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Research Article

Existence results for a Dirichlet boundary value problem through a local minimization principle

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Abstract

In this paper, a local minimum result for differentiable functionals is exploited in order to prove that a perturbed Dirichlet boundary value problem including a Lipschitz continuous non-linear term admits at least one non-trivial weak solution under an asymptotical behaviour of the nonlinear datum at zero. Some special cases and a concrete example of an application is then presented.

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1. Introduction

In this paper, we consider a bifurcation result for the following parametric Dirichlet boundary value problem on a bounded interval $[a,b]$ in $\mathbb{R}$

$$\begin{cases} -u'' + u\,h(x,u') = [\lambda f(x,u) + g(u)]h(x,u') & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{1.1}$$

where λ is a positive parameter, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2|,$$

for every $t_1, t_2 \in \mathbb{R}$, with $g(0) = 0$, and $h : [a,b] \times \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m := \inf_{(x,t)\in[a,b]\times\mathbb{R}} h(x,t) > 0$.

Due to the applications of Dirichlet boundary value problems in various fields of applied sciences such as control systems, economics, mechanical engineering, and biological or artificial neural networks, these problems have been extensively studied.

In this context, several existence and multiplicity results for solutions to second order ordinary differential nonlinear equations, with the nonlinearity dependent on the derivative and Dirichlet conditions at the ends, have been investigated using variational methods. For an overview on this subject, we cite the papers [1–5, 7–14, 16] and references therin.

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The pioneering work in this direction is the paper of Averna and Bonanno [7], where using a three-critical-points theorem [6], the existence of at least three classical solutions for the following quasilinear two-point boundary-value problem has been obtained

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(t,u)h(u') & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

where $p > 1$, λ is a positive parameter, $h : \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function such that $\inf h > 0$, and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

We also refer the reader to the paper [2] in which, by means of Ricceri's Variational Principle, the existence of infinitely many weak solutions for the following Dirichlet doubly eigenvalue boundary value problem

$$\begin{cases} -u'' + u\,h(x,u') = [\lambda f(x,u) + \mu g(x,u) + p(u)]h(x,u') & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

where λ is a positive parameter, μ is a non-negative parameter, $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $p(0) = 0$, and $h : [a,b] \times \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function, are ensured.

Further, we point out that Heidarkhani *et al.* in [11, Theorem 3.1], using the same variational setting, established the existence of at least one non-trivial classical solution for the quasilinear system

$$\begin{cases} -(p_i - 1)|u_i'(x)|^{p_i-2}u_i''(x) = \lambda F_{u_i}(x,u_1,\dots,u_n)h_i(x,u_i') & x \in (a,b), \\ u_i(a) = u_i(b) = 0, \qquad \text{for } 1 \le i \le n, \end{cases}$$

where $p_i > 1$ for $1 \le i \le n$, $\lambda > 0$, $h_i : [a,b] \times \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m_i := \inf_{(x,t)\in[a,b]\times\mathbb{R}} h_i(x,t) > 0$ for $1 \le i \le n$, $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is measurable with respect to x, for every $(t_1,\dots,t_n) \in \mathbb{R}^n$, continuously differentiable in $(t_1,\dots,t_n)$, for almost every $x \in [a,b]$, and $F_{t_i}(x,0,\dots,0) = 0$ for all $x \in [a,b]$ and for $1 \le i \le n$.

In the present paper, motivated by the above works and using a general critical point theorem (see Theorem 2.1 below), we study the existence of at least one non-trivial weak solution for problem (1.1) for small values of the parameter λ and requiring an additional asymptotical behaviour of the potential at zero if $f(x,0) \equiv 0$.

A special case of our main result reads as follows.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. If $f(0) = 0$, assume also that*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty.$$

Assume further that $1 \le L < 5$. Then, for each parameter

$$\lambda \in \left]0, \frac{5-L}{2}\left(\sup_{c>0} \frac{c^2}{\int_0^c f(t)dt}\right)\right[,$$

the following problem

$$\begin{cases} -u'' + u = \lambda f(u) + g(u) & \textit{in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

admits at least one non-trivial and non-negative classical solution u_λ. Moreover, we have

$$\lim_{\lambda\to 0^+} \int_0^1 |u_\lambda'(x)|^2 dx = 0,$$

and the real function

$$\lambda \mapsto \frac{1}{2}\int_0^1 \left(|u_\lambda(x)|^2 + |u_\lambda'(x)|^2\right)dx - \int_0^1 \left(\int_0^{u_\lambda(x)} (\lambda f(t) + g(t))\,dt\right)dx,$$

is negative and strictly decreasing in the interval

$$\left]0, \frac{5-L}{2}\left(\sup_{c>0}\frac{c^2}{\int_0^c f(t)dt}\right)\right[.$$

