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Research Article

On some geometric properties of the Le Roy-type Mittag-Leffler function

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Abstract

In this paper, we consider the Le Roy-type Mittag-Leffler function. Our main focus is to establish some sufficient conditions so that the normalized Le-Roy type Mittag-Leffler function posses some geometric properties such as starlikeness, convexity, close-to-convexity (univalency) and uniformly convexity inside the unit disk. Using these results, geometric properties of the normalized Mittag-Leffler function are derived as application. Results obtained in this paper are new. Interesting consequences, corollaries and examples are provided to support that these results are better and improve several results available in the literature.

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1. Introduction

1.1. Preliminaries

Suppose that $\mathbb{H}$ denotes the class of analytic functions in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Let $\mathbb{A}$ be the class of all functions $f \in \mathbb{H}$ such that $f(0) = f'(0) - 1 = 0$ with the following form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}.$$

Assume that $\mathbb{S}$ denotes the class of all functions in $\mathbb{A}$ which are univalent in the unit disc $\mathbb{D}$. A function $f \in \mathbb{A}$ is called starlike (with respect to the origin 0) in $\mathbb{D}$, if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a star-like domain with respect to 0 in $\mathbb{C}$. The class of starlike functions is denoted by $\mathbb{S}^*$. The analytic characterization of $\mathbb{S}^*$ can be found in [5], which is given below:

$$f \in \mathbb{S}^* \iff \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \ \forall z \in \mathbb{D}.$$

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Let $0 \leq \alpha < 1$. Then a function $f \in \mathbb{A}$ is called starlike function of order α, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad \forall z \in \mathbb{D}.$$

This class is denoted by $\mathbb{S}^*(\alpha)$. A function $f \in \mathbb{A}$ is called convex in $\mathbb{D}$ if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a convex domain in $\mathbb{C}$. The class of convex functions is denoted by $\mathbb{K}$. The analytic characterization of this class is given by:

$$f \in \mathbb{K} \iff \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ \forall z \in \mathbb{D}.$$

If in addition,

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad \forall z \in \mathbb{D},$$

where $\alpha \in [0, 1)$, then f is called convex function of order α. We denote the class of convex functions of order α by $\mathbb{K}(\alpha)$.

A function $f \in \mathbb{A}$ is close-to-convex in $\mathbb{D}$ if there exists a starlike function g in $\mathbb{D}$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}.$$

It is well known that every close-to-convex function in $\mathbb{D}$ is also univalent in the unit disk $\mathbb{D}$.

A function $f \in \mathbb{A}$ is called uniformly convex (starlike) if for any circular arc γ contained in $\mathbb{D}$ with center $\zeta \in \mathbb{D}$ the image arc $f(\gamma)$ is convex (starlike w.r.t. the image $f(\zeta)$). Let UCV (UST) denote the class of all uniformly convex (starlike) functions [26]. In [9,10], A. W. Goodman introduced these classes. In [26], F. Rønning introduced a class of starlike functions $\mathbb{S}_p$ in the following way.

$$\mathbb{S}_p := \{f : f(z) = zF'(z), \ F \in UCV\}.$$

For further details on geometric properties of analytic functions we refer to [5,13–16] and references cited therein.

1.2. Motivation

Problems for studying the geometric properties (including univalency, starlikeness or convexity) of family of analytic functions (in the unit disk) involving special functions have always been attracted by several researchers [2, 9, 10, 15, 16, 20, 21]. Mittag-Leffler functions are important special functions which play important role in fractional calculus, approximation theory and various branches of science and engineering. These functions also appear in the solution of fractional order differential equations or fractional order integral equations. In [25], application of Mittag-Leffler functions in fractional modeling has been discussed. In 1903, M. G. Mittag-Leffler [17,18] introduced the classical Mittag-Leffler function, defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \ z \in \mathbb{C}.$$

A famous generalization of $E_\alpha(z)$ with two parameters (i.e., two parametric Mittag-Leffler function) is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \ z \in \mathbb{C}.$$

In [2,20,21], several geometric properties of the normalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}(z)$, defined as

$$\mathbb{E}_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(\beta)z^{k+1}}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta>0,\ z\in\mathbb{C},$$

has been discussed inside the unit disk $\mathbb{D}$.

M. A. Al-Bassam and Yu. F. Luchko [1] introduced multi-index (also known as vector index) Mittag-Leffler functions of $2m$-parameters, defined as

$$E^{(m)}_{(\alpha,\beta)}(z) \equiv E^{(m)}_{(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{i=1}^{m}\Gamma(\alpha_i k+\beta_i)}, \quad \alpha_i,\beta_i>0, m\in\mathbb{N}, z\in\mathbb{C}, \tag{1.1}$$

to solve a Cauchy type problem for a fractional differential equation and obtained explicit solution in terms of $E^{(m)}_{(\alpha,\beta)}(z)$.

In [27], Le Roy function was introduced by É. Le Roy, defined as

$$R_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^\gamma}, \quad \gamma>0, z\in\mathbb{C},$$

to study asymptotic of certain power series. Recently, S. Gerhold [8] and independently R. Garra-F. Polito [6] introduced Le Roy-type Mittag-Leffler function, defined as

$$F^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k+\beta)]^\gamma}, \quad \alpha,\beta,\gamma>0,\ z\in\mathbb{C}. \tag{1.2}$$

It can be easily noted that $F^{(\gamma)}_{\alpha,\beta}(z)$ is a generalization of $E_{\alpha,\beta}(z)$, $F^{(\gamma)}(z)$ and various other special functions. For example,

$$\begin{aligned}
&F^{(1)}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), \quad F^{(\gamma)}_{1,1}(z) = R_\gamma(z)\\
&F^{(1)}_{2,2}(z) = \frac{\sinh\sqrt{z}}{z}, \quad F^{(1)}_{1,1}(z) = \exp(z)\\
&F^{(1)}_{1,2}(z) = \frac{\exp(z)-1}{z}, \quad F^{(1)}_{2,1}(z) = \cosh\sqrt{z}\\
&F^{(2)}_{1,1}(z) = J_0(2\sqrt{z}), \quad F^{(n)}_{\alpha,\beta}(z) = E^{(n)}_{(\alpha,\beta),\ldots,(\alpha,\beta)}(z), n\in\mathbb{N},\\
&F^{(\nu)}_{1,1}(\lambda) = Z(\lambda,\nu), \quad F^{(\alpha+1)}_{1,1}(z) = \mathfrak{e}_\alpha(z),
\end{aligned}$$

where $J_0(z)$, $Z(\lambda,\nu)$ and $\mathfrak{e}_\alpha(z)$ denote the Bessel function of the first kind [3], COM-Poisson renormalization constant [4] and αL-exponential function [6] respectively.

In [7], R. Garrappa, S. Rogosin and F. Mainardi derived integral representations, integral transforms and asymptotic expansion of $F^{(\gamma)}_{\alpha,\beta}(z)$. Moreover, they posed some open problems related to the complete monotonicity of $F^{(\gamma)}_{\alpha,\beta}(z)$ in [7]. These open problems [7] have been solved by K. Górska, A. Horzela and R. Garrappa in [11]. Definite integral representation of $F^{(\gamma)}_{\alpha,\beta}(z)$ and COM-Poisson renormalization constants integral forms have been established by T. K. Pogány in [23]. Recently, T. Simon [28] studied complete monotonicity property of $F^{(\gamma)}_{\alpha,\beta}(z)$ on the negative half-line and proposed some conjectures related to random variables. The above results motivate us to study the geometric properties of normalized form of $F^{(\gamma)}_{\alpha,\beta}(z)$.

Since, $F^{(\gamma)}_{\alpha,\beta}(z) \notin \mathbb{A}$, we consider the following normalization of $F^{(\gamma)}_{\alpha,\beta}(z)$:

$$\begin{aligned}\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) = z[\Gamma(\beta)]^{\gamma} F^{(\gamma)}_{\alpha,\beta}(z) &= \sum_{k=1}^{\infty}\left[\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right]^{\gamma} z^k, \quad \alpha,\beta,\gamma>0,\, z\in\mathbb{C} \\ &:= z+\sum_{k=2}^{\infty} a_k(\alpha,\beta,\gamma)z^k,\end{aligned} \tag{1.3}$$

where $a_k(\alpha,\beta,\gamma) = \left[\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right]^{\gamma}$. Although, the formula (1.3) holds for $\alpha,\beta,\gamma>0$ and $z\in\mathbb{C}$, in this article we will restrict our attention to the case of positive real valued α,β,γ and $z\in\mathbb{D}$.

