Comparison of improved class of priors for the analysis of the Burr type VII model under doubly censored samples

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Abstract

In recent years, the finite mixture lifetime models have frequently been used in chemical, physical, social science, biological and other fields due to their methodological development and practical applications. The Bayesian analysis of the mixture models has also developed a significant interest among the statisticians especially in the last decade. However, the most of these contributions are limited to the Bayes estimation for the parameters of lifetime models under singly type I censoring. This paper discusses the Bayesian estimation for the two-component mixture of the lifetime models under doubly censored samples with a particular case for the Burr type VII model. A class of improved priors has been proposed for the posterior estimation. The likelihood function, for doubly censored samples using two-component mixture of life time distributions, has been introduced. The hazard rate function for the mixture model has been compared for different parametric values. The performance of various estimators has been compared under a simulation study along with a real life example.

1. Introduction

Burr [5] introduced a family of Burr distribution including twelve life time distributions. These distributions can be used to model almost all kinds of failure time data. Among these distributions, the Burr type VII distribution has rarely been considered by the analysts, maybe due to its complex density function. This distribution can also be used for analysis of the life time data. Recently, Feroze and Aslam [10] discussed some properties of the distribution under a Bayesian framework using left, singly type II and doubly type II censored samples. The distribution still requires much attention to explore its applications in different real life situations. To deal with characteristics of such unfamiliar distributions is always of great interest for the researchers. The insight we can get from them can be beneficial to the professionals looking to use those distributions as models.

Among the other members from the family of Burr distributions; the Burr type X and XII have received the sizable attention of the researchers. For further details regarding these contributions, the readers may refer to: [11, 12, 28, 33, 40–44, 46, 49–51].

The finite mixture models are typically used to model data where each observation is assumed to have come from more than one groups (sub-populations), each group being suitably modeled by a density from some parametric family. The density representing each group is referred to as a component of the mixture. These components are mixed according to their proportions (relative frequencies) in the population. These mixture models provide a convenient and flexible family of distributions for estimating or approximating distributions which are not well modeled by any standard family, such as kernel density estimation. In recent years, the finite mixture lifetime models have frequently been used in chemical, physical, social science, biological and other fields due to their methodological development and practical applications. These mixture models can analyze different datasets when a population is categorized in number of sub-populations based on different causes of failures. For example, a population of lifetimes of certain electrical elements can be classified into number of sub-populations due to different causes of failures. Several papers have been appeared on classical analysis of the mixture of life time distributions under complete and censored data, these include: [1, 2, 8, 9, 19, 23, 24, 26, 27, 30–32, 45, 47].

The analysis of mixture models under Bayesian framework has developed a significant interest among the statisticians. The authors dealing with Bayesian analysis of mixture models include: [25, 36–39]. But these contributions are limited to the Bayes point estimation of the parameters under singly type I censored data. In singly type I censoring there is one test termination point; however an analyst may be interested to have two test termination times in situations where the equipment age is unknown at the start of a period of equipment monitoring and the tracking of the equipment is stopped after a predetermined observation interval. In such situation the use of doubly censored samples can effectively be used. Doubly censoring is the device used to describe the methodologies for analyzing duration times between two events. In this censoring, it is necessary to define two time points: the origin time corresponding to the time at which an original event occurs and the failure time corresponding to the time at which the final event occurs. For example, in the context of HIV/AIDS studies starting time may be taken as the infection time of a patient, the termination time can be the time of the AIDS diagnosis of the patients. The patients which at the end of the study have not developed the disease may be considered as right censored. The elapsed time between these two points is the AIDS latency time. For more illustrations on doubly type-II censoring, the reader may refer to: [12, 15, 29, 34, 35, 48].

In this paper, the two-component mixture of the Burr type VII has been considered under doubly type II censored samples. An extended version of the likelihood function under doubly type II censored samples has been introduced for two-component mixture of lifetime distributions. A class of improved informative priors has been proposed for the posterior analysis. The Bayes point and interval estimators along with posterior predictive intervals have been derived and evaluated. The proposed method can be extended for more than two component mixtures.

The organization of the paper is as follows: Section 2 contains of the model and the introduction of likelihood function for two-component mixture of Burr type VII distribution under doubly censored samples. The improved priors have been introduced and posterior distributions have been derived in Section 3. Bayes point estimation has been discussed in Section 4. The algorithm for prior elicitation has been given in Section 5. The results of the simulation study have been presented in Section 6. A real life example has been used to discuss the applicability of the results in Section 7. The hazard rate function for the mixture model has been discussed in Section 8. Finally, the conclusions regarding the study have been presented in Section 9.

2. The model and likelihood function

The probability density function (pdf) of the Burr type VII distribution is

$$f_j\left(x_{ji}\right) = \theta_j 2^{-\theta_j} sech^2\left(x_{ji}\right)\left\{1 + \tanh(x_{ji})\right\}^{\theta_j - 1},$$
$$-\infty < x_{ji} < \infty,\ \theta_j > 0,\ j = 1, 2\ i = 1, 2, ..., n. \quad (2.1)$$

The cumulative distribution function (CDF) of the distribution is

$$F_j\left(x_{ji}\right) = 2^{-\theta_j}\left\{1 + \tanh(x_{ji})\right\}^{\theta_j},$$
$$-\infty < x_{ji} < \infty, \theta_j > 0, j = 1, 2\ i = 1, 2, ..., n. \quad (2.2)$$

A density function for mixture of two components densities with mixing weights $(\pi, 1 - \pi)$ is

$$f\left(x\right) = \pi f_1\left(x\right) + \left(1 - \pi\right) f_2\left(x\right),\ \ 0 < \pi < 1. \quad (2.3)$$

The cumulative distribution function for the mixture model is

$$F\left(x\right) = \pi F_1\left(x\right) + \left(1 - \pi\right) F_2\left(x\right). \quad (2.4)$$

Consider a random sample of size n from Burr type VII distribution and let $x_r, x_r + 1, \ldots, x_s$ be the ordered observations that can only be observed. The remaining $r - 1$ smallest observations and the $n - s$ largest observations have been assumed to be censored. Now based on causes of failure, the failed items are assumed to come either from sub-population 1 or from sub-population 2; so that the $x_{1r_1}, \ldots, x_{1s_1}$ and $x_{2r_2}, \ldots, x_{2s_2}$ failed items come from first and second sub-populations respectively. The rest of the observations which are less than x_r and greater than x_s have been assumed to be censored from each component where $x_s = max\left(x_{1,s_1}, x_{2,s_2}\right)$ and $x_r = min\left(x_{1,r_1}, x_{2,r_2}\right)$. Therefore, $m_1 = s_1 - r_1 + 1$ and $m_2 = s_2 - r_2 + 1$ number of failed items can be observed from first and second sub-population respectively. The remaining $m = s - r + 1$ items are assumed to be censored observations where $r = r_1 + r_2$, $s = s_1 + s_2$ and $m = m_1 + m_2$. Then the likelihood function for the type II doubly censored sample x $= \{(x_{1r_1}, \ldots, x_{1s_1}), (x_{2r_2}, \ldots, x_{2s_2})\}$, assuming the causes of the failure of the left censored items are identified, can be written as

$$L\left(\theta_1, \theta_2, \pi \mid \text{x}\right) \propto \{F\left(x_{1,r_1}\right)\}^{r_1 - 1}\{F\left(x_{2,r_2}\right)\}^{r_2 - 1}\{1 - F\left(x_s\right)\}^{n-s}$$
$$\times \left\{\prod_{i=r_1}^{s_1} f\left(x_{1,i}\right)\right\}\left\{\prod_{i=r_2}^{s_2} f\left(x_{2,i}\right)\right\}. \quad (2.5)$$

Putting vales in Equation (2.5) and simplifying, we get

$$L\left(\theta_1, \theta_2, \pi \mid \text{x}\right) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\pi^{k_1+k_3+k_4}$$
$$\times (1-\pi)^{r_1+r_2+k_3-k_1-k_2-k_4-2}\theta_1^{m_1}e^{-\theta_1\Omega(x_{1i})}\theta_2^{m_2}e^{-\theta_2\Omega(x_{2i})}, \quad (2.6)$$

where
$$\Omega\left(x_{1i}\right) = -\sum_{i=r_1}^{s_1}\ln\left\{\frac{1+\tanh(x_{1i})}{2}\right\} - k_1\ln\left\{\frac{1+\tanh(x_{1,r_1})}{2}\right\} - k_2\ln\left\{\frac{1+\tanh(x_{2,r_2})}{2}\right\} - k_4\ln\left\{\frac{1+\tanh(x_s)}{2}\right\}$$

and
$$\Omega\left(x_{2i}\right) = -\sum_{i=r_2}^{s_2}\ln\left\{\frac{1+\tanh(x_{2i})}{2}\right\} - (r_1-k_1-1)\ln\left\{\frac{1+\tanh(x_{1,r_1})}{2}\right\} - (r_2-k_2-1)\ln\left\{\frac{1+\tanh(x_{2,r_2})}{2}\right\} - (k_3-k_4)\ln\left\{\frac{1+\tanh(x_s)}{2}\right\}.$$