The plan of the paper is as follows. In Section 2 we introduce our notations and a suitable abstract setting (see Theorem 2.1). In Section 3 our main result (see Theorem 3.1) is presented, while Section 4 is devoted to some comments on the results of the paper. Finally, a concrete example of an application is exhibited in Example 4.5.

2. Preliminaries

First we here recall for the reader's convenience the following version of Ricceri's variational principle [15, Theorem 2.1] which is our main tool to prove the results.

Theorem 2.1. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive in X, and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and for any $r > \inf_X \Phi$, let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup_{v\in\Phi^{-1}((-\infty,r))}\Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for any $r > \inf_X \Phi$ and any $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional I_λ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

Now, assume that the functions f, g and h hold in the conditions given in the following of problem (1.1). Corresponding to f, g and h we introduce the functions $F : [a, b] \times \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ and $H : [a, b] \times \mathbb{R} \to [0, +\infty)$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)d\xi, \qquad G(t) := -\int_0^t g(\xi)d\xi,$$

and

$$H(x,t) := \int_0^t \left(\int_0^\tau \frac{1}{h(x,\delta)}d\delta\right)d\tau,$$

for all $x \in [a, b]$ and $t \in \mathbb{R}$.

In the continuation of this paper, we let X be the Sobolev space $W_0^{1,2}([a, b])$ equipped with the norm

$$\|u\| := \left(\int_a^b |u'(x)|^2 dx\right)^{1/2}.$$

We say that a function $u \in X$ is a *weak solution* of problem (1.1) if

$$\int_a^b \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}d\tau\right)v'(x)dx + \int_a^b u(x)v(x)dx - \lambda\int_a^b f(x,u(x))v(x)dx - \int_a^b g(u(x))v(x)dx = 0,$$

for all $v \in X$.

If f is a continuous function, then by standard regularity results, weak solutions of problem (1.1) are classical solutions, i.e., they belong to $C^2([a, b])$.

We recall the following inequality of Sobolev type (see, for instance, the paper [17])

$$\max_{x\in[a,b]} |u(x)| \le \frac{(b-a)^{1/2}}{2}\|u\|, \qquad \forall u \in X. \tag{2.1}$$

In the following, let $M := \sup_{(x,t)\in[a,b]\times\mathbb{R}} h(x,t)$ and suppose that the Lipschitz constant $L > 0$ of the function g satisfies the following condition:

(A_0) $L \geq 1$ and $M(L-1)(b-a)^2 < 4$.

Now, put

$$\alpha_1 := \frac{4 + m(1+L)(b-a)^2}{8m},$$
$$\alpha_2 := \frac{4 + M(1-L)(b-a)^2}{2M}.$$

We introduce the energy functional $I_\lambda(u) : X \to \mathbb{R}$ associated with (1.1),

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \qquad \forall u \in X,$$

where

$$\Phi(u) := \int_a^b H(x, u'(x))dx + \frac{1}{2}\int_a^b |u(x)|^2 dx + \int_a^b G(u(x))dx,$$

and

$$\Psi(u) := \int_a^b F(x, u(x))dx,$$

for every $u \in X$. By standard arguments, one has that Φ is continuously Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_a^b \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}\, d\tau\right) v'(x)dx + \int_a^b u(x)v(x)dx - \int_a^b g(u(x))v(x)dx,$$

for every $v \in X$. Since g is Lipschitz continuous and satisfies $g(0) = 0$, while h is bounded away from zero, we have from (2.1) that

$$\Phi(u) \geq \frac{\alpha_2}{4}\|u\|^2, \tag{2.2}$$

for all $u \in X$, and so Φ is coercive.

On the other hand, the fact that X is compactly embedded into $C^0([a,b])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and sequentially weakly (upper) continuous, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_a^b f(x, u(x))v(x)dx,$$

for every $v \in X$; for more details, see the proof of [2, Theorem 3.1].

