)\rangle=0$, we get

$$-t_{g_1}\langle\boldsymbol{y}_1(t,0),\boldsymbol{n}_2(t,0)\rangle-t_{g_2}\langle\boldsymbol{n}_1(t,0),\boldsymbol{y}_2(t,0)\rangle=0. \tag{3.44}$$

That is

$$-t_{g_1}\langle\boldsymbol{y}_1(t),\boldsymbol{n}_2(t)\rangle=t_{g_2}\langle\boldsymbol{n}_1(t),\boldsymbol{y}_2(t)\rangle. \tag{3.45}$$

As a result, if $t_{g_1}=0$ (resp. $t_{g_2}=0$), then $t_{g_2}=0$ (resp. $t_{g_1}=0$). And this completes the proof. □

4. Examples of ruled surfaces constructed by curves on $\mathbb{S}^2_{\mathbb{D}_1}$

Example 4.1. Let us take the hyper-dual curve $\tilde{\Gamma}(t)=\hat{A}(t)+\varepsilon^*\hat{A}^*(t)$, where $\hat{A}(t)=\boldsymbol{a}_0(t)+\varepsilon\boldsymbol{a}_1(t)$ and $\hat{A}^*(t)=\boldsymbol{a}_2(t)+\varepsilon\boldsymbol{a}_3(t)$. Here;

$$\boldsymbol{a}_0(t)=(\cos t\cos 2t,\cos t\sin 2t,\sin t), \tag{4.1}$$

$$\boldsymbol{a}_1(t)=(\sin t\sin 2t,-\sin t\cos 2t,0), \tag{4.2}$$

$$\boldsymbol{a}_2(t)=(\sin t\cos 2t,\sin t\sin 2t,-\cos t), \tag{4.3}$$

$$\boldsymbol{a}_3(t)=(-\cos t\sin 2t,\cos t\cos 2t,0). \tag{4.4}$$

Since $\left|\hat{A}(t)\right|_D=\left|\hat{A}^*(t)\right|_D=1$ and $\left\langle\hat{A}(t),\hat{A}^*(t)\right\rangle_D=0$; $\tilde{\Gamma}(t)$ is a hyper-dual curve on $\mathbb{S}^2_{\mathbb{D}_1}$, and $\hat{A}(t)$ and $\hat{A}^*(t)$ are dual curves on unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$. Using Eqs. (3.23) and

(3.24), the ruled surfaces corresponding to the dual curves $\hat{A}(t) = \boldsymbol{a}_0(t) + \varepsilon \boldsymbol{a}_1(t)$ and $\hat{A}^*(t) = \boldsymbol{a}_2(t) + \varepsilon \boldsymbol{a}_3(t)$ are obtained, respectively, as

$$\begin{aligned} \Phi_1(t, u_1) = &\left(\sin^2 t \cos 2t, \sin^2 t \sin 2t, -\sin t \cos t\right) \\ &+ u_1 \left(\cos t \cos 2t, \cos t \sin 2t, \sin t\right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} \Phi_2(t, u_2) = &\left(\cos^2 t \cos 2t, \cos^2 t \sin 2t, \sin t \cos t\right) \\ &+ u_2 \left(\sin t \cos 2t, \sin t \sin 2t, -\cos t\right), \end{aligned} \tag{4.6}$$

where $t \in I = (0, \pi)$ and $u_1, u_2 \in \mathbb{R}$. For $t = t_0$, $\Phi_1(t_0, u)$ and $\Phi_2(t_0, u)$ represent the lines $m_{t_0}(u_1)$ and $n_{t_0}(u_2)$, respectively. Moreover, $m_{t_0}(u_1)$ is a line corresponding to the unit dual vector $\hat{A}(t_0) = \boldsymbol{a}_0(t_0) + \varepsilon \boldsymbol{a}_1(t_0)$, and $n_{t_0}(u_2)$ is a line corresponding to the unit dual vector $\hat{A}^*(t_0) = \boldsymbol{a}_2(t_0) + \varepsilon \boldsymbol{a}_3(t_0)$.