This can further be written as

$$L\left(\theta_1, \theta_2, \pi \mid \text{x}\right) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\prod_{w=1}^{2}(-1)^{k_3}$$
$$\times\pi^{k_1+k_3+k_4}(1-\pi)^{r_1+r_2+k_3-k_1-k_2-k_4-2}\theta_w^{m_w}e^{-\theta_w\Omega(x_{wi})}. \quad (2.7)$$

3. Prior and posterior distributions

There are several rules for defining a non-informative prior, such as the principle of indifference and the Jeffreys rule. The principle of indifference is undoubtedly the simplest and oldest rule. Using the principle of indifference, Laplace [20] has proposed one of the most widely used non-informative prior, namely uniform prior. The expressions of uniform prior for the parameters of the mixture model can be given as: $g(\theta_1, \theta_2, \pi) \propto 1$. But it seems to be inadequate to the specific situation. As prior is often used when the parameter θ is defined on $(-\infty, \infty)$. If the parametric range is restricted to be positive, then a non-informative prior satisfying the Jeffreys rule is often considered as: $g(\theta) \propto \theta^{-1}$. Whereas, if θ is defined in $(0,1)$, the non-informative prior is usually defined as: $g(\theta) \propto \theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}}$. Whenever a specific rule is not defined and the resulting non-informative prior cannot explicitly be derived, the following vague joint prior can be used for the parameters of the mixed Burr type VII distribution:

$$g(\theta_1, \theta_2, \pi) \propto \theta_1^{-1}\theta_2^{-1}\pi^{-1/2}(1-\pi)^{-1/2} \ ; \theta_1, \theta_2 > 0 \tag{3.1}$$

The current literature regarding Bayesian analysis of the mixture models mostly includes the assumption of non-informative uniform prior for the mixing parameter along with informative priors for the parameters representing the different components of the model (see [25, 36–39]). We have proposed the informative beta prior for the mixing parameter, which has been combined with other priors to produce the improved informative priors for the posterior analysis. The description of these priors has been given in the following.

Let $\theta_1 \sim Gamma(a_1, b_1)$ and $\theta_2 \sim Gamma(a_2, b_2)$ are the gamma priors for each parameter and $\pi \sim Beta(a_3, b_3)$ is the beta prior for mixing parameter π. Under the assumption of independence, these priors have been combined to produce a joint improved gamma prior for parameter as

$$g(\theta_1, \theta_2, \pi) \propto \theta_1^{a_1-2}\theta_2^{a_2-2}e^{-(\theta_1 b_1 + \theta_2 b_2)}\pi^{a_3-1}(1-\pi)^{b_3-1},$$
$$\theta_1, \theta_2 > 0, \ 0 < \pi < 1, \ a_1, a_2, a_3, b_1, b_2, b_3 > 0 \tag{3.2}$$

where a_1, a_2, a_3, b_1, b_2 and b_3 are hyper-parameters. Another informative prior is the Jeffreys gamma prior suggested by [18]. Let $\theta_1 \sim JeffreysGamma(a_1, b_1)$ and $\theta_2 \sim JeffreysGamma(a_2, b_2)$ are the Jeffreys gamma priors for each parameter and $\pi \sim Beta(a_3, b_3)$ is the beta prior for mixing parameter π. Under the assumption of independence, these priors have been combined to produce a joint improved Jeffreys gamma prior for parameter as

$$g(\theta_1, \theta_2, \pi) \propto \theta_1^{a_1-2}\theta_2^{a_2-2}e^{-(\theta_1 b_1 + \theta_2 b_2)}\pi^{a_3-1}(1-\pi)^{b_3-1},$$
$$\theta_1, \theta_2 > 0, \ 0 < \pi < 1, \ a_1, a_2, a_3, b_1, b_2, b_3 > 0 \tag{3.3}$$

where a_1, a_2, a_3, b_1, b_2 and b_3 are hyper-parameters. Suppose $\theta_1 \sim InverseLevy(b_4)$, $\theta_2 \sim InverseLevy(b_5)$ and $\pi \sim Beta(a_3, b_3)$ then these marginal priors can be united to introduce a joint improved inverse levy prior as

$$g(\theta_1, \theta_2, \pi) \propto (\theta_1\theta_2)^{-0.5}e^{-0.5(b_4\theta_1 + b_5\theta_2)}\pi^{a_3-1}(1-\pi)^{b_3-1},$$
$$\theta_1, \theta_2 > 0, \ 0 < \pi < 1, \ a_3, b_3, b_4, b_5 > 0 \tag{3.4}$$

where a_3, b_3, b_4 and b_5 are hyper-parameters. Further, suppose $\theta_1 \sim Exponential(d_1)$, $\theta_2 \sim Exponential(d_2)$ and $\pi \sim Beta(a_3, b_3)$ then these marginal priors can be combined to introduce a joint improved exponential prior as

$$g(\theta_1, \theta_2, \pi) \propto e^{-(d_1\theta_1 + d_2\theta_2)}\pi^{a_3-1}(1-\pi)^{b_3-1},$$
$$\theta_1, \theta_2 > 0, \ 0 < \pi < 1, \ a_3, d_1, d_2 > 0 \tag{3.5}$$

where a_3, d_1 and d_2 are hyper-parameters. Finally, consider $\theta_1 \sim ChiSquare\,(d_3)$ and $\pi \sim Beta(a_3, b_3)$ then these marginal priors can be combined to introduce a joint improved chi square prior as

$$g\left(\theta_1, \theta_2, \pi\right) \propto \theta_1^{(d_3/2)-1} \theta_2^{(d_4/2)-1} e^{-0.5(\theta_1+\theta_2)} \pi^{a_3-1} (1-\pi)^{b_3-1},$$
$$\theta_1, \theta_2 > 0,\ 0 < \pi < 1,\ a_3, b_3, d_3, d_4 > 0 \tag{3.6}$$

where a_3, b_3, d_3 and d_4 are hyper-parameters. Using the prior distributions presented in Equations (3.1)–(3.6) and the likelihood function in Equation (2.7), the posterior distributions can respectively be obtained as

$$h_1\left(\theta_1, \theta_2, \pi \,|\mathrm{x}\right) \propto \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4}$$
$$\times \pi^{k_1+k_3+k_4-\frac{1}{2}} (1-\pi)^{r_1+r_2+k_3-k_1-k_2-k_4-\frac{5}{2}} \theta_w^{m_w-1} e^{-\theta_w \Omega(x_{wi})}, \tag{3.7}$$

$$h_1\left(\theta_1, \theta_2, \pi \,|\mathrm{x}\right) \propto \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4}$$
$$\times \pi^{k_1+k_3+k_4-a_3-1} (1-\pi)^{r_1+r_2+k_3-k_1-k_2-k_4-2+b_3-1} \theta_w^{m_w+a_w-1}$$
$$\times e^{-\theta_w \{\Omega(x_{wi})+b_w\}}, \tag{3.8}$$

$$h_1\left(\theta_1, \theta_2, \pi \,|\mathrm{x}\right) \propto \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4}$$
$$\times \pi^{k_1+k_3+k_4-a_3-1} (1-\pi)^{r_1+r_2+k_3-k_1-k_2-k_4-2+b_3-1} \theta_w^{m_w+a_w-2}$$
$$\times e^{-\theta_w \{\Omega(x_{wi})+b_w\}}, \tag{3.9}$$

$$h_1\left(\theta_1, \theta_2, \pi \,|\mathrm{x}\right) \propto \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4}$$
$$\times \pi^{k_1+k_3+k_4-a_3-1} (1-\pi)^{r_1+r_2+k_3-k_1-k_2-k_4-2+b_3-1} \theta_w^{m_w-0.5}$$
$$\times e^{-\theta_w \{\Omega(x_{wi})+0.5b_w+3\}}, \tag{3.10}$$

$$h_1\left(\theta_1, \theta_2, \pi \,|\mathrm{x}\right) \propto \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4}$$
$$\times \pi^{k_1+k_3+k_4-a_3-1} (1-\pi)^{r_1+r_2+k_3-k_1-k_2-k_4-2+b_3-1} \theta_w^{m_w}$$
$$\times e^{-\theta_w \{\Omega(x_{wi})+d_w\}}, \tag{3.11}$$

$$h_1\left(\theta_1, \theta_2, \pi \,|\mathrm{x}\right) \propto \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4}$$
$$\times \pi^{k_1+k_3+k_4-a_3-1} (1-\pi)^{r_1+r_2+k_3-k_1-k_2-k_4-2+b_3-1} \theta_w^{m_w+0.5d_w+2}$$
$$\times e^{-\theta_w \{\Omega(x_{wi})+0.5\}}. \tag{3.12}$$