Note that the weak solutions of (1.1) are exactly the critical points of I_λ.

3. The main result

In this section, we prove our existence result that reads as follows.

Theorem 3.1. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. In addition, if $f(x,0) \equiv 0$, assume that there are a non-empty open set $D \subseteq (a,b)$ and a set $B \subseteq D$ of positive Lebesgue measure such that*

$$\limsup_{t\to 0^+} \frac{\operatorname{ess\,inf}_{x\in B} F(x,t)}{t^2} = +\infty, \tag{3.1}$$

and

$$\liminf_{t\to 0^+} \frac{\operatorname{ess\,inf}_{x\in D} F(x,t)}{t^2} > -\infty. \tag{3.2}$$

Then, for every

$$\lambda \in \Lambda := \left]0, \frac{\alpha_2}{b-a}\left(\sup_{c>0} \frac{c^2}{\int_a^b \max_{|t|\leq c} F(x,t)dx}\right)\right[,$$

problem (1.1) *admits at least one non-trivial weak solution* $u_\lambda \in X$. *Moreover, we have*

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0,$$

and the real function $\lambda \mapsto I_\lambda(u_\lambda)$ *is negative and strictly decreasing in the open interval* Λ.

Proof. Our goal is to apply Theorem 2.1 to problem (1.1). For this, let Φ, Ψ and I_λ are the functionals introduced in Section 2. As seen before, the functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 2.1. Now, we try to prove the existence of critical points for the functional I_λ in X. Consider λ as given in the theorem. Hence, there exists $\bar{c} > 0$ such that

$$\frac{(b-a)\lambda}{\alpha_2} < \frac{\bar{c}^2}{\int_a^b \max_{|t|\le\bar{c}} F(x,t)dx}.$$

Set

$$r := \frac{\alpha_2}{b-a}\bar{c}^2.$$

Then, for all $u \in X$ with $\Phi(u) < r$, according to inequalities (2.1) and (2.2), we have $\|u\|_\infty \le \bar{c}$. Hence, one has

$$\Psi(u) = \int_a^b F(x,u(x))\,dx \le \int_a^b \max_{|t|\le\bar{c}} F(x,t)\,dx,$$

for every $u \in X$ such that $\Phi(u) < r$. Then,

$$\sup_{\Phi(u)<r} \Psi(u) \le \int_a^b \max_{|t|\le\bar{c}} F(x,t)\,dx.$$

Taking into account the above calculations and the definition of $\varphi(r)$, since $0 \in \Phi^{-1}((-\infty,r))$ and $\Phi(0) = \Psi(0) = 0$, we have

$$\begin{aligned}
\varphi(r) &= \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup\limits_{v\in\Phi^{-1}((-\infty,r))} \Psi(v) - \Psi(u)}{r-\Phi(u)} \\
&\le \frac{\sup\limits_{v\in\Phi^{-1}((-\infty,r))} \Psi(v)}{r} \\
&\le \frac{(b-a)}{\alpha_2}\frac{\int_a^b \max_{|t|\le\bar{c}} F(x,t)\,dx}{\bar{c}^2} < \frac{1}{\lambda}.
\end{aligned}$$

Hence, set

$$\lambda^\star := \frac{\alpha_2}{(b-a)}\frac{\bar{c}^2}{\int_a^b \max_{|t|\le\bar{c}} F(x,t)\,dx} \in (0,+\infty].$$

So, thanks to Theorem 2.1, for every $\lambda \in (0,\lambda^\star) \subseteq (0, 1/\varphi(r))$, the functional I_λ has at least one critical point (local minima) $u_\lambda \in \Phi^{-1}((-\infty,r))$.

Now, we establish that for each fixed $\lambda \in (0,\lambda^\star)$ the solution u_λ found above is not the trivial function and the map

$$(0,\lambda^\star) \ni \lambda \mapsto I_\lambda(u_\lambda),$$

is negative. If $f(x,0) \ne 0$ for some $x \in (a,b)$, then it follows that $u_\lambda \not\equiv 0_X$, since the trivial function does not solve problem (1.1). We consider the case when $f(x,0) \equiv 0$. First, we show that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{3.3}$$

Due to the hypotheses (3.1) and (3.2), we can fix a sequence $\{t_n\} \subset \mathbb{R}^+$ converging to zero and two constant σ, and κ (with $\sigma > 0$) such that

$$\lim_{n\to\infty} \frac{\operatorname{ess\,inf}_{x\in B} F(x,t_n)}{t_n^2} = +\infty,$$

and

$$\operatorname{ess\,inf}_{x\in D} F(x,t) \geq \kappa\, t^2,$$

for every $t \in [0,\sigma]$.