1.3. Main contributions and methodologies

The main focus of this paper is to study certain geometric properties of $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$. The main contributions along with methodologies are listed below:

- Derive sufficient conditions so that $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) \in \mathbb{S}^*$ in $\mathbb{D}$.
  To solve this problem, we will use the classical definition of stralikeness and the following lemmas:

  Lemma 1.1. [12] *For any positive real number s, the digamma function (psi function) $\psi(s)=\frac{\Gamma'(s)}{\Gamma(s)}$ satisfies the following inequality:*

  $$\log(s)-\frac{1}{s} < \psi(s) < \log(s)-\frac{1}{2s}. \tag{1.4}$$

  Lemma 1.2. [19] *Let $f(z)\in\mathbb{A}$ and $|f'(z)-1|<2/\sqrt{5}$ $\forall\, z\in\mathbb{D}$. Then $f(z)$ is a starlike function in $\mathbb{D}$.*

- Obtain sufficient conditions so that $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) \in \mathbb{S}^*$ in $\mathbb{D}_{1/2}$.
  We will use Lemma 1.1 and the following lemma to solve this problem.

  Lemma 1.3. [13] *Let $f\in\mathbb{A}$ and $|(f(z)/z)-1|<1$ for each $z\in\mathbb{D}$, then f is univalent and starlike in $\mathbb{D}_{1/2}=\{z:|z|<1/2\}$.*

- Establish sufficient conditions so that $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) \in \mathbb{K}$ in $\mathbb{D}$.
  To solve this problem, we will use the fact that a function $f(z)\in\mathbb{K}$ in $\mathbb{D}$ if and only if $zf'(z)\in\mathbb{S}^*$ in $\mathbb{D}$. Moreover, with the help of Lemma 1.1 and classical definition of starlikeness, we will obtain the required result.
- Find sufficient conditions so that $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) \in \mathbb{K}$ in $\mathbb{D}_{1/2}$.
  Lemma 1.1 and the following lemma will be applied to solve this problem.

  Lemma 1.4. [14] *Let $f\in\mathbb{A}$ and $|f'(z)-1|<1$ for each $z\in\mathbb{D}$, then f is convex in $\mathbb{D}_{1/2}=\{z:|z|<1/2\}$.*

- Obtain sufficient conditions so that $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ belongs to UCV and $\mathbb{S}_p$ inside the unit disk.
  Using Lemma 1.1 and the following lemma, we will derive the required result.

  Lemma 1.5. [24] *Let $f\in\mathbb{A}$.*

  (i) *If* $\left|\dfrac{zf''(z)}{f'(z)}\right| < \dfrac{1}{2}$, *then* $f\in UCV$.

  (ii) *If* $\left|\dfrac{zf'(z)}{f(z)}-1\right| < \dfrac{1}{2}$, *then* $f\in\mathbb{S}_p$.

- Derive sufficient conditions so that $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is close-to-convex with respect to certain functions.
  Lemma 1.1 and the following lemmas will be used to solve this problem.

  Lemma 1.6. [22] *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k z^k$. *If* $1 \leq 2A_2 \leq \cdots \leq nA_n \leq (n+1)A_{n+1} \leq \cdots \leq 2$, *or* $1 \geq 2A_2 \geq \cdots \geq nA_n \geq (n+1)A_{n+1} \geq \cdots \geq 0$, *then* f *is close-to-convex with respect to* $-\log(1-z)$.

  Lemma 1.7. [22] *Let* $f(z) = z + \sum_{k=2}^{\infty} A_{2k-1} z^{2k-1}$ *be analytic in* $\mathbb{D}$. *If* $1 \geq 3A_3 \geq \cdots \geq (2k-1)A_{2k-1} \geq \cdots \geq 0$ *or* $1 \leq 3A_3 \leq \cdots \leq (2k-1)A_{2k-1} \leq \cdots \leq 2$, *then* f *is univalent in* $\mathbb{D}$.

- Discuss the geometric properties of $\mathbb{E}_{\alpha,\beta}(z)$ as application and show that the results obtained in this paper are better and improve several results available in the literature.
  To do so, we will use numerical computation with the help of mathematical software.

2. Starlikness of normalized Le Roy-type Mittag-Leffler function

Theorem 2.1. *Let* α, β, γ *be positive real numbers such that* $\alpha\gamma \geq 1$ *and* $\alpha^2\gamma \geq \beta$ *and the following relation holds:*

$$\alpha\gamma \log(\alpha+\beta) - \log(2) - \frac{3}{4} - \frac{\alpha\gamma}{\alpha+\beta} > 0.$$

(i) *If* $2(e-1)[\Gamma(\beta)]^{\gamma} < [\Gamma(\alpha+\beta)]^{\gamma}$, *then the function* $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ *is starlike in* $\mathbb{D}$.

(ii) *If* $3(e-1)[\Gamma(\beta)]^{\gamma} < [\Gamma(\alpha+\beta)]^{\gamma}$, *then* $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z) \in \mathbb{S}_p$.

Proof. (i) By definition, to prove that the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is starlike in $\mathbb{D}$, it is enough to show that

$$\left| \left[z \left(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z) \right)' / \left(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z) \right) \right] - 1 \right| < 1,$$

for all $z \in \mathbb{D}$. Thus, we have

$$\begin{aligned} \left(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z) \right)' - \frac{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}{z} &= \sum_{k=1}^{\infty} \frac{k[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}} z^k \\ &= \sum_{k=1}^{\infty} \frac{b_k(\alpha,\beta,\gamma) z^k}{k!}, \end{aligned} \tag{2.1}$$

where $(b_k)_{k\geq}$ is defined by

$$b_k(\alpha,\beta,\gamma) = \frac{k\Gamma(k+1)[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}},\ k \geq 1. \tag{2.2}$$

We consider the function $f_1(s)$ defined by

$$f_1(s) = \frac{s\Gamma(s+1)[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha s+\beta)]^{\gamma}}, s \geq 1. \tag{2.3}$$

Therefore,

$$f_1'(s) = f_1(s) f_2(s), \tag{2.4}$$

where $f_2(s)$ is defined as

$$f_2(s) = \frac{1}{s} + \psi(s+1) - \alpha\gamma\psi(\alpha s+\beta), s \geq 1.$$

With the help of Lemma 1.1, we get

$$f_2(s) < f_3(s) := \log(s+1) - \alpha\gamma\log(\alpha s+\beta) + \frac{s+2}{2s(s+1)} + \frac{\alpha\gamma}{\alpha s+\beta}. \tag{2.5}$$

By differentiation, we get

$$f_3'(s) = \frac{\alpha(1-\alpha\gamma)s + (\beta-\alpha^2\gamma)}{(\alpha s+\beta)(s+1)} - \frac{s^2+4s+2}{2s^2(s+1)^2} - \frac{\alpha^2\gamma}{(\alpha s+\beta)^2}. \tag{2.6}$$

This implies that the function $f_3(s)$ is decreasing on $(0,\infty)$ if $\alpha\gamma \geq 1$ and $\alpha^2\gamma \geq \beta$. In addition, it can be verified that $f_3(1) < 0$, under the given conditions. Therefore, $f_3(s)$ is negative on $[1,\infty)$, which implies that $f_2(s) < 0$ for all $s \geq 1$. Consequently, the function $f_1(s)$ is decreasing on $[1,\infty)$. Hence, the sequence $(b_k)_{k\geq 1}$ is decreasing. Thus, in view of (2.1), we obtain

$$\begin{aligned} \left| \left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - \frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z} \right| &< \sum_{k=1}^{\infty} \frac{b_k(\alpha,\beta,\gamma)}{k!} \\ &\leq \sum_{k=1}^{\infty} \frac{b_1(\alpha,\beta,\gamma)}{k!} \\ &= b_1(\alpha,\beta,\gamma)(e-1). \end{aligned} \tag{2.7}$$

On the other hand, we have

$$\left| \frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z} \right| > 1 - \sum_{k=1}^{\infty} \frac{c_k(\alpha,\beta,\gamma)}{k!}, \quad z \in \mathbb{D}, \tag{2.8}$$

where the sequence $(c_k)_{k\geq 1}$ is defined by

$$c_k = \frac{\Gamma(k+1)[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha k+\beta)]^\gamma}, \; k \geq 1. \tag{2.9}$$

Since, the sequence $(b_k)_{k\geq 1}$ is decreasing, the sequence $(c_k)_{k\geq 1}$ is also decreasing. Then, by (2.8) we obtain

$$\left| \frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z} \right| > 1 - c_1(\alpha,\beta,\gamma)(e-1), \quad z \in \mathbb{D}. \tag{2.10}$$

In virtue of (2.7) and (2.10) we get

$$\left| \left[\frac{z\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'}{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)} \right] - 1 \right| < \frac{b_1(\alpha,\beta,\gamma)(e-1)}{1 - c_1(\alpha,\beta,\gamma)(e-1)}, \; z \in \mathbb{D}. \tag{2.11}$$

The above inequality needs to be less than 1, this gives the conditions $2[\Gamma(\beta)]^\gamma(e-1) < [\Gamma(\alpha+\beta)]^\gamma$. Hence,

$$\Re\left(\frac{z(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'}{(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)} \right) > 0$$

for all $z \in \mathbb{D}$. This implies that the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike on $\mathbb{D}$. The proof of part (i) of Theorem 2.1 is complete.