The posterior distributions given in Equations (3.7) and (3.12) be combined to a single equation as

$$h_1\left(\theta_1, \theta_2, \pi\,|\text{x}\right) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}$$

$$\times \pi^{k_1+k_3+k_4-l_1(a_3-1)-\frac{1}{2}}(1-\pi)^{r_1+r_2+k_3-k_1-k_2-k_4+l_1(b_3-1)-\frac{5}{2}}$$

$$\times \theta_w^{m_w-l_2+l_3(a_w-1-l_4)-0.5l_5+l_7(0.5d_{2+w}-1)}$$

$$\times e^{-\theta_w\{\Omega(x_{wi})+l_3b_w+0.5l_5b_{w+3}+l_6d_w+0.5l_7\}}. \tag{3.13}$$

Here, the posterior distributions given in Equations (3.7)–(3.12) can be derived as a special case of the above density Equation (3.13) by putting $l_1 = l_2 = l_3 = l_4 = l_5 = l_6 = l_7 = 0$; $l_1 = l_3 = 1, l_2 = l_4 = l_5 = l_6 = l_7 = 0$; $l_1 = l_3 = 1, l_2 = l_4 = l_5 = l_6 = l_7 = 0$; $l_1 = l_5 = 1, l_2 = l_3 = l_4 = l_6 = l_7 = 0$; $l_1 = l_6 = 1, l_2 = l_3 = l_4 = l_5 = l_7 = 0$; and $l_1 = l_7 = 1, l_2 = l_3 = l_4 = l_5 = l_6 = 0$, respectively.

Let $A_1 = k_1+k_3+k_4+l_1\left(a_3-1\right)-1/2$, $A_2 = r_1+r_2+k_3-k_1-k_2-k_4+l_1\left(b_3-1\right)-5/2$, $M_w = m_w - l_2 + l_3\left(a_w-1-l_4\right) - 0.5l_5 + l_7\left(0.5d_{2+w}-1\right)$ and $\phi\left(x_{wi}\right) = \Omega\left(x_{wi}\right) + l_3b_w + 0.5l_5b_{w+3} + l_6d_w + 0.5l_7$. Therefore, the model given in Equation (3.13) can be presented as

$$h\left(\theta_1, \theta_2, \pi\,|\text{x}\right) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}$$

$$\times \pi^{A_1}(1-\pi)^{A_2}\theta_w^{M_w}exp\left\{-\theta_w\phi\left(x_{wi}\right)\right\}. \tag{3.14}$$

Consider that

$$\int_0^{\infty}\int_0^1\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\pi^{A_1}(1-\pi)^{A_2}\theta_w^{M_w}e^{-\theta_w\phi(x_{wi})}d\pi d\theta_1 d\theta_2$$

$$=\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\times\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\pi^{A_1}(1-\pi)^{A_2}\theta_w^{M_w}e^{-\theta_w\phi(x_{wi})}d\pi d\theta_1 d\theta_2$$

$$=\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\frac{B\left(A_1+1,A_2+1\right)\Gamma\left(M_w+1\right)}{\left\{\phi\left(x_{wi}\right)\right\}^{M_w+1}},$$

where $B(x,y)$ is a standard Beta function and $\Gamma(x)$ is the gamma function. Therefore the complete density for the posterior distributions given in Equation (3.14) can be obtained as

$$h\left(\theta_1, \theta_2, \pi\,|\text{x}\right) = \frac{\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\pi^{A_1}(1-\pi)^{A_2}\theta_w^{M_w}exp\{-\theta_w\phi(x_{wi})\}}{\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\frac{B(A_1+1,A_2+1)\Gamma(M_w+1)}{\{\phi(x_{wi})\}^{M_w+1}}}. \tag{3.15}$$

Let

$$D = \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\frac{B\left(A_1+1,A_2+1\right)\Gamma\left(M_w+1\right)}{\left\{\phi\left(x_{wi}\right)\right\}^{M_w+1}}.$$

Therefore, Equation (3.15) can be presented as

$$h\left(\theta_1, \theta_2, \pi\,|\text{x}\right) = \frac{1}{D}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}$$

$$\times\pi^{A_1}(1-\pi)^{A_2}\theta_w^{M_w}e^{-\theta_w\phi(x_{wi})}. \tag{3.16}$$

The graphs for the marginal posterior distributions for the parameters of the mixture density, given in Equation (2.3), under different priors are presented in Figures 1 and 2. The graphs are based on the simulated data from the mixture model using a sample of size 50. The legends in the graphs contain following abbreviations: NIP: Non-informative prior; GP: Gamma prior; JGP: Jeffreys Gamma prior; ILP: Inverse levy prior; EP: Exponential prior; CSP: Chi square prior.
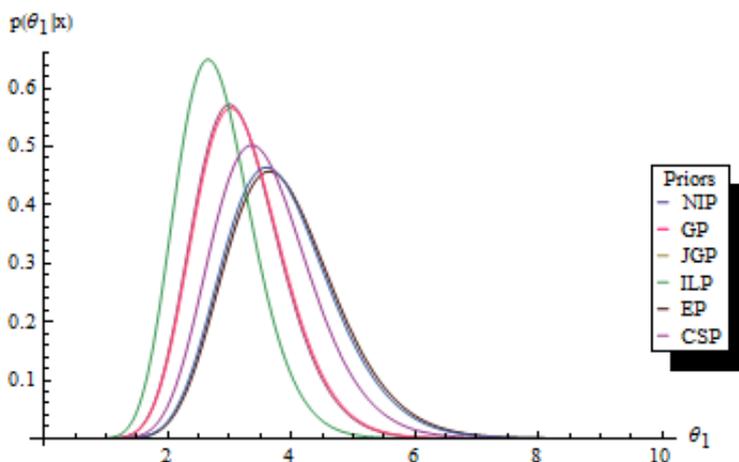
Figure 1. Graph for the marginal posterior distribution of θ_1 using different priors.
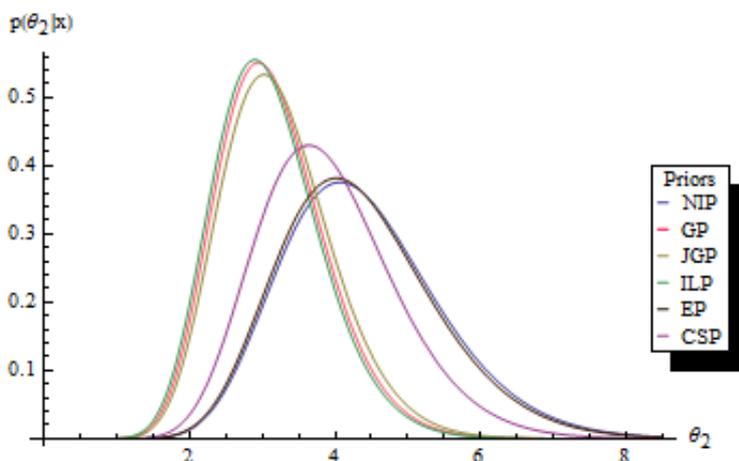
Figure 2. Graph for the marginal posterior distribution of θ_2 using different priors.

The graphs indicate that the marginal posterior distributions for the parameters θ_1 and θ_2 are slightly positively skewed. The tendency of skewness is higher for the second component. In case of first component, it can be observed that height of posterior distribution under inverse levy prior (ILP) is the most. The shape of posterior distributions under gamma and Jeffreys gamma priors are very close to each other. Similarly, the shape of posterior distributions under non-informative prior and the exponential priors are alike. In case second component, the posterior distributions under non-informative prior, gamma prior and Jeffreys gamma prior are having similar shapes. Finally, the posterior distributions based on exponential prior and chi square prior are identical.

4. Loss functions and Bayes estimators

The decision theory suggests that in order to select the best estimator, a loss function must be specified and used to estimate the risk associated with each of the possible estimates. Since, there is no definite analytical process that allows us to identify the proper loss function to be used, most of the analysts uses some symmetric loss functions, these associate equal importance to the losses due to overestimation and underestimation of equal magnitude and obtain the posterior mean as Bayesian estimate. In situations where the loss is asymmetric, we have to employ some asymmetric loss functions. Therefore, we have included both symmetric and asymmetric loss functions for posterior estimation. The description of the both loss function is given as the followings.