Now, fix a set $C \subset B$ of positive measure and a function $v \in X$ such that:

i) $v(x) \in [0,1]$, for every $x \in [a,b]$;
ii) $v(x) = 1$, for every $x \in C$;
iii) $v(x) = 0$, for every $x \in (a,b) \setminus D$.

Hence, fix $N > 0$ and consider a real positive number η with

$$N < \frac{\eta \operatorname{meas}(C) + \kappa \int_{D\setminus C} (v(x))^2\, dx}{\alpha_1 \|v\|^2},$$

where $\operatorname{meas}(C)$ denotes the Lebesgue measure of the set C. Then, there is $\nu \in \mathbb{N}$ such that $t_n < \sigma$ and

$$\operatorname{ess\,inf}_{x\in B} F(x,t_n) \geq \eta\, t_n^2,$$

for every $n > \nu$.

Now, for every $n > \nu$, according to the properties of the function v ($0 \leq t_n v(x) < \sigma$ for n sufficiently large), we have

$$\begin{aligned} \frac{\Psi(t_n v)}{\Phi(t_n v)} &= \frac{\int_C F(x,t_n)\, dx + \int_{D\setminus C} F(x, t_n v(x))\, dx}{\Phi(t_n v)} \\ &\geq \frac{\eta \operatorname{meas}(C) + \kappa \displaystyle\int_{D\setminus C} (v(x))^2\, dx}{\alpha_1 \|v\|^2} > N. \end{aligned}$$

Since N could be arbitrarily large, it follows that

$$\lim_{n\to\infty} \frac{\Psi(t_n v)}{\Phi(t_n v)} = +\infty,$$

so, the relation (3.3) follows.

Let $w_n := t_n v$ for any $n \in \mathbb{N}$. We have

$$\|w_n\| = |t_n| \|v\| \to 0,$$

as $n \to +\infty$, so that for n large enough,

$$\|w_n\| < \sqrt{\frac{\alpha_2}{\alpha_1(b-a)}}\, \bar{c}.$$

According to the above inequality and taking into account the relation $\Phi(w_n) \leq \alpha_1 \|w_n\|^2$, one has

$$w_n \in \Phi^{-1}((-\infty, r)), \tag{3.4}$$

provided n is large enough. Also, by (3.3) and the fact that $\lambda > 0$,

$$I_\lambda(w_n) = \Phi(w_n) - \lambda \Psi(w_n) < 0, \tag{3.5}$$

for n sufficiently large.

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}((-\infty, r))$, by (3.4) and (3.5) we deduce that

$$I_\lambda(u_\lambda) \leq I_\lambda(w_n) < 0 = I_\lambda(0), \tag{3.6}$$

so, $u_\lambda \not\equiv 0_X$. Therefore, u_λ is a non-trivial solution of problem (1.1). Moreover, from (3.6) we get that the map

$$(0, \lambda^\star) \ni \lambda \mapsto I_\lambda(u_\lambda), \tag{3.7}$$

is negative.

Now, we claim that $\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$.

Because Φ is coercive and for every $\lambda \in (0, \lambda^\star)$, one has $u_\lambda \in \Phi^{-1}((-\infty, r))$, so there exists a positive constant L such that $\|u_\lambda\| \le L$, for every $\lambda \in (0, \lambda^\star)$. Then, we have

$$\left| \int_a^b f(x, u_\lambda(x)) u_\lambda(x)\, dx \right| \le \frac{(b-a)^{3/2} L}{2} \sup_{|s| \le \frac{\sqrt{b-a}}{2} L} |f(x,s)|, \tag{3.8}$$

for every $\lambda \in (0, \lambda^\star)$.