For all $t \in I$, the intersection point of the lines $m_t(u_1)$ and $n_t(u_2)$ will be obtained as

$$k(t) = (\cos 2t, \sin 2t, 0), \tag{4.7}$$

where $u_1 = \cos t$ and $u_2 = \sin t$. Using Eqs. (3.28) and (3.29), these ruled surfaces can be expressed as

$$\Phi_1(t, v_1) = (\cos 2t, \sin 2t, 0) + v_1 (\cos t \cos 2t, \cos t \sin 2t, \sin t), \tag{4.8}$$

$$\Phi_2(t, v_2) = (\cos 2t, \sin 2t, 0) + v_2 (\sin t \cos 2t, \sin t \sin 2t, -\cos t), \tag{4.9}$$

where $v_1, v_2 \in \mathbb{R}$. From Eqs. (4.8) and (4.9), it can be seen that the ruled surfaces $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ have a common base curve $k(t) = (\cos 2t, \sin 2t, 0)$. Using Eqs. (4.1) and (4.3), we get $\langle \boldsymbol{a}_0(t), \boldsymbol{a}_2(t) \rangle = 0$. Thus, the position vectors of the director curves $\boldsymbol{a}_0(t)$ and $\boldsymbol{a}_2(t)$ of the surfaces $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ are perpendicular.

The velocity vector $k'(t) = (-2 \sin 2t, 2 \cos 2t, 0)$ is perpendicular to $\boldsymbol{a}_0(t)$ and $\boldsymbol{a}_2(t)$. Thus, according to Theorem 3.5 the normal vectors of $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ are perpendicular along $k(t)$.

$\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ represent Möbius strips. For intervals $0 \leq t \leq \pi$, $-0.3 \leq v_1 \leq 0.3$ and $-0.3 \leq v_2 \leq 0.3$, these two Möbius strips can be drawn as in Fig. 3.

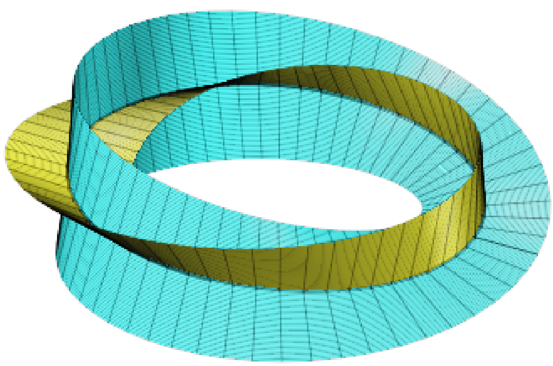

Figure 3. Geometric representation of two Möbius strips in $\mathbb{R}^3$ corresponding to the hyper-dual curve $\tilde{\Gamma}(t)$ on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$.

Example 4.2. Let us take the hyper-dual curve $\tilde{\Gamma}(t) = \hat{A}(t) + \varepsilon^* \hat{A}^*(t)$, where $\hat{A}(t) = \boldsymbol{a}_0(t) + \varepsilon \boldsymbol{a}_1(t)$ and $\hat{A}^*(t) = \boldsymbol{a}_2(t) + \varepsilon \boldsymbol{a}_3(t)$. Here;

$$\boldsymbol{a}_0(t) = (0, 0, 1)\,, \tag{4.10}$$
$$\boldsymbol{a}_1(t) = (\sin t, -\cos t, 0)\,, \tag{4.11}$$
$$\boldsymbol{a}_2(t) = (\cos t, \sin t, 0)\,, \tag{4.12}$$
$$\boldsymbol{a}_3(t) = (-t \sin t, t \cos t, 0)\,. \tag{4.13}$$