4.1. Squared error loss function (SELF)

The squared error loss function proposed by [14] and [22] is defined as

$$L\left(\theta, \theta_{SELF}\right) = \left(\theta - \theta_{SELF}\right)^2.$$

The Bayes estimator under this loss function is

$$\theta_{SELF} = E\left(\theta\right),$$

where θ is the parameter. The risk associated with this loss function can be viewed as variance of the estimator and is related to classical least square theory.

4.2. Squared logarithmic loss function (SLLF)

Another loss function due to Brown [4] is called squared logarithmic loss function. It can be defined as

$$L\left(\theta_{SLLF}, \theta\right) = \left(\log\theta_{SLLF} - \log\theta\right)^2.$$

The Bayes estimate under SLLF is

$$\theta_{SLLF} = \exp\left\{E\left(\log\theta\right)\right\}.$$

The generalized expressions for Bayes estimators and posterior risks under these loss functions have been presented in the following.

The Bayes estimator of parameter under SELF, denoted by $\left(\hat{\theta}_1\right)_{SELF}$, can be obtained as

$$
\begin{aligned}
\left(\hat{\theta}_1\right)_{SELF} = E\left(\theta_1\right) &= \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\quad \times \int_0^\infty \int_0^\infty \int_0^1 \pi^{A_1}(1-\pi)^{A_2} \theta_1^{M_1+1} \theta_2^{M_2} exp\left\{-\theta_1 \phi\left(x_{1i}\right)\right\} \\
&\quad \times exp\left\{-\theta_2 \phi\left(x_{2i}\right)\right\} d\pi d\theta_1 d\theta_2 \\
&= \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\quad \times \frac{B\left(A_1+1, A_2+1\right) \Gamma\left(M_1+2\right) \Gamma\left(M_2+1\right)}{\left\{\phi\left(x_{1i}\right)\right\}^{M_1+2} \left\{\phi\left(x_{2i}\right)\right\}^{M_2+1}}.
\end{aligned}
\tag{4.1}
$$

The Bayes estimator of parameter θ_2 under SELF, denoted by $\left(\hat{\theta}_2\right)_{SELF}$, can be derived as

$$
\begin{aligned}
\left(\hat{\theta}_2\right)_{SELF} = \mathrm{E}\left(\theta_1\right) &= \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\times \int_0^\infty \int_0^\infty \int_0^1 \pi^{A_1}(1-\pi)^{A_2} \theta_1^{M_1+1} \theta_2^{M_2+1} exp\left\{-\theta_1 \phi\left(x_{1i}\right)\right\} \\
&\times exp\left\{-\theta_2 \phi\left(x_{2i}\right)\right\} d\pi d\theta_1 d\theta_2 \\
&= \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\times \frac{B\left(A_1+2, A_2+1\right) \Gamma\left(M_1+1\right) \Gamma\left(M_2+1\right)}{\left\{\phi\left(x_{1i}\right)\right\}^{M_1+1} \left\{\phi\left(x_{2i}\right)\right\}^{M_2+1}}.
\end{aligned} \tag{4.2}
$$

The Bayes estimator of parameter π under SELF, denoted by $(\hat{\pi})_{SELF}$, can be obtained as

$$
\begin{aligned}
(\hat{\pi}_2)_{SELF} = \mathrm{E}\left(\pi\right) &= \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\times \int_0^\infty \int_0^\infty \int_0^1 \pi^{A_1+1}(1-\pi)^{A_2} \theta_1^{M_1} \theta_2^{M_2} exp\left\{-\theta_1 \phi\left(x_{1i}\right)\right\} \\
&\times exp\left\{-\theta_2 \phi\left(x_{2i}\right)\right\} d\pi d\theta_1 d\theta_2 \\
&= \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\times \frac{B\left(A_1+2, A_2+1\right) \Gamma\left(M_1+1\right) \Gamma\left(M_2+1\right)}{\left\{\phi\left(x_{1i}\right)\right\}^{M_1+1} \left\{\phi\left(x_{2i}\right)\right\}^{M_2+1}}.
\end{aligned} \tag{4.3}
$$

Similarly, the generalized expressions of the estimator for the posterior risk under SELF, using the formula $\rho(\theta)_{SELF} = \mathrm{E}\left(\theta^2\right) - \left\{\mathrm{E}\left(\theta\right)\right\}^2$, have been presented in the followings.

The Bayes estimator of posterior risk for parameter θ_1 under SELF, denoted by $\rho(\theta_1)_{SELF}$, can be obtained as

$$
\begin{aligned}
\rho(\theta_1)_{SELF} &= \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\times \frac{B\left(A_1+1, A_2+1\right) \Gamma\left(M_1+3\right) \Gamma\left(M_2+1\right)}{\left\{\phi\left(x_{1i}\right)\right\}^{M_1+3} \left\{\phi\left(x_{2i}\right)\right\}^{M_2+1}} \\
&- \left[\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \right. \\
&\left. \times \frac{B(A_1+1, A_2+1)\Gamma(M_1+2)\Gamma(M_2+1)}{\left\{\phi(x_{1i})\right\}^{M_1+2}\left\{\phi(x_{2i})\right\}^{M_2+1}}\right]^2.
\end{aligned} \tag{4.4}
$$

The Bayes estimator of posterior risk for parameter θ_2 under SELF, denoted by $\rho(\theta_2)_{SELF}$, can be obtained as

$$
\begin{aligned}
\rho(\theta_2)_{SELF} = &\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\times \frac{B(A_1+1, A_2+1)\,\Gamma(M_1+1)\,\Gamma(M_2+3)}{\{\phi(x_{1i})\}^{M_1+1}\{\phi(x_{2i})\}^{M_2+3}} \\
&- \left[\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \right. \\
&\left. \times \frac{B(A_1+1, A_2+1)\Gamma(M_1+1)\Gamma(M_2+2)}{\{\phi(x_{1i})\}^{M_1+2}\{\phi(x_{2i})\}^{M_2+1}} \right]^2.
\end{aligned}
\tag{4.5}
$$

Similarly, the Bayes estimator of posterior risk for parameter under SELF, denoted by $\rho(\pi)_{SELF}$, can be presented as

$$
\begin{aligned}
\rho(\pi)_{SELF} = &\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\times \frac{B(A_1+1, A_2+1)\,\Gamma(M_1+3)\,\Gamma(M_2+1)}{\{\phi(x_{1i})\}^{M_1+1}\{\phi(x_{2i})\}^{M_2+1}} \\
&- \left[\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \right. \\
&\left. \times \frac{B(A_1+1, A_2+1)\Gamma(M_1+1)\Gamma(M_2+1)}{\{\phi(x_{1i})\}^{M_1+1}\{\phi(x_{2i})\}^{M_2+1}} \right]^2.
\end{aligned}
\tag{4.6}
$$

Again, the generalized expression for the Bayes estimators and posterior risks based on SLLF are:

The Bayes estimator of under SLLF, can be obtained by using following formula.

$$
\left(\hat{\theta}_1\right)_{SLLF} = \exp\{E(\log\theta_1)\}
$$

Consider that

$$
\begin{aligned}
E(log\theta_1) = &\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\times \int_0^\infty \int_0^\infty \int_0^1 \pi^{A_1}(1-\pi)^{A_2} \theta_1^{M_1+1} \theta_2^{M_2} exp\{-log\theta_1 \phi(x_{1i})\} \\
&\times exp\{-\theta_2 \phi(x_{2i})\}\, d\pi d\theta_1 d\theta_2.
\end{aligned}
\tag{4.7}
$$

Since $\int_0^\infty \log(x) x^{a-1} e^{-bx} dx = \frac{\Gamma(a)[\psi(a) - \log(b)]}{b^a}$,

$$
\begin{aligned}
E(\log\theta_1) = &\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \\
&\times \frac{B(A_1+1, A_2+1)(T_1)\,\Gamma(M_w+1)}{\{\phi(x_{wi})\}^{M_w+1}},
\end{aligned}
\tag{4.8}
$$

where $T_1 = \psi(M_1 + 1) - \ln\{\phi(x_{1i})\}$ and $\psi(x)$ is digamma function. Hence, the Bayes estimator for θ_1 under SLLF, denoted by $\left(\hat{\theta}_1\right)_{SLLF}$, can be written as

$$\left(\hat{\theta}_1\right)_{SLLF} = exp\left\{\frac{1}{D}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\right\}$$
$$\times exp\left\{\frac{B(A_1+1, A_2+1)(T_1)\Gamma(M_w+1)}{\{\phi(x_{wi})\}^{M_w+1}}\right\}. \tag{4.9}$$

Similarly, the Bayes estimator for θ_2 under SLLF, denoted by $\left(\hat{\theta}_2\right)_{SLLF}$, can be obtained as

$$\left(\hat{\theta}_1\right)_{SLLF} = exp\left\{\frac{1}{D}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\right\}$$
$$\times exp\left\{\frac{B(A_1+1, A_2+1)(T_2)\Gamma(M_w+1)}{\{\phi(x_{wi})\}^{M_w+1}}\right\}, \tag{4.10}$$

where $T_2 = \psi(M_2 + 1) - \ln\{\phi(x_{2i})\}$.