Since u_λ is a critical point of I_λ , then $I'_\lambda(u_\lambda)(v) = 0$, for any $v \in X$ and every $\lambda \in (0, \lambda^\star)$. Specially, $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_a^b f(x, u_\lambda(x)) u_\lambda(x)\, dx, \tag{3.9}$$

for every $\lambda \in (0, \lambda^\star)$. According to the inequality (2.1) we will have

$$\begin{aligned} \Phi'(u_\lambda)(u_\lambda) &= \int_a^b \left(\int_0^{u'_\lambda(x)} \frac{1}{h(x,\tau)}\, d\tau \right) u'_\lambda(x) dx + \int_a^b |u_\lambda(x)|^2 dx - \int_a^b g(u_\lambda(x)) u_\lambda(x) dx \\ &\ge \frac{1}{M} \|u_\lambda\|^2 + (1-L) \|u_\lambda\|^2_{L^2([a,b])} \\ &\ge \frac{\alpha_2}{2} \|u_\lambda\|^2 \ge 0. \end{aligned}$$

Then, by (3.9), it follows that

$$0 \le \frac{\alpha_2}{2} \|u_\lambda\|^2 \le \Phi'(u_\lambda)(u_\lambda) = \lambda \int_a^b f(x, u_\lambda(x)) u_\lambda(x)\, dx,$$

for any $\lambda \in (0, \lambda^\star)$. Letting $\lambda \to 0^+$, by (3.8), we get $\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$. By the inequality (2.1), we have $\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = 0$.

Finally, we show that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^\star)$. For this, for any $u \in X$, one has

$$I_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right). \tag{3.10}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^\star$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi^{-1}\big((-\infty, r)\big)$ for $i = 1, 2$.

Also, let

$$m_{\lambda_i} := \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}\left((-\infty, r)\right)} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v) \right),$$

for every $i = 1, 2$.

Clearly, the positivity of λ together with (3.7) and (3.10) show that

$$m_{\lambda_i} < 0, \quad \text{for} \ \ i = 1, 2. \tag{3.11}$$

Moreover,

$$m_{\lambda_2} \le m_{\lambda_1}, \tag{3.12}$$

since $0 < \lambda_1 < \lambda_2$. Then, by (3.10)–(3.12) and by the fact that $0 < \lambda_1 < \lambda_2$, we have

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^\star)$. The arbitrariness of $\lambda < \lambda^\star$ shows that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^\star)$. This concludes the proof. $\square$

Corollary 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(\xi) := \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$. In addition, if $f(0) = 0$, assume also that*

$$\lim_{t\to 0^+} \frac{F(t)}{t^2} = +\infty.$$

Then, for every

$$\lambda \in \left]0, \frac{\alpha_2}{(b-a)^2}\left(\sup_{c>0} \frac{c^2}{\max_{|t|\le c} F(t)}\right)\right[,$$

the problem

$$\begin{cases} -u'' + u\, h(x,u') = [\lambda f(u) + g(u)]h(x,u') & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

admits at least one non-trivial classical solution in X.

4. Some comments

In this section, we give some remarks and a concrete example of application of our results.

Remark 4.1. If in Theorem 3.1 one has $f(x,s) \ge 0$ for a.e. $x \in [a,b]$ and every $s < 0$, then the obtained weak solution is non-negative. Indeed, arguing by a contradiction, let u be a critical point of I_λ and that the open set

$$S := \{x \in [a,b] \ : \ u(x) < 0\},$$

is of positive Lebesgue measure. Set $v := \min\{0, u\}$. Clearly, $v \in X$ and, since u is a critical point, by (2.1) and the sign assumption on f, we have

$$\begin{aligned}
0 &= \Phi'(u)(v) - \lambda\Psi'(u)(v) \\
&= \int_a^b \Big(\int_0^{u'(x)} \frac{1}{h(x,\tau)}\, d\tau\Big) v'(x)dx + \int_a^b u(x)v(x)dx \\
&\quad - \int_a^b g(u(x))v(x)dx - \lambda \int_a^b f(x,u(x))v(x)dx \\
&\ge \int_S \Big(\int_0^{u'(x)} \frac{1}{h(x,\tau)}\, d\tau\Big) u'(x)dx + \int_S |u(x)|^2 dx - \int_S g(u(x))u(x)dx \\
&\ge \frac{1}{M}\int_S |u'(x)|^2 dx + (1-L)\int_S |u(x)|^2 dx \\
&\ge \frac{4 + M(1-L)\,(\mathrm{meas}(S))^2}{4M} \|u\|_S^2 \\
&\ge \frac{\alpha_2}{2}\|u\|_S^2 .
\end{aligned}$$

Hence, since $u|_S \in W_0^{1,2}(S)$, one has $u \equiv 0$ on S, which is a contradiction. So, our claim is proved.