(ii) Proceeding similarly as part (i) and applying part (ii) of Lemma 1.5 in (2.11), part (ii) of Theorem 2.1 can be proved.

□

Corollary 2.2. *If* $\frac{-1+\sqrt{1+4\sqrt{2(e-1)}}}{2} < \beta \leq 8$, *then the function* $\mathbb{F}^{(2)}_{2,\beta}(z)$ *is starlike in* $\mathbb{D}$.

Proof. Letting $\alpha = \gamma = 2$ in Theorem 2.1, we get that the function $\mathbb{F}^{(2)}_{2,\beta}(z)$ is starlike in $\mathbb{D}$, if $\beta(\beta+1) - \sqrt{2(e-1)} > 0$ such that $0 < \beta \le 8$ and the function $f(\beta)$ defined by

$$f(\beta) = 4\log(2+\beta) - \log(2) - \frac{3}{4} - \frac{4}{\beta+2},$$

is positive. We observe that the function $f(\beta)$ is increasing on $(0,\infty)$ such that

$$f\left(\frac{-1+\sqrt{1+4\sqrt{2(e-1)}}}{2}\right) \approx 1.53 > 0.$$

This implies that the function $f(\beta) > 0$ if

$$\frac{-1+\sqrt{1+4\sqrt{2(e-1)}}}{2} < \beta \le 8,$$

which completes the proof. $\square$

Similarly, we can derive the following corollary.

Corollary 2.3. *If* $\frac{-1+\sqrt{1+4\sqrt{3(e-1)}}}{2} < \beta \le 8$, *then the function* $\mathbb{F}^{(2)}_{2,\beta}(z) \in \mathbb{S}_p$.

Example 2.4. (i) The function $\mathbb{F}^{(2)}_{2,\frac{20}{21}}(z)$ is starlike in $\mathbb{D}$.

(ii) $\mathbb{F}^{(2)}_{2,\frac{11}{10}}(z) \in \mathbb{S}_p$.

On setting $\gamma = 1$ in Theorem 2.1, we get the following result as follows:

Corollary 2.5. *Let α, β be positive real numbers such that $\alpha \ge 1$, $\alpha^2 \ge \beta$ and the following condition holds:*

$$\alpha\log(\alpha+\beta) - \log(2) - \frac{3}{2} - \frac{\alpha}{\alpha+\beta} > 0.$$

(i) *If $2(e-1)\Gamma(\beta) < \Gamma(\alpha+\beta)$, then the function $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}$.*
(ii) *If $3(e-1)\Gamma(\beta) < \Gamma(\alpha+\beta)$, then $\mathbb{E}_{\alpha,\beta}(z) \in \mathbb{S}_p$.*

Example 2.6. The function $\mathbb{E}_{2,\frac{19}{10}}(z)$ is starlike in $\mathbb{D}$ and $\mathbb{E}_{2,\frac{19}{10}}(z) \in \mathbb{S}_p$.

Remark 2.7. Using Corollary 2.5 and proceeding similarly as Corollary 2.2, we can compute that $\mathbb{E}_{2,\beta}(z)$ and $\mathbb{E}_{\alpha,\frac{1}{2}}(z)$ are starlike in $\mathbb{D}$ if $1.9 \le \beta \le 4$ and $\alpha \ge 3.52$ respectively. Furthermore, with the help of numerical computation, we observe that for any $\alpha \ge 2.55$, there exists $\beta \in (0,1)$ such that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}$. Hence, Corollary 2.5 can discusses the case when $0 < \beta < 1$. In [2, Theorem 2.2], it is proved that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}$ if $\alpha \ge 1$ and $\beta \ge \frac{(3+\sqrt{17})}{2} \approx 3.56155$. In [21, Theorem 2], it is shown that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}$ if $\alpha \ge 2.67$ and $\beta \ge 1$. Moreover, it is also proved in [21, Theorem 6] that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}$ if $\alpha \ge 1$ and $\beta \ge 3.214319744$. The results in [2, 21] discuss the starlikeness of $\mathbb{E}_{\alpha,\beta}(z)$ when $\alpha, \beta \ge 1$ but Corollary 2.5 can also consider the case when $0 < \beta < 1$ and also provide the sharper lower bound for β. Hence, Corollary 2.5 improves the results available in [2, Theorem 2.2] and [21, Theorem 2, Theorem 6].

Theorem 2.8. *Let $\alpha, \beta, \gamma > 0$ such that $\alpha\gamma \ge 1, \gamma \ge \frac{1}{2}, \beta \le \min(\alpha\sqrt{2\gamma}, \alpha^2\gamma)$ and $(e-1)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma$. Also, if*

$$\alpha\gamma\log(\alpha+\beta) - \log(2) + \frac{1}{4} - \frac{\alpha\gamma}{\alpha+\beta} > 0,$$

then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}_{1/2}$.

Proof. A simple computation gives

$$\left|\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))/z-1\right| < \sum_{k=1}^{\infty}\frac{c_k(\alpha,\beta,\gamma)}{k!},\ z\in\mathbb{D}, \tag{2.12}$$

where $(c_k)_{k\geq 1}$ is defined in (2.9). Now, we define the function $g_1(s)$ by

$$g_1(s)=\frac{\Gamma(s+1)}{[\Gamma(\alpha s+\beta)]^{\gamma}}, s\geq 1.$$

Then, we have

$$g_1'(s)=g_1(s)g_2(s), \tag{2.13}$$

where

$$g_2(s)=\psi(s+1)-\alpha\gamma\psi(\alpha s+\beta).$$

Again, applying Lemma 1.1, we get

$$g_2(s)<g_3(s):=\log(s+1)-\alpha\gamma\log(\alpha s+\beta)-\frac{1}{2(s+1)}+\frac{\alpha\gamma}{\alpha s+\beta}. \tag{2.14}$$

Therefore,

$$g_3'(s)=\frac{\alpha(1-\alpha\gamma)s+\beta-\alpha^2\gamma}{(s+1)(\alpha s+\beta)}+\frac{[\alpha(1-\sqrt{2\gamma})s+\beta-\sqrt{2\gamma}\alpha]\,[\alpha(1+\sqrt{2\gamma})s+\beta+\sqrt{2\gamma}\alpha]}{2(s+1)^2(\alpha s+\beta)^2}<0,$$

under the given hypotheses. This implies that the function $g_3(s)$ is decreasing on $[1,\infty)$ with $g_3(1)<0$. So, $g_3(s)<0$ for all $s\geq 1$. Consequently, the function $g_1(s)$ is decreasing with the aid of (2.13) and (2.14). Hence, the sequence $(c_k)_{k\geq 1}$ is decreasing. Therefore, using (2.12), we obtain

$$\left|\frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z}-1\right|<c_1(e-1)<1,\ z\in\mathbb{D},$$

under the given conditions of Theorem 2.8. Finally, using Lemma 1.3, the desired result can be obtained. □

Corollary 2.9. *If $\sqrt{e-1}<\beta\leq 2$, then the function $\mathbb{F}^{(2)}_{1,\beta}$ is starlike in $\mathbb{D}_{1/2}$.*

Proof. We set $\alpha=1$ and $\gamma=2$ in Theorem 2.8, we deduce that the function $\mathbb{F}^{(2)}_{1,\beta}$ is starlike in $\mathbb{D}_{1/2}$ if $\sqrt{e-1}<\beta\leq 2$ and the function $g(\beta)$ defined by

$$g(\beta)=2\log(1+\beta)-\log(2)+\frac{1}{4}-\frac{2}{\beta+1},$$

is positive. Since the function $g(\beta)$ is increasing on $(0,\infty)$ and $g(\sqrt{e-1})\approx 0.37>0$, we get the desired result. □

Example 2.10. The function $\mathbb{F}^{(2)}_{1,3/2}$ is starlike in $\mathbb{D}_{1/2}$.