Similarly, the Bayes estimator for π under SLLF, denoted by $(\hat{\pi})_{SLLF} = \exp\{E(\log \pi)\}$, can be obtained as

$$E(\log \pi) = \frac{1}{D}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}$$
$$\times \int_0^\infty\int_0^\infty\int_0^1 \log(\pi)\,\pi^{A_1}(1-\pi)^{A_2}\theta_1^{M_1}\theta_2^{M_2}exp\{-\theta_1\phi(x_{1i})\}$$
$$\times exp\{-\theta_2\phi(x_{2i})\}\,d\pi d\theta_1 d\theta_2. \tag{4.11}$$

Since $\int_0^1 \log(x)x^{a-1}(1-x)^{b-1}dx = B(a,b)\{\psi(a) - \psi(a+b)\}$,

$$E(\log \pi) = \frac{1}{D}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}$$
$$\times \frac{B(A_1+1, A_2+1)(W_1)\Gamma(M_w+1)}{\{\phi(x_{wi})\}^{M_w+1}}, \tag{4.12}$$

where $W_1 = \psi(A_1) - \psi(A_1 + A_2)$ and $\psi(x)$ is the digamma function. Hence, the Bayes estimator for π under SLLF, denoted by $(\hat{\pi})_{SLLF}$, can be given as

$$(\hat{\pi})_{SLLF} = exp\left\{\frac{1}{D}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\sum_{k_4=0}^{k_3}\prod_{w=1}^{2}(-1)^{k_3}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4}\right.$$
$$\left.\times \frac{B(A_1+1, A_2+1)(W_1)\Gamma(M_w+1)}{\{\phi(x_{wi})\}^{M_w+1}}\right\}. \tag{4.13}$$

The formula for the derivation of estimator of posterior risk for the parameter θ under SLLF is $\rho(\theta)_{SLLF} = E(\log \theta)^2 - \{E(\log \theta)\}^2$.

Accordingly, the Bayes estimator of posterior risk for parameter under SLLF can be obtained as $\rho(\theta_1)_{SLLF}$. Consider that

$$
\begin{aligned}
\mathrm{E}(\log \theta_1)^2 = \frac{1}{D} &\sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4} \\
&\times \int_0^\infty \int_0^\infty \int_0^1 \pi^{A_1}(1-\pi)^{A_2} \{\log(\theta_1)\}^2 \theta_1^{M_1}\theta_2^{M_2} exp\{-\theta_1\phi(x_{1i})\} \\
&\times exp\{-\theta_2\phi(x_{2i})\}\, d\pi d\theta_1 d\theta_2.
\end{aligned}
\tag{4.14}
$$

Since $\int_0^\infty \{\log(x)\}^2 x^{a-1} e^{-bx} dx = \frac{\Gamma(a)\left[\{\psi(a)-\log(b)\}^2-\psi_1(a)\right]}{b^a}$,

$$
\begin{aligned}
\mathrm{E}(\log \theta_1)^2 = \frac{1}{D} &\sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4} \\
&\times \frac{B(A_1+1,A_2+1)\left[\{T_1\}^2+\psi_1(M_1+1)\right]}{\{\Gamma(M_w+1)\}^{-1}\{\phi(x_{wi})\}^{M_w+1}}
\end{aligned}
\tag{4.15}
$$

where $\psi_1(x)$ is a trigamma function. Hence, the estimator for θ_1 posterior risk of under SLLF can be given as

$$
\begin{aligned}
\rho(\theta_1)_{SLLF} = \frac{1}{D} &\sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4} \\
&\times \frac{B(A_1+1,A_2+1)\left[\{T_1\}^2+\psi_1(M_1+1)\right]}{\{\Gamma(M_w+1)\}^{-1}\{\phi(x_{wi})\}^{M_w+1}} \\
&- \left[\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4} \right. \\
&\left. \times \frac{B(A_1+1,A_2+1)(T_1)\Gamma(M_w+1)}{\{\phi(x_{wi})\}^{M_w+1}} \right]^2.
\end{aligned}
\tag{4.16}
$$

Similarly, by little modifications, the estimator for θ_2 posterior risk of under SLLF, denoted by $\rho(\theta_2)_{SLLF}$ can be presented as

$$
\begin{aligned}
\rho(\theta_2)_{SLLF} = \frac{1}{D} &\sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4} \\
&\times \frac{B(A_1+1,A_2+1)\left[\{T_2\}^2+\psi_1(M_2+1)\right]}{\{\Gamma(M_w+1)\}^{-1}\{\phi(x_{wi})\}^{M_w+1}} \\
&- \left[\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\binom{k_3}{k_4} \right. \\
&\left. \times \frac{B(A_1+1,A_2+1)(T_2)\Gamma(M_w+1)}{\{\phi(x_{wi})\}^{M_w+1}} \right]^2.
\end{aligned}
\tag{4.17}
$$

The Bayes estimator of posterior risk for π under SLLF can be derived by using the formula $\rho(\pi)_{SLLF} = \mathrm{E}(\log \pi)^2 - \{\mathrm{E}(\log \pi)\}^2$. Consider that

$$\mathrm{E}(\log \pi)^2 = \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4}$$

$$\times \int_0^\infty \int_0^\infty \int_0^1 \pi^{A_1} (1-\pi)^{A_2} \{\log(\pi)\}^2 \theta_1^{M_1} \theta_2^{M_2}$$

$$\times \exp\{-\theta_1 \phi(x_{1i})\} \exp\{-\theta_2 \phi(x_{2i})\} \, d\pi d\theta_1 d\theta_2. \tag{4.18}$$

Since $\int_0^1 \{\log(x)\}^2 x^{a-1} (1-x)^{b-1} dx = B(a,b) \left[\{\psi(a) - \psi(a+b)\}^2 + \{\psi_1(a) - \psi_1(a+b)\} \right]$,

$$\mathrm{E}(\log \pi)^2 = \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4}$$

$$\times \frac{B(A_1+1, A_2+1) \left[\{W_1\}^2 + W_2 \right]}{\{\Gamma(M_w+1)\}^{-1} \{\phi(x_{wi})\}^{M_w+1}} \tag{4.19}$$

where $W_1 = \psi(A_1) - \psi(A_1 + A_2)$, $W_2 = \psi_1(A_1) - \psi_1(A_1 + A_2)$; $B(x,y)$ is a standard beta function; $\psi(x)$ is a digamma function and $\psi_1(x)$ is a trigamma function. Hence, the estimator for posterior risk of can be obtained as

$$\rho(\theta_2)_{SLLF} = \frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4}$$

$$\times \frac{B(A_1+1, A_2+1) \left[\{W_2\}^2 + W_2 \right]}{\{\Gamma(M_w+1)\}^{-1} \{\phi(x_{wi})\}^{M_w+1}}$$

$$- \left[\frac{1}{D} \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} \sum_{k_4=0}^{k_3} \prod_{w=1}^{2} (-1)^{k_3} \binom{r_1-1}{k_1} \binom{r_2-1}{k_2} \binom{n-s}{k_3} \binom{k_3}{k_4} \right.$$

$$\left. \times \frac{B(A_1+1, A_2+1)(W_1)}{\{\Gamma(M_w+1)\}^{-1} \{\phi(x_{wi})\}^{M_w+1}} \right]^2. \tag{4.20}$$

5. Prior elicitation

The elicitation is method to formulate the prior beliefs regarding some quantities into a probabilistic model. Under Bayesian inference it can be regarded as a technique to specify the values of hyper-parameters in a prior distribution for one or more parameters of the sampling distribution. It is not easy to have an accurate elicitation because in many real life situations, the experts are often not familiar with the concept of probabilities. Even when the expert is familiar with probabilities and their concept, it is by no means straightforward to evaluate a probability value for an event exactly. In such cases, elicitation encourages the expert and the facilitator to consider the meaning of the parameters being elicited. This has two helpful consequences. First, it brings the analysis closer to the application by demanding attention to what is being modeled and what is reasonable to believe about it. Second, it helps to make the posterior distributions, once calculated, into meaningful quantities. Some methods of prior elicitation can be seen from [3, 6, 7, 13, 16, 17].