Remark 4.2. We note that Theorem 3.1 is a bifurcation result, because $\lambda = 0$ is a bifurcation point for problem (1.1), in the sense that the pair $(0,0)$ belongs to the closure of the set

$$\big\{(u_\lambda, \lambda) \in X \times (0,+\infty) \ : \ u_\lambda \text{ is a non-trivial weak solution of (1.1)}\big\},$$

in $X \times \mathbb{R}$. Indeed, by Theorem 3.1 we have that

$$\|u_\lambda\| \to 0 \quad \text{as} \quad \lambda \to 0^+.$$

Hence, there exists two sequences $\{u_j\}$ in X and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here, $u_j := u_{\lambda_j}$) such that

$$\lambda_j \to 0^+ \quad \text{and} \quad \|u_j\| \to 0,$$

as $j \to +\infty$.

Further, we want to point out that for any $\lambda_1, \lambda_2 \in \Lambda$, with $\lambda_1 \neq \lambda_2$, the solutions u_{λ_1} and u_{λ_2}, given by Theorem 3.1 are different, since the map

$$\Lambda \ni \lambda \mapsto I_\lambda(u_\lambda),$$

is strictly decreasing.

Remark 4.3. Let the hypotheses of Corollary 3.2 be satisfied. Assume also that f is non-negative, and

$$\sup_{c>0} \frac{c^2}{F(c)} > \frac{(b-a)^2}{\alpha_2}.$$

Then, the problem

$$\begin{cases} -u'' + u\,h(x,u') = [f(u) + g(u)]h(x,u') & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

admits at least one non-trivial and non-negative classical solution in X.

Remark 4.4. Theorem 1.1 in the Introduction immediately follows from Corollary 3.2 and Remark 4.1, setting $h(x,t) \equiv 1$ for all $(x,t) \in [0,1] \times \mathbb{R}$.

Example 4.5. Consider the following parametric problem

$$\begin{cases} -u'' = \lambda e^u & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{4.1}$$

Obviously, we have $L = 1$. Then, due to Theorem 1.1, for each parameter

$$\lambda \in \Lambda' := \left]0, 2\sup_{c>0} \frac{c^2}{e^c - 1}\right[=]0, 1.2952[,$$

problem (4.1) admits at least one non-trivial and non-negative classical solution u_λ. Moreover, one has that

$$\lim_{\lambda \to 0^+} \int_0^1 |u_\lambda'(x)|^2 dx = 0,$$

and the real function

$$\lambda \mapsto \frac{1}{2}\int_0^1 |u_\lambda'(x)|^2 dx - \lambda\left(\int_0^1 e^{u_\lambda(x)} dx - 1\right),$$

is negative and strictly decreasing in the interval Λ'. Since $1 \in \Lambda'$, the problem

$$\begin{cases} -u'' = e^u & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{4.2}$$

admits at least one non-trivial and non-negative classical solution. Further, we prove that problem (4.2) has a unique positive solution. Let $-u'' = \sigma$. With the boundary conditions $u(0) = u(1) = 0$, we have $u(t) = \int_0^1 g(t,s)\sigma(s)ds$, where

$$g(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t < s \le 1. \end{cases}$$

Then, $\sigma \ge 0$ implies $u \ge 0$. Moreover, $\sigma \not\equiv 0$ on any subinterval. Then, $u(t) > 0$ for any $t \in (0,1)$. If u is a positive solution of problem (4.2), then $u(t) = \int_0^1 g(t,s)e^{u(s)}ds$, $u(0) = 0$ and $u'(0) > 0$ (Let $u'(0) \le 0$. Since $u'' < 0$, then u' is decreasing, and so u can not be positive). Now, let there exist two solutions u_1 and u_2 for problem (4.2). Assume also that $u_1'(0) < u_2'(0)$. If $u_2(\xi) = u_1(\xi)$ for some $\xi \in (0,1]$, and $u_2(t) > u_1(t)$ for every $t \in (0,\xi)$, then

$$0 = u_2(\xi) - u_1(\xi) = \int_0^1 g(\xi,s)\left[e^{u_2(s)} - e^{u_1(s)}\right] ds > 0,$$

which is a contradiction. Thus, our claim is proved.

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