Upon setting $\gamma=1$ in Theorem (2.8), we establish the following result:

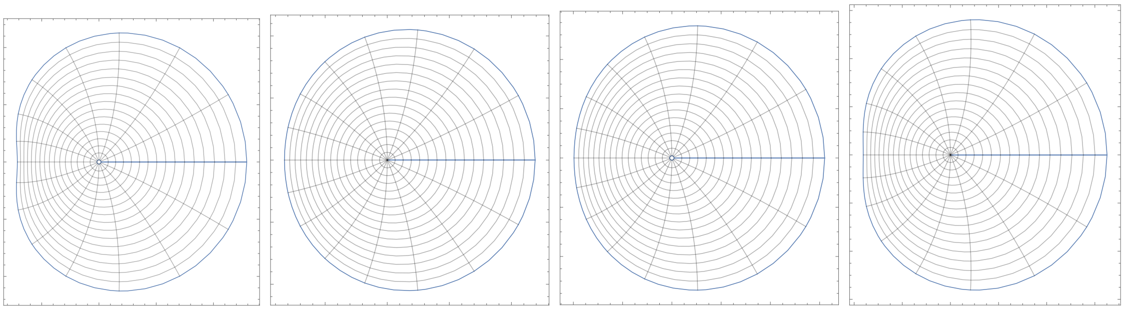
Corollary 2.11. *Let $\alpha,\beta>0$ such that $\alpha\geq 1,\beta\leq\min(\alpha\sqrt{2},\alpha^2)$ and $(e-1)\Gamma(\beta)<\Gamma(\alpha+\beta)$. Also, if*

$$\alpha\log(\alpha+\beta)-\log(2)+\frac{1}{4}-\frac{\alpha}{\alpha+\beta}>0,$$

then the function $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}_{1/2}$.

Example 2.12. The function $\mathbb{E}_{2,\frac{10}{11}}(z)$ is starlike in $\mathbb{D}_{1/2}$.

Remark 2.13. Using Corollary 2.11 and following similar techniques as Corollary 2.9, we can verify that $\mathbb{E}_{2,\beta}(z)$ is starlike in $\mathbb{D}_{1/2}$ if $\frac{10}{11} \approx 0.909091 \le \beta \le 2\sqrt{2}$. Moreover, using numerical computation we can verify that for any $\alpha \ge 2.4$, there exists $\beta \in (0,1)$ such that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}_{1/2}$. In [2, Theorem 2.4], it is shown that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}_{1/2}$ if $\alpha \ge 1$ and $\beta \ge \frac{(1+\sqrt{5})}{2} \approx 1.61803$. Moreover, the results proved in [2, Theorem 2.4] consider the case $\alpha, \beta \ge 1$. But Corollary 2.11 discusses the case for $0 < \beta < 1$ and also provides sharper lower bound for β. Hence, Corollary 2.11 improves the results available in [2, Theorem 2.4].



(a) Mapping of $\mathbb{F}^{(2)}_{2,\frac{20}{21}}(z)$ over $\mathbb{D}$ (b) Mapping of $\mathbb{E}_{2,\frac{19}{10}}(z)$ over $\mathbb{D}$ (c) Mapping of $\mathbb{F}^{(2)}_{1,\frac{3}{2}}(z)$ over $\mathbb{D}_{1/2}$ (d) Mapping of $\mathbb{E}_{2,\frac{10}{11}}(z)$ over $\mathbb{D}_{1/2}$

Figure 1. Mapping $\mathbb{F}^{(2)}_{\alpha,\beta}(z)$ and $\mathbb{E}_{\alpha,\beta}(z)$ over $\mathbb{D}$ and $\mathbb{D}_{1/2}$.

3. Convexity of normalized Le Roy-type Mittag-Leffler function

Theorem 3.1. *Let $\alpha, \beta, \gamma > 0$ be such that $\alpha\gamma \ge 1$ and $2\alpha^2\gamma \ge \beta$. If*

$$\alpha\gamma \log(\alpha+\beta) - \log(3) - \frac{5}{6} - \frac{\alpha\gamma}{\alpha+\beta} > 0 \text{ and } 4(e-1)[\Gamma(\beta)]^{\gamma} < [\Gamma(\alpha+\beta)]^{\gamma},$$

then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is convex in $\mathbb{D}$.

Proof. It is well known that a function $f(z)$ is convex in $\mathbb{D}$ if and only if $zf'(z)$ is starlike in $\mathbb{D}$. So, in order to prove $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is convex it is sufficient to prove that the function

$$\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z) := z\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'$$

is starlike in $\mathbb{D}$. Therefore,

$$\left(\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)\right)' - \mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)/z = \sum_{k=1}^{\infty} \frac{d_k(\alpha,\beta,\gamma) z^k}{k!}, \tag{3.1}$$

where $(d_k)_{k\ge1}$ is defined by

$$d_k := d_k(\alpha,\beta,\gamma) = \frac{[\Gamma(\beta)]^{\gamma} k\Gamma(k+2)}{[\Gamma(\alpha k+\beta)]^{\gamma}},\ k \ge 1.$$

Let us define the function $h_1(s)$ by

$$h_1(s) = \frac{s\Gamma(s+2)}{[\Gamma(\alpha s+\beta)]^{\gamma}}.$$

Then, we have

$$h_1'(s) = h_1(s)h_2(s), s \ge 1,$$

where

$$h_2(s) = \frac{1}{s} + \psi(s+2) - \alpha\gamma\psi(\alpha s+\beta).$$

Using Lemma 1.1, we have

$$h_2(s) < h_3(s) := \log(s+2) - \alpha\gamma\log(\alpha s+\beta) + \frac{s+4}{2s(s+2)} + \frac{\alpha\gamma}{\alpha s+\beta}.$$

Hence,

$$h_3'(s) = \frac{\alpha(1-\alpha\gamma)s+\beta-2\alpha^2\gamma}{(s+2)(\alpha s+\beta)} + \frac{s(s+2)-2(s+1)(s+4)}{2s^2(s+2)^2} - \frac{\alpha^2\gamma}{(\alpha s +_b eta)^2} < 0$$

for all $s \geq 1, \alpha\gamma \geq 1$ and $2\alpha^2\beta \geq \beta$. This implies that the function $h_3(s)$ is decreasing on $[1,\infty)$. As $h_3(1) < 0$, we deduce that the function $h_1(s)$ is decraesing on $[1,\infty)$ and consequently, the sequence $(d_k)_{k\geq 1}$ is decreasing. Then for all $z \in \mathbb{D}$ we get

$$\left|\left(\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)\right)' - \mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)/z\right| < \sum_{k=1}^{\infty} \frac{d_1(\alpha,\beta,\gamma)z^k}{k!} = d_1(e-1). \tag{3.2}$$

However, for $z \in \mathbb{D}$, we obtain

$$\left|\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)/z\right| = \left|\left(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)\right)'\right| \geq 1 - \sum_{k=1}^{\infty} \frac{e_k(\alpha,\beta,\gamma)z^k}{k!}, \tag{3.3}$$

where $(e_k)_{k\geq 1}$ is defined by

$$e_k := e_k(\alpha,\beta,\gamma) = \frac{d_k(\alpha,\beta,\gamma)}{k}, k \geq 1. \tag{3.4}$$

We observe that the sequence $(e_k)_{k\geq 1}$ is decreasing because $(d_k)_{k\geq 1}$ is decreasing. Therefore, using (4.2) we have

$$\left|\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)/z\right| > 1 - e_1(\alpha,\beta,\gamma)(e-1). \tag{3.5}$$

In view of (3.2) and (3.5), we have

$$\left|\frac{z\left(\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)\right)'}{\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)} - 1\right| \leq \frac{d_1(\alpha,\beta,\gamma)(e-1)}{1-e_1(\alpha,\beta,\gamma)(e-1)} \leq 1. \tag{3.6}$$

This shows that the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is convex in $\mathbb{D}$. □

Corollary 3.2. *If* $(-1+\sqrt{1+8\sqrt{e-1}})/2 < \beta \leq 16$, *then the function* $\mathbb{F}_{2,\beta}^{(2)}(z)$ *is convex in* $\mathbb{D}$.