We have used the method suggested by [3] for the prior elicitation. This method uses the prior predictive probabilities for elicitation. It compares the prior predictive distribution with expert assessment about this distribution and then to choose the hyper-parameters

that make the assessment agree closely with the member of the family. The prior predictive distribution can be defined as

$$g\left(y\right) = \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^1 g(\theta_1, \theta_2, \pi) f\left(y \,|\theta_1, \theta_2, \pi\right) d\pi d\theta_1 d\theta_2. \tag{5.1}$$

where $g(\theta_1, \theta_2, \pi)$ is the prior distribution, $f\left(y\,|\theta_1, \theta_2, \pi\right)$ is mixture density for future observation from Equation (2.3). According to Equation (5.1) the prior predictive distribution under improved gamma prior, presented in Equation (2.1), is

$$g\left(y\right) = \frac{sech^2\left(y\right)}{B(a_3, b_3)\left\{1 + \tanh(y)\right\}}$$
$$\times \left[\frac{a_1 b_1^{a_1} B(a_3 + 1, b_3)}{\left\{b_1 - \ln\left\{0.5\left(1 + \tanh(y)\right)\right\}\right\}^{a_1+1}} + \frac{a_2 b_2^{a_2} B(a_3, b_3 + 1)}{\left\{b_2 - \ln\left\{0.5\left(1 + \tanh(y)\right)\right\}\right\}^{a_2+1}} \right] \tag{5.2}$$

As we have to elicit six hyper-parameters so we have to consider six integrals. The set of hyper-parameters with minimum values has been chosen to be the elicited values of the hyper-parameters. By considering the prior predictive distribution in Equation (5.2), we have assumed the experts probabilities to be 0.10 for each integral. We considered the following integrals:

$$\int\limits_{-30}^{-20} g\left(y\right) = 0.10, \qquad \int\limits_{-19.9}^{-10.0} g\left(y\right) = 0.10, \qquad \int\limits_{-9.9}^{0.0} g\left(y\right) = 0.10,$$

$$\int\limits_{0.1}^{10.0} g\left(y\right) = 0.10, \qquad \int\limits_{10.1}^{20} g\left(y\right) = 0.10, \qquad \int\limits_{20.1}^{30.0} g\left(y\right) = 0.10.$$

In order to solve these integrals simultaneously, a program has been framed in SAS package using the PROC SYSLIN command and the elicited values of the hyper-parameters have been found to be

$$(a_1, b_1, a_2, b_2, a_3, b_3) = (0.8545, 1.3184, 0.9426, 1.6789, 0.0216, 0.0321).$$

Same integrals have been assumed for the elicitation of the hyper-parameters for the remaining priors. However, the above mentioned program has been slightly modified according to the expressions of the respective prior predictive distribution. The prior predictive distribution for improved Jeffreys gamma prior using Equations (2.3) and (3.2) is

$$g\left(y\right) = \frac{b_1^{a_1} b_2^{a_2} sech^2\left(y\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)B(a_3, b_3)\left\{1 + \tanh(y)\right\}}$$
$$\times \left[\frac{\Gamma(a_1)\Gamma(a_2 - 1)B(a_3 + 1, b_3)}{\left\{b_1 - \ln\{0.5(1 + \tanh(y))\}\right\}^{a_1} b_2^{a_2-1}} + \frac{\Gamma(a_1 - 1)\Gamma(a_2)B(a_3, b_3 + 1)}{b_1^{a_1-1}\left\{b_2 - \ln\{0.5(1 + \tanh(y))\}\right\}^{a_2}} \right]. \tag{5.3}$$

The elicited values of the hyper-parameters in Equation (5.3) are

$$(a_1, b_1, a_2, b_2, a_3, b_3) = (0.7317, 1.2183, 0.7962, 1.4841, 0.0196, 0.0275).$$

Similarly, the prior predictive distribution for improved inverse levy prior using Equations (2.3) and (3.3) is

$$g(y) = \frac{(b_4 b_5)^{\frac{1}{2}} sech^2(y)}{4B(a_3, b_3)\{1 + \tanh(y)\}}$$

$$\times \left[\left\{ \frac{b_4}{2} - \ln\left\{ \frac{1 + \tanh(y)}{2} \right\} \right\}^{-\frac{3}{2}} \left(\frac{b_5}{2} \right)^{-\frac{1}{2}} B(a_3 + 1, b_3) \right.$$

$$\left. + \left(\frac{b_4}{2} \right)^{-\frac{1}{2}} \left\{ \frac{b_5}{2} - \ln\left\{ \frac{1 + \tanh(y)}{2} \right\} \right\}^{-\frac{3}{2}} B(a_3, b_3 + 1) \right]. \tag{5.4}$$

The elicited values of the hyper-parameters in Equation (5.4) are

$$(a_3, b_3, b_4, b_5) = (0.06315, 0.0825, 1.0174, 0.8526).$$

Using Equations (2.3) and (3.4), the prior predictive distribution for improved exponential prior is

$$g(y) = \frac{sech^2(y)}{B(a_3, b_3)\{1 + \tanh(y)\}}$$

$$\times \left[\frac{d_1 B(a_3 + 1, b_3)}{\{d_1 - \ln\{0.5(1 + \tanh(y))\}\}^2} + \frac{d_2 B(a_3, b_3 + 1)}{\{d_2 - \ln\{0.5(1 + \tanh(y))\}\}^2} \right]. \tag{5.5}$$

The elicited values of the hyper-parameters in Equation (5.5) are

$$(a_3, b_3, d_1, d_2) = (0.0316, 0.0418, 0.0745, 0.0587).$$

Finally, Equations (2.3) and (3.5), the prior predictive distribution for improved chi square prior is:

$$g(y) = \frac{sech^2(y)}{2^{d_3/2} 2^{d_4/2} 2 B(a_3, b_3)\{1 + \tanh(y)\}}$$

$$\times \left[\frac{0.5 d_3 B(a_3 + 1, b_3)}{\{1.19 - \ln\{1 + \tanh(y)\}\}^{0.5 d_3 + 1}(0.5)^{0.5 d_4}} \right.$$

$$\left. + \frac{0.5 d_4 B(a_3, b_3 + 1)}{(0.5)^{0.5 d_3}\{1.19 - \ln\{1 + \tanh(y)\}\}^{0.5 d_4 + 1}} \right]. \tag{5.6}$$

The elicited values of the hyper-parameters in Equation (5.6) are

$$(a_3, b_3, d_3, d_4) = (0.0295, 0.0364, 1.8169, 1.7852).$$

6. Simulation study

The simulation study is a useful technique to assess the behavior and performance of a proposed system. The results from the study often mimic the real life situations. Therefore, in order to assess and compare the performance of the Bayes estimators with respect to different priors and loss functions and to investigate the impact of sample size on the estimators, a simulation study has been conducted using the parametric space: $(\theta_1, \theta_2, \pi) \in (1, 2, 0.45), (1, 2, 0.55), (2, 4, 0.45), (2, 4, 0.55)$. The probabilistic mixing have been used to draw samples of sizes n = 20, 50, 100, 150 and 200 from two components mixture of Burr type VII distribution under 10000 replications. The observations in the respective sample have been assumed to be 20% censored. In each case, only failures were

identified to be a member of either first or second sub-population of the mixture. These samples have been drawn by following the listed steps:

(1) Draw samples of size n from the mixture model.
(2) Generate a uniform random no. μ for each observation.
(3) If $\mu \leq \pi$, the take the observation from first sub-population otherwise from the second sub-population.
(4) Determine the test termination points on left and right, that is, determine the values of x_r and x_s.
(5) The observations which are $< x_r$ and $> x_s$ have been considered to be censored from each component.
(6) Use the remaining observations from each component for the analysis.

The mathematica software was used to obtain the numerical estimates. The amounts of posterior risks associated with each Bayes estimate have been given in the parenthesis in the tables. The abbreviations used in the tables are: PS: Parametric Set, PS1: $\theta_1 = 1$, $\theta_2 = 2$ and $\pi = 0.45$; PS2: $\theta_1 = 1$, $\theta_2 = 2$ and $\pi = 0.55$; PS3: $\theta_1 = 2$, $\theta_2 = 4$ and $\pi = 0.45$; PS4: $\theta_1 = 2$, $\theta_2 = 4$ and $\pi = 0.55$; LL: lower limit; UL: upper limit; NP: Non-informative prior; IGP: Improved Gamma Prior; IJP: Improved Jeffreys Prior; IILP: Improved Inverse Levy Prior; IEP: Improved Exponential Prior; ICSP: Improved Chi Square Prior.