Since $\left|\hat{A}(t)\right|_D = \left|\hat{A}^*(t)\right|_D = 1$ and $\left\langle \hat{A}(t), \hat{A}^*(t) \right\rangle_D = 0$; $\tilde{\Gamma}(t)$ is a curve on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$, and $\hat{A}(t)$ and $\hat{A}^*(t)$ are dual curves on unit dual sphere $\mathbb{S}^2_{\mathbb{D}}$. Using Eqs. (3.23) and (3.24), the ruled surfaces corresponding to the dual curves $\hat{A}(t) = \boldsymbol{a}_0(t) + \varepsilon \boldsymbol{a}_1(t)$ and $\hat{A}^*(t) = \boldsymbol{a}_2(t) + \varepsilon \boldsymbol{a}_3(t)$ are obtained, respectively, as

$$\Phi_1(t, u_1) = (\cos t, \sin t, 0) + u_1\,(0, 0, 1)\,, \tag{4.14}$$
$$\Phi_2(t, u_2) = (0, 0, t) + u_2\,(\cos t, \sin t, 0)\,, \tag{4.15}$$

where $t \in I = (0, \pi)$ and $u_1, u_2 \in \mathbb{R}$. From the Theorem 3.4, the ruled surfaces $\Phi_1(t, u_1)$ and $\Phi_2(t, u_2)$ can be also obtained, respectively, as

$$\Phi_1(t, v_1) = (\cos t, \sin t, t) + v_1(0, 0, 1), \qquad v_1 \in \mathbb{R} \tag{4.16}$$
$$\Phi_2(t, v_2) = (\cos t, \sin t, t) + v_2(\cos t, \sin t, 0), \qquad v_2 \in \mathbb{R} \tag{4.17}$$

where $k(t) = (\cos t, \sin t, t)$ is a common base curve of $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$. Since $\langle \boldsymbol{a}_0(t), \boldsymbol{a}_2(t) \rangle = 0$, the position vectors of the director curves $\boldsymbol{a}_0(t) = (0, 0, 1)$ and $\boldsymbol{a}_2(t) = (\cos t, \sin t, 0)$ are perpendicular.

The velocity vector $k'(t) = (-\sin t, \cos t, 1)$ is perpendicular to $\boldsymbol{a}_2(t)$. Thus, according to Theorem 3.5 the normal vectors of $\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ are perpendicular along $k(t)$.

$\Phi_1(t, v_1)$ and $\Phi_2(t, v_2)$ represent, respectively, cylindrical and helicoid surfaces. They intersect along a helix curve $k(t) = (\cos t, \sin t, t)$. For intervals $-\pi \leq t \leq \pi$, $-10 \leq v_1 \leq 10$ and $-10 \leq v_2 \leq 10$, these surfaces can be drawn as in Fig. 4.

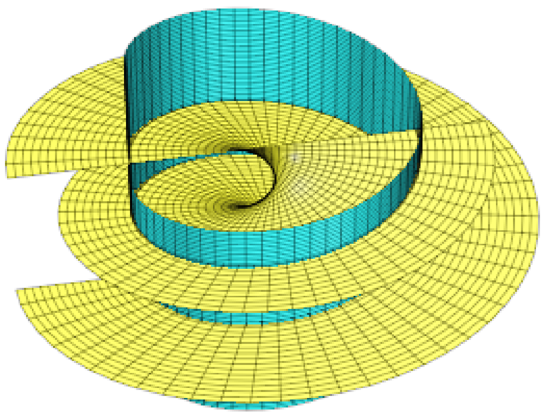

Figure 4. Geometric representation of two ruled surfaces in $\mathbb{R}^3$ corresponding to the hyper-dual curve $\tilde{\Gamma}(t)$ on $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$.

5. Conclusions

In this paper, some basic concepts of hyper-dual numbers are given by using dual numbers. Using these concepts, we have given the definition of a set $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$, which is a subset of unit hyper-dual sphere $\mathbb{S}^2_{\tilde{\mathbb{D}}}$. We show that there exists a one-to-one correspondence between the points of $\mathbb{S}^2_{\tilde{\mathbb{D}}_1}$ and any two intersecting perpendicular directed lines in $\mathbb{R}^3$.

Moreover, we show that each hyper-dual curve on $\mathbb{S}^2_{\mathbb{D}_1}$ represents two ruled surfaces in $\mathbb{R}^3$ such that these ruled surfaces intersect along a common base curve.

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