Proof. Setting $\alpha = \gamma = 2$ in Theorem 3.1, we observe that the function $\mathbb{F}_{2,\beta}^{(2)}(z)$ is convex in $\mathbb{D}$, if $\beta \leq 16, \beta^2+\beta-2\sqrt{e-1} > 0$ and the function $h(\beta)$ defined by

$$h(\beta) = 4\log(2+\beta) - \log(3) - \frac{5}{6} - \frac{4}{2+\beta},$$

is positive. The condition $\beta^2+\beta-2\sqrt{e-1} > 0$ holds true if $\beta \geq \beta_1 := \frac{-1+\sqrt{1+8\sqrt{e-1}}}{2}$. In addition, we see that the function $h(\beta)$ is increasing on $(0,\infty)$ such that $h(\beta_1) > 0$. This completes the proof of Corollary 3.2. □

Example 3.3. The function $\mathbb{F}_{2,\frac{6}{5}}^{(2)}(z)$ is convex in $\mathbb{D}$.

Taking $\gamma = 1$ in Theorem 3.1, we obtain the following result:

Corollary 3.4. *Let $\alpha, \beta > 0$ such that $\alpha \geq 1$ and $2\alpha^2 \geq \beta$. Also, if*

$$\alpha \log(\alpha+\beta) - \log(3) - \frac{5}{6} - \frac{\alpha}{\alpha+\beta} > 0 \text{ and } 4(e-1)\Gamma(\beta) < \Gamma(\alpha+\beta),$$

then the function $\mathbb{E}_{\alpha,\beta}(z)$ is convex in $\mathbb{D}$.

Setting $\alpha = 2$ in Corollary 3.4, we get the following result:

Corollary 3.5. *If $(-1+\sqrt{16e-15})/2 < \beta \leq 8$, then the function $\mathbb{E}_{2,\beta}(z)$ is convex in $\mathbb{D}$.*

Example 3.6. The function $\mathbb{E}_{2,\frac{9}{4}}(z)$ is convex in $\mathbb{D}$.

Remark 3.7. In [21, Theorem 7], it is proved that $\mathbb{E}_{\alpha,\beta}(z)$ is convex in $\mathbb{D}$ if $\alpha \geq 1$ and $\beta \geq 3.56155281$. From Corollary 3.5, we can verify that $\mathbb{E}_{2,\beta}(z)$ is convex in $\mathbb{D}$ if $2.16893 < \beta \leq 8$. Further, setting $\beta = 1/2$ in Corollary 3.5, we can see that $\mathbb{E}_{\alpha,\frac{1}{2}}(z)$ is convex in $\mathbb{D}$ if $\alpha \geq 4.04$. Moreover, numerical computation shows that for any $\alpha \geq 3.2$, there exists $\beta \in (0,1)$ such that $\mathbb{E}_{\alpha,\beta}(z)$ is convex in $\mathbb{D}$. Therefore, Corollary 3.5 can also discuss the case when $0 < \beta < 1$. Hence, Corollary 3.5 improves the results in [21, Theorem 7].

Theorem 3.8. *Let $\alpha, \beta > 0, \gamma \geq \frac{1}{2}$ such that $\alpha\gamma \geq 1$ and $\beta \leq \min(2\alpha^2\gamma, 2\sqrt{2\gamma}\alpha)$ and the following condition holds:*

$$\alpha\gamma \log(\alpha+\beta) - \log(3) + \frac{1}{6} - \frac{\alpha\gamma}{\alpha+\beta} > 0.$$

(i) *If $2[\Gamma(\beta)]^\gamma(e-1) < [\Gamma(\alpha+\beta)]^\gamma$, then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is convex in $\mathbb{D}_{1/2}$.*

(ii) *If $\sqrt{5}[\Gamma(\beta)]^\gamma(e-1) < [\Gamma(\alpha+\beta)]^\gamma$, then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}$.*

Proof. Direct computation gives

$$\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1 = \sum_{k=1}^{\infty} \frac{e_k(\alpha,\beta,\gamma)z^k}{k!}, \tag{3.7}$$

where $(e_k)_{k\geq 1}$ is defined in (3.4). We define the function $I_1(s)$ by

$$I_1(s) = \frac{\Gamma(s+2)}{[\Gamma(\alpha s+\beta)]^\gamma}, s \geq 1.$$

Then, we get

$$I_1'(s) = I_1(s)I_2(s) =: I_1(s)[\psi(s+2) - \alpha\gamma\psi(\alpha s+\beta)].$$

Hence, by Lemma 1.1, we have

$$I_2(s) < I_3(s) := \log(s+2) - \alpha\gamma\log(\alpha s+\beta) - \frac{1}{2(s+2)} + \frac{\alpha\gamma}{\alpha s+\beta}.$$

Then

$$I_3'(s) = \frac{\alpha(1-\alpha\gamma)s + \beta - 2\alpha^2\gamma}{(s+2)(\alpha s+\beta)} + \frac{(\alpha s+\beta)^2 - 2\alpha^2\gamma(s+2)^2}{2(s+2)^2(\alpha s+\beta)^2}.$$

It follows that the function $I_3(s)$ is decreasing on $[1,\infty)$ such that $I_3(1) < 0$. This implies that the function $I_1(s)$ is decreasing on $[1,\infty)$ and consequently the sequence $(e_k)_{k\geq 1}$ is decreasing. Thus, for all $z \in \mathbb{D}$, we obtain

$$\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1\right| < e_1(\alpha,\beta,\gamma)(e-1). \tag{3.8}$$

Under the given condition (i), we have

$$\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1\right| < 1.$$

Now, using Lemma 1.4, part (i) of this theorem can be proved.

Again, under the given condition (ii), we get

$$\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1\right| < \frac{2}{\sqrt{5}}.$$

Finally, using Lemma 1.2, the desired result can be obtained. □

Corollary 3.9. *If $\sqrt{2(e-1)} < \beta \leq 4$, then the function $\mathbb{F}^{(2)}_{1,\beta}(z)$ is convex in $\mathbb{D}_{1/2}$.*

Proof. Setting $\alpha = 1$ and $\gamma = 2$ in part (i) of Theorem 3.8, we observe that $\mathbb{F}^{(2)}_{1,\beta}(z)$ is convex in $\mathbb{D}_{1/2}$ if $\sqrt{2(e-1)} < \beta \leq 4$ and $T(\beta) := 2\log(\beta+1) - \log 3 + \frac{1}{6} - \frac{2}{\beta+1} > 0$.

It can be easily seen that $T(\beta)$ is increasing on $(0,\infty)$ and $T\left(\sqrt{2(e-1)}\right) > 0$, which completes the proof of the corollary. □

Example 3.10. The function $\mathbb{F}^{(2)}_{1,\frac{19}{10}}(z)$ is convex in $\mathbb{D}_{1/2}$.

If we set $\gamma = 1$ in Theorem 3.8, we conclude the following result:

Corollary 3.11. *Let $\alpha, \beta > 0$ such that $\alpha \geq 1$, $\beta \leq \min(2\alpha^2, 2\sqrt{2}\alpha)$ and following condition holds:*

$$\alpha\log(\alpha+\beta) - \log(3) + \frac{1}{6} - \frac{\alpha}{\alpha+\beta} > 0.$$

(i) *If $2\Gamma(\beta)(e-1) < \Gamma(\alpha+\beta)$, then the function $\mathbb{E}_{\alpha,\beta}(z)$ is convex in $\mathbb{D}_{1/2}$.*
(ii) *If $\sqrt{5}\Gamma(\beta)(e-1) < \Gamma(\alpha+\beta)$, then the function $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}$.*

Corollary 3.12. *If $(-1+\sqrt{8e-7})/2 < \beta \leq 4\sqrt{2}$, then the function $\mathbb{E}_{2,\beta}(z)$ is convex in $\mathbb{D}_{1/2}$.*

Proof. Setting $\alpha = 2$ in part (i) of Corollary 3.11, we compute that the function $\mathbb{E}_{2,\beta}(z)$ is convex in $\mathbb{D}_{1/2}$ if $(-1+\sqrt{8e-7})/2 < \beta \leq 4\sqrt{2}$ and $L(\beta) := 2\log(\beta+2) - \log 3 + \frac{1}{6} - \frac{2}{\beta+2} > 0$.