Tables 1-4 represent the amounts of the Bayes estimates and posterior risks for the parameters of the mixture of Burr type VII distribution under doubly censored samples using different priors. From the analysis, it can be assessed that the parameter representing the first component (θ_1) has been overestimated, while the parameter representing the second component (θ_2) has been underestimated. The mixing parameter (π) has been under estimated for most of the cases. Nonetheless, the Bayes estimates (either under estimated or over estimated) converge to the true value of the parameter as sample size increases. The rate of convergence is bit slower for larger values of the parameters. However, the increase in the values of the mixing parameter leads to relative faster convergence of the estimates (towards the true values of the parameter) of the concerned component. The estimates under the assumption of improved informative priors have faster convergence than those under non-informative prior. On the whole, the rate of convergence is the most under the assumption of improved inverse levy prior. On the other hand, the SLLF provides better convergence than SELF. To be more specific, the estimates under improved inverse levy prior using SLLF are associated with the highest convergence rate. It is interesting to note that the amounts of posterior risks associated with each Bayes estimate decreases by increasing the sample size. This simply indicates that the estimates are consistent. The amounts of posterior risks increase for larger choice of true parametric values; however, the moderate to large sample sizes can face off this problem. In comparison of priors, it has been found that the improve informative priors work better than non-informative prior under both loss functions, as the amounts of posterior risks associated with estimates based on improved informative priors are smaller than those under non-informative prior. The minimum amounts of posterior risks have been observed for the estimates on the basis of improved inverse levy prior. In order of efficiency (amounts of posterior risks), the performance of the priors can be ranked as: improved inverse levy prior (1), improved Jeffreys gamma prior (2), improved chi square prior (3), improved gamma prior (4), improved exponential prior (5), non-informative prior (6) with some exceptions. In contrast, the estimates under SLLF are superior to those under SELF. Hence, for the point estimation of the parameters of the mixed Burr type VII model, the use of inverse levy prior under SLLF may be preferred.

Table 1. The Bayes estimates and posterior risks based on NP, IGP and IJP under SLLF using PS1, PS2, PS3 and PS4, respectively.

n	Non-informative Prior			Improved Gamma Prior			Improved Jeffreys Gamma Prior		
	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$
20	1.36678	1.65857	0.43948	1.36102	1.64905	0.43769	1.33324	1.64487	0.43591
	(0.00627)	(0.00706)	(0.00490)	(0.00524)	(0.00664)	(0.00449)	(0.00501)	(0.00602)	(0.00578)
50	1.17319	1.74046	0.44222	1.16824	1.73046	0.44042	1.14439	1.72608	0.43862
	(0.00570)	(0.00643)	(0.00434)	(0.00477)	(0.00604)	(0.00398)	(0.00449)	(0.00548)	(0.00511)
100	1.08085	1.80232	0.44428	1.07630	1.79196	0.44247	1.05433	1.78742	0.44066
	(0.00513)	(0.00549)	(0.00371)	(0.00428)	(0.00517)	(0.00340)	(0.00403)	(0.00469)	(0.00438)
150	1.05023	1.89440	0.44644	1.04579	1.88351	0.44462	1.02445	1.87875	0.44280
	(0.00419)	(0.00439)	(0.00296)	(0.00349)	(0.00412)	(0.00271)	(0.00329)	(0.00374)	(0.00349)
200	1.01715	1.94293	0.45011	1.01286	1.93178	0.44827	1.01699	1.92688	0.44644
	(0.00315)	(0.00342)	(0.00231)	(0.00263)	(0.00322)	(0.00212)	(0.00247)	(0.00291)	(0.00272)
20	1.34688	1.63262	0.52737	1.34119	1.62324	0.52523	1.31382	1.61914	0.52308
	(0.00568)	(0.00738)	(0.00562)	(0.00474)	(0.00694)	(0.00514)	(0.00454)	(0.00629)	(0.00662)
50	1.15610	1.71323	0.53066	1.15122	1.70339	0.52850	1.12773	1.69908	0.52634
	(0.00517)	(0.00672)	(0.00497)	(0.00432)	(0.00632)	(0.00455)	(0.00407)	(0.00573)	(0.00586)
100	1.06511	1.77411	0.53313	1.06061	1.76393	0.53096	1.03897	1.75946	0.52880
	(0.00464)	(0.00575)	(0.00425)	(0.00388)	(0.00541)	(0.00390)	(0.00366)	(0.00491)	(0.00502)
150	1.03494	1.86475	0.53572	1.03056	1.85404	0.53354	1.00954	1.84935	0.53136
	(0.00379)	(0.00457)	(0.00339)	(0.00316)	(0.00431)	(0.00310)	(0.00298)	(0.00391)	(0.00399)
200	1.02584	1.91253	0.54013	1.02150	1.90155	0.53793	1.02568	1.89673	0.53572
	(0.00285)	(0.00357)	(0.00265)	(0.00238)	(0.00336)	(0.00242)	(0.00224)	(0.00305)	(0.00312)
20	2.53951	3.32142	0.42825	2.67466	3.47754	0.43408	2.50496	3.28757	0.43991
	(0.00759)	(0.00870)	(0.00805)	(0.00957)	(0.01163)	(0.00787)	(0.00576)	(0.00740)	(0.00576)
50	2.17980	3.48540	0.43092	2.29581	3.64923	0.43678	2.15015	3.44987	0.44264
	(0.00690)	(0.00792)	(0.00713)	(0.00871)	(0.01059)	(0.00696)	(0.00524)	(0.00674)	(0.00511)
100	2.12875	3.60927	0.43293	2.24203	3.77893	0.43881	2.09978	3.57249	0.44470
	(0.00620)	(0.00678)	(0.00611)	(0.00783)	(0.00906)	(0.00596)	(0.00471)	(0.00577)	(0.00437)
150	2.08642	3.79367	0.43503	2.19746	3.97199	0.44095	2.05803	3.75501	0.44686
	(0.00506)	(0.00541)	(0.00486)	(0.00639)	(0.00722)	(0.00475)	(0.00384)	(0.00459)	(0.00348)
200	2.03753	3.89086	0.43860	2.14597	4.07375	0.44457	2.02991	3.85121	0.45054
	(0.00381)	(0.00422)	(0.00380)	(0.00480)	(0.00564)	(0.00371)	(0.00289)	(0.00359)	(0.00272)
20	2.50253	3.26946	0.51390	2.63571	3.42313	0.52089	2.46848	3.23614	0.52789
	(0.00687)	(0.00910)	(0.00923)	(0.00867)	(0.01215)	(0.00901)	(0.00522)	(0.00774)	(0.00660)
50	2.14806	3.43086	0.51711	2.26237	3.59213	0.52414	2.11883	3.39591	0.53118
	(0.00625)	(0.00830)	(0.00817)	(0.00789)	(0.01107)	(0.00798)	(0.00475)	(0.00705)	(0.00585)
100	2.09774	3.55280	0.51950	2.20938	3.71981	0.52657	2.06920	3.51659	0.53364
	(0.00562)	(0.00709)	(0.00700)	(0.00709)	(0.00948)	(0.00683)	(0.00427)	(0.00603)	(0.00501)
150	2.05603	3.73432	0.52204	2.16545	3.90985	0.52914	2.02806	3.69626	0.53624
	(0.00459)	(0.00564)	(0.00557)	(0.00578)	(0.00755)	(0.00544)	(0.00348)	(0.00480)	(0.00399)
200	2.05495	3.83000	0.52632	2.16430	4.01002	0.53348	2.04725	3.79096	0.54064
	(0.00344)	(0.00442)	(0.00435)	(0.00435)	(0.00589)	(0.00425)	(0.00262)	(0.00375)	(0.00311)

Table 2. The Bayes estimates and posterior risks based on NP, IGP and IJP under SELF using PS1, PS2, PS3 and PS4 respectively .