A simple computation shows that $L(\beta)$ is increasing on $(0,\infty)$ and $L\left((-1+\sqrt{8e-7})/2\right) > 0$, which completes the proof of this corollary. □

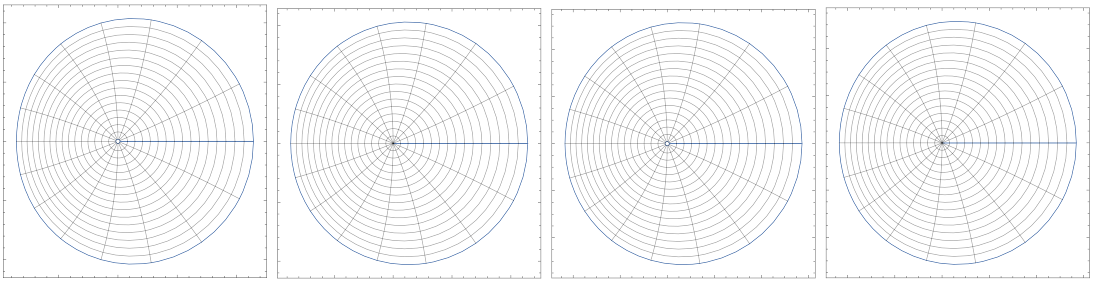
Similarly, we can prove the following corollary.

Corollary 3.13. (i) *If $\left(-1+\sqrt{1+4\sqrt{5}(e-1)}\right)/2 < \beta \leq 4\sqrt{2}$, then the function $\mathbb{E}_{2,\beta}(z)$ is starlike in $\mathbb{D}$.*
(ii) *If $0.78 \leq \beta \leq 6\sqrt{2}$, then the function $\mathbb{E}_{3,\beta}(z)$ is starlike in $\mathbb{D}$.*
(iii) *If $0.73 \leq \beta \leq 6\sqrt{2}$, then the function $\mathbb{E}_{3,\beta}(z)$ is convex in $\mathbb{D}_{1/2}$.*

Example 3.14. (i) The function $\mathbb{E}_{2,\frac{3}{2}}(z)$ is convex in $\mathbb{D}_{1/2}$.
(ii) The function $\mathbb{E}_{2,\frac{8}{5}}(z)$ is starlike in $\mathbb{D}$.

Remark 3.15. In [2, Theorem 2.4], it is shown that $\mathbb{E}_{\alpha,\beta}(z)$ is convex in $\mathbb{D}_{1/2}$ if $\alpha \geq 1$ and $\beta \geq \frac{(3+\sqrt{17})}{2} \approx 3.56155$. Moreover, In [2, Example 2.1], it is proved that $\mathbb{E}_{2,\beta_2}$ and $\mathbb{E}_{3,\beta_3}$ are starlike in $\mathbb{D}$ if $\beta_2 \geq \frac{(-1+\sqrt{17})}{2} \approx 1.56155$ and $\beta_3 \geq 1$ respectively. In [21, Theorem 2], it is shown that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}$ if $\alpha \geq 2.67$ and $\beta \geq 1$. Moreover, it is also proved in [21, Theorem 6] that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathbb{D}$ if $\alpha \geq 1$ and $\beta \geq 3.214319744$. Also, starlikeness of $\mathbb{E}_{\alpha,\beta}(z)$ in $\mathbb{D}$ is discussed in [2, 21] for the case $\alpha \geq 1$ and $\beta \geq 1$. From Corollary 3.12 and Corollary 3.13, we see that $\mathbb{E}_{2,\beta_2}(z)$ and $\mathbb{E}_{3,\beta_3}(z)$ are convex in $\mathbb{D}_{1/2}$ if $1.43 \leq \beta_2 \leq 4\sqrt{2}$ and $0.73 \leq \beta_3 \leq 6\sqrt{2}$ respectively; and starlike in $\mathbb{D}$ if $1.53 \leq \beta_2 \leq 4\sqrt{2}$ and $0.78 \leq \beta_3 \leq 6\sqrt{2}$ respectively. Furthermore, setting $\beta = \frac{1}{2}$ in Corollary 3.11, we observe that $\mathbb{E}_{\alpha,\frac{1}{2}}(z)$ is convex in $\mathbb{D}_{1/2}$ if $\alpha \geq 3.55$. Moreover, using numerical computation, we can show that for any $\alpha_1 \geq 2.55$ and $\alpha_2 \geq 2.66$, there exist

$\beta_1, \beta_2 \in (0,1)$ such that $\mathbb{E}_{\alpha_1,\beta_1}(z)$ is convex in $\mathbb{D}_{1/2}$ and $\mathbb{E}_{\alpha_2,\beta_2}(z)$ is starlike in $\mathbb{D}$. Therefore, Corollary 3.11 can discusses the case when $0<\beta<1$ and provides sharper lower bound for β. Hence, Corollary 3.11 improves the results available in [2, Theorem 2.1, Theorem 2.2, Theorem 2.4] and [21, Theorem 2, Theorem 6].



(a) Mapping of $\mathbb{F}^{(2)}_{2,\frac{6}{5}}(z)$ over $\mathbb{D}$ (b) Mapping of $\mathbb{E}_{2,\frac{9}{4}}(z)$ over $\mathbb{D}$ (c) Mapping of $\mathbb{F}^{(2)}_{1,\frac{19}{10}}(z)$ over $\mathbb{D}_{1/2}$ (d) Mapping of $\mathbb{E}_{2,\frac{3}{2}}(z)$ over $\mathbb{D}_{1/2}$

Figure 2. Mapping $\mathbb{F}^{(2)}_{\alpha,\beta}(z)$ and $\mathbb{E}_{\alpha,\beta}(z)$ over $\mathbb{D}$.

Theorem 3.16. *Let $\alpha,\beta>0, \gamma\geq \frac{1}{2}$ be such that $\alpha\gamma\geq 1$ and $\beta \leq \min(2\alpha^2\gamma, 2\sqrt{2\gamma}\alpha)$. In addition, if*

$$\alpha\gamma\log(\alpha+\beta)-\log(3)-\frac{5}{6}-\frac{\alpha\gamma}{\alpha+\beta}>0 \ \text{ and } \ 3(e-1)[\Gamma(\beta)]^{\gamma}<[\Gamma(\alpha+\beta)]^{\gamma},$$

then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is uniformly convex in $\mathbb{D}$.

Proof. Simple computation gives

$$z\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)''=\sum_{k=1}^{\infty}\frac{x_k(\alpha,\beta,\gamma)z^k}{k!}, \tag{3.9}$$

where the sequence $(x_k)_{k\geq 0}$ is defined by

$$x_k=\frac{k[\Gamma(\beta)]^{\gamma}\Gamma(k+2)}{[\Gamma(\alpha k+\beta)]^{\gamma}},\ k\geq 1.$$

We define the function $J_1(s)$ by

$$J_1(s)=\frac{s\Gamma(s+2)}{[\Gamma(\alpha s+\beta)]^{\gamma}},\ s\geq 1.$$

Therefore, we have

$$J_1'(s)=J_1(s)\left[\frac{1}{s}+\psi(s+2)-\alpha\gamma\psi(\alpha s+\beta)\right]:=J_1(s)J_2(s).$$

In view of Lemma 1.1, we obtain

$$J_2(s)<J_3(s):=\log(s+2)-\alpha\gamma\log(\alpha s+\beta)+\frac{1}{s}-\frac{1}{2(s+2)}+\frac{\alpha\gamma}{\alpha s+\beta}.$$

By differentiation, we get

$$J_3'(s)=\frac{\alpha(1-\alpha\gamma)s+\beta-2\alpha^2\gamma}{(s+2)(\alpha s+\beta)}+\frac{(\alpha s+\beta)^2-2\alpha^2\gamma(s+2)^2}{2(s+2)^2(\alpha s+\beta)^2}-\frac{1}{s^2}.$$

This leads to

$$J_3'(s)\leq 0 \ \text{ and } \ J_3(s)\leq J_3(1)<0,$$

which implies that the function $J_3(s)$ is decreasing on $[1,\infty)$, and consequently the sequence $(x_k)_{k\geq 1}$ is decreasing. Hence, for $z\in\mathbb{D}$, we have

$$\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)''\right| < \sum_{k=1}^{\infty}\frac{x_1(\alpha,\beta,\gamma)}{k!} = (e-1)x_1(\alpha,\beta,\gamma). \tag{3.10}$$

Moreover, for all $z\in\mathbb{D}$, we have

$$\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' > 1-\sum_{k=1}^{\infty}\frac{y_k(\alpha,\beta,\gamma)}{k!}, \tag{3.11}$$

where $y_k = x_k/k, k\geq 1$. Since the sequence $(x_k)_{k\geq 1}$, we conclude that the sequences $(y_k)_{k\geq 1}$ is decreasing. Then, by (3.11) we have

$$(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))' > 1-y_1(\alpha,\beta,\gamma)(e-1),\ z\in\mathbb{D}. \tag{3.12}$$

In view of (3.10) and (3.12), we obtain

$$\left|\frac{z\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)''}{\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'}\right| < \frac{(e-1)x_1(\alpha,\beta,\gamma)}{1-(e-1)y_1(\alpha,\beta,\gamma)},$$

and this is less or equal $\frac{1}{2}$ under the given hypotheses. By means of Lemma 1.5, we deduce that the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is uniformly convex in $\mathbb{D}$. □

Corollary 3.17. *If $\frac{5}{2}\leq\beta\leq 4$, then the function $\mathbb{F}^{(2)}_{1,\beta}(z)$ is uniformly convex in $\mathbb{D}$.*

Proof. Setting $\alpha=1$ and $\gamma=2$ in Theorem 3.16, we observe that $\mathbb{F}^{(2)}_{1,\beta}(z)$ is uniformly convex in $\mathbb{D}$ if $\sqrt{3(e-1)}<\beta\leq 4$ and $p(\beta)=2\log(\beta+1)-\log 3-\frac{5}{6}-\frac{2}{\beta+1}>0$. It can be easily shown that $p(\beta)$ is increasing in $(0,\infty)$ and $p(5/2)>0$. But $p(\sqrt{3(e-1)})<0$ and $\sqrt{3(e-1)}<5/2$, which leads to the required result. □

Example 3.18. The function $\mathbb{F}^{(2)}_{1,\frac{5}{2}}(z)$ is uniformly convex in $\mathbb{D}$.

Upon setting $\gamma=1$ in Theorem 3.16, we compute the following result:

Corollary 3.19. *Let $\alpha,\beta>0$ such that $\alpha\geq 1$ and $\beta\leq\min(2\alpha^2,2\sqrt{2}\alpha)$. In addition, if*

$$\alpha\log(\alpha+\beta)-\log(3)-\frac{5}{6}-\frac{\alpha}{\alpha+\beta}>0\ \text{ and }\ 3(e-1)\Gamma(\beta)<\Gamma(\alpha+\beta),$$

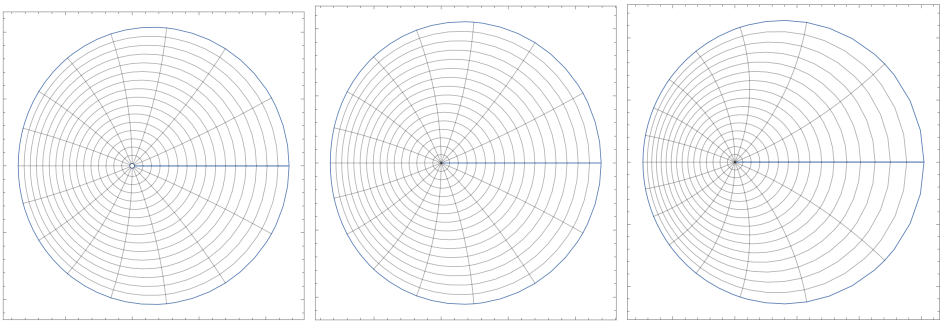
then the function $\mathbb{E}_{\alpha,\beta}(z)$ is uniformly convex in $\mathbb{D}$.

Corollary 3.20. *If $\beta\in\left[(-1+\sqrt{12e-11})/2,4\sqrt{2}\right]$, then the function $\mathbb{E}_{2,\beta}(z)$ is uniformly convex in $\mathbb{D}$.*

Proof. Assuming $\alpha=2$ in Corollary 3.19, we see that $\mathbb{E}_{2,\beta}(z)$ is uniformly convex in $\mathbb{D}$ if $(-1+\sqrt{12e-11})/2\leq\beta\leq 4\sqrt{2}$ and $q(\beta)=2\log(2+\beta)-\log(3)-\frac{5}{6}-\frac{2}{\beta+2}>0$. A simple computation shows that $q(\beta)$ is increasing in $(0,\infty)$ with $q((-1+\sqrt{12e-11})/2)>0$, which yields the required result. □

Example 3.21. The function $\mathbb{E}_{2,\frac{17}{9}}(z)$ is uniformly convex in $\mathbb{D}$.

Remark 3.22. In [20, Theorem 2.6], it is proved that $\mathbb{E}_{\alpha,\beta}(z)$ uniformly convex in $\mathbb{D}$ if $\alpha\geq 1$ and $\beta\geq 9.1112597744$. From Corollary 3.19, we see that $\mathbb{E}_{2,\beta}(z)$ is uniformly convex in $\mathbb{D}$ if $\frac{17}{9}\leq\beta\leq 4$. Moreover, setting $\beta=\frac{1}{2}$ in Corollary 3.19, we observe that $\mathbb{E}_{\alpha,\frac{1}{2}}(z)$ uniformly convex in $\mathbb{D}$ if $\alpha\geq 3.88$. Numerical computation shows that for any $\alpha\geq 2.9$, there exists $\beta\in(0,1)$ such that $\mathbb{E}_{\alpha,\beta}(z)$ uniformly convex in $\mathbb{D}$. Therefore, Corollary 3.19 can also discusses the case $0<\beta<1$. Hence, Corollary 3.19 improves the result available in [20, Theorem 2.6].



(a) Mapping of $\mathbb{F}^{(2)}_{1,\frac{5}{2}}(z)$ over $\mathbb{D}$ (b) Mapping of $\mathbb{E}_{2,\frac{17}{9}}(z)$ over $\mathbb{D}$ (c) Mapping of $\mathbb{E}_{\frac{1}{2},\frac{17}{2}}(z)$ over $\mathbb{D}$

Figure 3. Mapping $\mathbb{F}^{(2)}_{\alpha,\beta}(z)$ and $\mathbb{E}_{\alpha,\beta}(z)$ over $\mathbb{D}$.

4. Close-to-convexity of normalized Le Roy-type Mittag-Leffler function with respect to certain starlike functions

Theorem 4.1. *Suppose that α, β and γ are positive real numbers such that the following condition holds:*

$$\alpha\gamma\log(\beta) - \frac{\alpha\gamma}{\beta} - 1 > 0.$$

Then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is close-to-convex with respect to the function $-\log(1-z)$ in $\mathbb{D}$.

Proof. We will use Lemma 1.6 to prove this theorem. To show that $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is close-to-convex with respect to the function $-\log(1-z)$ in $\mathbb{D}$, it is sufficient to prove that the sequence $\{ka_k\}_{k\geq 1}$ is decreasing, where $(a_k)_{k\geq 1}$ is defined in (1.3).

Now, we define the function $K_1(s)$ by

$$K_1(s) = \frac{s}{[\Gamma(\alpha s + \beta - \alpha)]^{\gamma}}, s \geq 1.$$

Differentiation gives

$$K_1'(s) = K_1(s)K_2(s) := K_1(s)\left[\frac{1}{s} - \alpha\gamma\psi(\alpha s + \beta - \alpha)\right].$$

In view of Lemma 1.1, we obtain

$$K_2(s) < K_3(s) := \frac{1}{s} + \frac{\alpha\gamma}{(\alpha s + \beta - \alpha)} - \alpha\gamma\log(\alpha s + \beta - \alpha).$$

Thus, we have

$$K_3'(s) = -\frac{1}{s^2} - \frac{\alpha^2\gamma}{(\alpha s + \beta - \alpha)^2}\left\{\alpha(s-1) + (\beta+1)\right\}.$$

We observe that the function $K_3(s)$ is decreasing on $[1,\infty)$ for any $s \geq 1$, under the given condition. According to the given hypothesis $K_3(1) < 0$, which yields $K_3(s) < 0$ for all $s \geq 1$. This implies that the function $K_1(s)$ is deceasing on $[1,\infty)$ for any $s \geq 1$. Consequently, the sequence $(ka_k)_{k\geq 1}$ is decreasing. Hence, Lemma 1.6 completes the proof of Theorem 4.1. □

Corollary 4.2. *Let $\alpha, \beta > 0$. If $\alpha\gamma = 1$ and $\beta \geq \frac{18}{5}$, then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is close-to-convex with respect to the function $-\log(1-z)$ in $\mathbb{D}$.*

Corollary 4.3. *Let $\alpha, \beta > 0$. If $\beta \geq \frac{5}{2}$, then the function $\mathbb{F}^{(2)}_{1,\beta}(z)$ is close-to-convex with respect to the function $-\log(1-z)$ in $\mathbb{D}$.*

Setting $\gamma = 1$ in Theorem 4.1, we obtain the following result.

Corollary 4.4. *Let α and β be positive real numbers such that $\beta > e^{\frac{\alpha+\beta}{\alpha\beta}}$. Then the function $\mathbb{E}_{\alpha,\beta}(z)$ is close-to-convex with respect to the function $-\log(1-z)$ in $\mathbb{D}$.*

Example 4.5. The function $\mathbb{E}_{\frac{1}{2},\frac{17}{2}}(z)$ is close-to-convex with respect to the function $-\log(1-z)$ in $\mathbb{D}$.

Remark 4.6. In [21, Theorem 4], it is proved that $\mathbb{E}_{\alpha,\beta}(z)$ is close-to-convex with respect to the function $-\log(1-z)$ if $\alpha \geq 1$ and $\beta \geq 1$. Using Corollary 4.4, with the help of numerical computation, we can show that for any $\beta \geq 3.6$, there exists $\alpha \in (0,1)$ such that $\mathbb{E}_{\alpha,\beta}(z)$ is close-to-convex with respect to the function $-\log(1-z)$. Hence, Corollary 4.4 consider the case $0 < \alpha < 1$ and consequently, improves the results in [21, Theorem 4].

Theorem 4.7. *Assume that the hypotheses of Theorem 2.1 are valid. Further, suppose that $\alpha \geq 1, \beta^\gamma > 2(e-1)$ and $\beta > 1-x^* \approx 0.55$, where x^* is the abscissa of the minimum of the Gamma function. Then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is close-to-convex with respect to starlike function $\mathbb{F}^{(\gamma)}_{1,\beta}(z)$ in $\mathbb{D}$.*

Proof. From Theorem 2.1, we observe that the function $\mathbb{F}^{(\gamma)}_{1,\beta}(z)$ is starlike in $\mathbb{D}$. Then, from the definition, we need to show that

$$\Re\left(\left[z\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\right]\Big/\left(\mathbb{F}^{(\gamma)}_{1,\beta}(z)\right)\right) > 0, \text{ for all } z \in \mathbb{D},$$

which is equivalent to

$$\left|\left[z\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\right]\Big/\left(\mathbb{F}^{(\gamma)}_{1,\beta}(z)\right) - 1\right| < 1, \text{ for all } z \in \mathbb{D}.$$

In view of (2.10), we have

$$\left|\frac{\mathbb{F}^{(\gamma)}_{1,\beta}(z)}{z}\right| > \beta^{-\gamma}(\beta^\gamma - (e-1)). \tag{4.1}$$

Moreover, we have

$$\begin{aligned}\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - \frac{\mathbb{F}^{(\gamma)}_{1,\beta}(z)}{z}\right| &< \sum_{k=1}^{\infty}[\Gamma(\beta)]^\gamma\left|\frac{k+1}{[\Gamma(\alpha k+\beta)]^\gamma} - \frac{1}{[\Gamma(k+\beta)]^\gamma}\right| \\ &\leq \sum_{k=1}^{\infty}\frac{k[\Gamma(\beta)]^\gamma}{[\Gamma(k+\beta)]^\gamma} \\ &= \sum_{k=1}^{\infty}\frac{b_k(1,\beta,\gamma)}{k!},\end{aligned} \tag{4.2}$$

where the sequence $(b_k)_{k\geq1}$ is defined in (2.2). Since, the sequence $(b_k)_{k\geq1}$ is monotonically decreasing under the given hypotheses. Therefore,

$$\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - \frac{\mathbb{F}^{(\gamma)}_{1,\beta}(z)}{z}\right| < \frac{(e-1)}{\beta^\gamma}, \quad \text{for all } z \in \mathbb{D}. \tag{4.3}$$

Combining the above inequality with (4.1), we obtain the following bound

$$\left|\left[z\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\right]\Big/\left(\mathbb{F}^{(\gamma)}_{1,\beta}(z)\right) - 1\right| < \frac{(e-1)}{\beta^\gamma - e + 1} < 1, \quad \text{for all } z \in \mathbb{D}.$$

This completes the proof of Theorem 4.7. □

Setting $\gamma = 1$ in Theorem 4.7, we obtain the following result.

Corollary 4.8. *Suppose that the hypotheses of Corollary 2.5 are valid. Also assume that* $\alpha \geq 1, \beta > 2(e-1)$*. Then the function* $\mathbb{E}_{\alpha,\beta}(z)$ *is close-to-convex with respect to starlike function* $\mathbb{E}_{1,\beta}(z)$ *in* $\mathbb{D}$*.*

Theorem 4.9. *Let* α, β *and* γ *be positive real numbers such that the following condition holds:*

$$\alpha\gamma \log(\beta) - \frac{\alpha\gamma}{\beta} - 2 > 0.$$

Then the function $\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z) = z\Gamma^{\gamma}(\beta) F^{(\gamma)}_{\alpha,\beta}(z^2)$ *is close-to-convex with respect to the function* $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ *in* $\mathbb{D}$*.*

Proof. We have,

$$\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z) = z + \sum_{k=2}^{\infty} \left(\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^{\gamma} z^{2k-1} := \sum_{k=1}^{\infty} A_{2k-1} z^{2k-1}.$$

First, we show that $\{(2k-1)A_{2k-1}\}_{k\geq 1}$ is a decreasing sequence. To do so, let us consider the function

$$L_1(s) = \frac{(2s-1)}{\{\Gamma(\alpha s + \beta - \alpha)\}^{\gamma}}, \quad s \geq 1.$$

Differentiation gives us

$$L_1'(s) = L_1(s)\left[\frac{2}{2s-1} - \alpha\gamma\psi(\alpha s + \beta - \alpha)\right] := L_1(s)L_2(s).$$

Using Lemma 1.1, we obtain

$$L_2(s) < L_3(s) := \frac{2}{2s-1} + \frac{\alpha\gamma}{(\alpha s + \beta - \alpha)} - \alpha\gamma\log(\alpha s + \beta - \alpha),$$

which leads to

$$L_3'(s) = -\frac{4}{(2s-1)^2} - \frac{\alpha^2\gamma}{(\alpha s + \beta - \alpha)^2} - \frac{\alpha^2\gamma}{(\alpha s + \beta - \alpha)} < 0, \quad s \geq 1.$$

Therefore, $L_3(s)$ is decreasing on $[1, \infty)$ for $s \geq 1$ with $L_3(1) < 0$, under given hypothesis. This shows that $L_3(s) < 0$ for any $s \geq 1$, which yields that $L_1(s)$ is decreasing on $[1, \infty)$ for $s \geq 1$. Consequently, $\{(2k-1)A_{2k-1}\}_{k\geq 1}$ is a decreasing sequence. Hence, the hypothesis of Lemma 1.7 is satisfied. It is well-known that [3, p. 55] if a function $f : \mathbb{D} \to \mathbb{C}$ satisfies the hypothesis of Lemma 1.7, then it is close-to-convex with respect to the function $\dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$. This completes the proof of the theorem. □

Setting $\gamma = 1$ in Theorem 4.9, we obtain the following result.

Corollary 4.10. *Let* α *and* β *be positive real numbers such that the following condition holds:*

$$\alpha\log(\beta) - \frac{\alpha}{\beta} - 2 > 0.$$

Then the function $\mathcal{E}_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z^2)$ *is close-to-convex with respect to the function* $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$*.*

5. Conclusion

In this paper, normalized Le Roy type Mittag-Leffler function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ has been considered and several geometric properties such as starlikeness, convexity, close-to-convexity (univalency) and uniformly convexity have been studied inside the unit disk for positive real numbers α, β and γ. It can be noted that $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ reduces to the normalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}(z)$ for $\gamma = 1$. As applications, geometric properties of $\mathbb{E}_{\alpha,\beta}(z)$ are also obtained. In literature, several geometric properties of $\mathbb{E}_{\alpha,\beta}(z)$ are discussed [2,20,21] with the hypothesis that $\alpha \geq 1$ and $\beta \geq 1$. Results obtained in this paper can discuss certain geometric properties of $\mathbb{E}_{\alpha,\beta}(z)$ for the cases $0 < \alpha < 1$ and $0 < \beta < 1$ with sharper lower bounds of α and β. Interesting consequences and examples are provided to support that these results are better and improve several results available in the literature.

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