n	Non-informative Prior			Improved Gamma Prior			Improved Jeffreys Gamma Prior		
	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$
20	1.39412	1.62540	0.43377	1.38824	1.61607	0.43200	1.37842	1.61197	0.43024
	(0.01037)	(0.01230)	(0.00803)	(0.00919)	(0.01223)	(0.00780)	(0.00853)	(0.01110)	(0.01003)
50	1.19665	1.70565	0.43647	1.19160	1.69586	0.43470	1.18317	1.69156	0.43292
	(0.00944)	(0.01121)	(0.00710)	(0.00836)	(0.01114)	(0.00691)	(0.00777)	(0.01011)	(0.00889)
100	1.10247	1.76627	0.43850	1.09782	1.75612	0.43672	1.09005	1.75167	0.43493
	(0.00849)	(0.00957)	(0.00608)	(0.00752)	(0.00954)	(0.00591)	(0.00698)	(0.00866)	(0.00761)
150	1.07123	1.85651	0.44064	1.06671	1.84584	0.43884	1.05917	1.84117	0.43705
	(0.00693)	(0.00765)	(0.00484)	(0.00613)	(0.00760)	(0.00471)	(0.00569)	(0.00689)	(0.00606)
200	1.03749	1.90407	0.44426	1.03312	1.89314	0.44244	1.02580	1.88834	0.44064
	(0.00521)	(0.00596)	(0.00378)	(0.00461)	(0.00593)	(0.00368)	(0.00428)	(0.00538)	(0.00473)
20	1.37381	1.59997	0.52052	1.36801	1.59078	0.51840	1.35834	1.58675	0.51628
	(0.00940)	(0.01287)	(0.00920)	(0.00832)	(0.01279)	(0.00894)	(0.00773)	(0.01161)	(0.01150)
50	1.17922	1.67896	0.52377	1.17424	1.66932	0.52163	1.16594	1.66509	0.51950
	(0.00856)	(0.01171)	(0.00815)	(0.00757)	(0.01165)	(0.00791)	(0.00703)	(0.01057)	(0.01018)
100	1.08641	1.73863	0.52620	1.08182	1.72865	0.52405	1.07418	1.72427	0.52192
	(0.00768)	(0.01002)	(0.00697)	(0.00681)	(0.00997)	(0.00678)	(0.00632)	(0.00905)	(0.00871)
150	1.05564	1.82746	0.52876	1.05117	1.81696	0.52661	1.04375	1.81236	0.52445
	(0.00627)	(0.00797)	(0.00555)	(0.00555)	(0.00794)	(0.00539)	(0.00516)	(0.00720)	(0.00694)
200	1.04635	1.87428	0.53311	1.04193	1.86352	0.53093	1.03457	1.85880	0.52876
	(0.00471)	(0.00623)	(0.00433)	(0.00417)	(0.00620)	(0.00421)	(0.00388)	(0.00562)	(0.00542)
20	2.59030	3.25499	0.42268	2.72816	3.40799	0.42843	2.55506	3.22182	0.43419
	(0.01255)	(0.01518)	(0.01319)	(0.01680)	(0.02143)	(0.01367)	(0.01011)	(0.01365)	(0.01002)
50	2.22340	3.41569	0.42531	2.34173	3.57624	0.43111	2.19315	3.38088	0.43689
	(0.01142)	(0.01381)	(0.01168)	(0.01529)	(0.01952)	(0.01210)	(0.00920)	(0.01243)	(0.00887)
100	2.17132	3.53709	0.42730	2.28687	3.70335	0.43311	2.14178	3.50104	0.43892
	(0.01026)	(0.01183)	(0.01000)	(0.01374)	(0.01671)	(0.01036)	(0.00827)	(0.01064)	(0.00760)
150	2.12815	3.71779	0.42937	2.24141	3.89255	0.43521	2.09919	3.67991	0.44106
	(0.00838)	(0.00943)	(0.00796)	(0.01121)	(0.01331)	(0.00825)	(0.00674)	(0.00847)	(0.00605)
200	2.07828	3.81305	0.43290	2.18889	3.99228	0.43879	2.07051	3.77419	0.44468
	(0.00630)	(0.00735)	(0.00622)	(0.00843)	(0.01039)	(0.00644)	(0.00507)	(0.00662)	(0.00472)
20	2.55258	3.20407	0.50722	2.68842	3.35467	0.51412	2.51785	3.17141	0.52102
	(0.01136)	(0.01586)	(0.01511)	(0.01521)	(0.02241)	(0.01566)	(0.00915)	(0.01427)	(0.01148)
50	2.19102	3.36225	0.51039	2.30762	3.52029	0.51733	2.16121	3.32799	0.52427
	(0.01034)	(0.01447)	(0.01338)	(0.01384)	(0.02041)	(0.01387)	(0.00833)	(0.01299)	(0.01016)
100	2.13969	3.48175	0.51275	2.25356	3.64541	0.51973	2.11058	3.44626	0.52671
	(0.00929)	(0.01236)	(0.01146)	(0.01244)	(0.01747)	(0.01187)	(0.00749)	(0.01113)	(0.00870)
150	2.09715	3.65963	0.51525	2.20876	3.83165	0.52226	2.06862	3.62233	0.52927
	(0.00759)	(0.00984)	(0.00912)	(0.01015)	(0.01391)	(0.00945)	(0.00611)	(0.00886)	(0.00693)
200	2.09604	3.75340	0.51948	2.20758	3.92982	0.52655	2.08819	3.71514	0.53361
	(0.00570)	(0.00771)	(0.00712)	(0.00763)	(0.01086)	(0.00738)	(0.00459)	(0.00691)	(0.00541)

Table 3. The Bayes estimates and posterior risks based on IILP, IEP and ICSP under SLLF using PS1, PS2, PS3 and PS4 respectively.

n	Non-informative Prior			Improved Gamma Prior			Improved Jeffreys Gamma Prior		
	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$
20	1.35139	1.73396	0.43833	1.34733	1.63605	0.43655	1.34433	1.63191	0.43477
	(0.00486)	(0.00563)	(0.00391)	(0.00591)	(0.00681)	(0.00490)	(0.00526)	(0.00668)	(0.00527)
50	1.15997	1.81957	0.44106	1.15649	1.71682	0.43927	1.15391	1.71247	0.43748
	(0.00443)	(0.00500)	(0.00343)	(0.00529)	(0.00604)	(0.00429)	(0.00471)	(0.00592)	(0.00461)
100	1.06868	1.88424	0.44311	1.06547	1.77783	0.44131	1.06309	1.77333	0.43951
	(0.00398)	(0.00427)	(0.00293)	(0.00476)	(0.00517)	(0.00367)	(0.00423)	(0.00507)	(0.00395)
150	1.03840	1.90301	0.44527	1.03528	1.79554	0.44346	1.03298	1.79100	0.44165
	(0.00324)	(0.00340)	(0.00233)	(0.00388)	(0.00412)	(0.00292)	(0.00345)	(0.00404)	(0.00314)
200	1.00569	1.95177	0.44894	1.02774	1.84155	0.44710	1.02545	1.83689	0.44528
	(0.00244)	(0.00265)	(0.00182)	(0.00292)	(0.00322)	(0.00228)	(0.00260)	(0.00315)	(0.00246)
20	1.33171	1.70684	0.52600	1.32770	1.61045	0.52386	1.32474	1.60637	0.52172
	(0.00440)	(0.00588)	(0.00448)	(0.00535)	(0.00712)	(0.00561)	(0.00476)	(0.00698)	(0.00604)
50	1.14308	1.79111	0.52928	1.13965	1.68996	0.52713	1.13710	1.68568	0.52498
	(0.00401)	(0.00522)	(0.00392)	(0.00479)	(0.00632)	(0.00491)	(0.00427)	(0.00619)	(0.00528)
100	1.05312	1.85476	0.53174	1.04995	1.75002	0.52958	1.04761	1.74559	0.52741
	(0.00360)	(0.00447)	(0.00336)	(0.00431)	(0.00541)	(0.00420)	(0.00383)	(0.00530)	(0.00452)
150	1.02328	1.87323	0.53433	1.02020	1.76746	0.53215	1.01793	1.76298	0.52998
	(0.00294)	(0.00355)	(0.00268)	(0.00351)	(0.00431)	(0.00335)	(0.00313)	(0.00422)	(0.00360)
200	1.01428	1.92123	0.53872	1.03652	1.81274	0.53652	1.03420	1.80814	0.53433
	(0.00221)	(0.00277)	(0.00209)	(0.00264)	(0.00336)	(0.00261)	(0.00235)	(0.00329)	(0.00281)
20	2.47719	3.47239	0.42714	2.64776	3.45012	0.43295	2.49187	3.26165	0.43876
	(0.00606)	(0.00695)	(0.00643)	(0.01079)	(0.01194)	(0.00858)	(0.00623)	(0.00821)	(0.00526)
50	2.12630	3.64383	0.42980	2.27272	3.62046	0.43565	2.13891	3.42268	0.44149
	(0.00543)	(0.00617)	(0.00563)	(0.00968)	(0.01059)	(0.00751)	(0.00558)	(0.00728)	(0.00460)
100	2.11568	3.77333	0.43180	2.26137	3.74914	0.43767	2.12822	3.54432	0.44355
	(0.00488)	(0.00529)	(0.00482)	(0.00870)	(0.00906)	(0.00643)	(0.00502)	(0.00623)	(0.00394)
150	2.05573	3.81092	0.43390	2.19729	3.78648	0.43980	2.06792	3.57963	0.44571
	(0.00398)	(0.00420)	(0.00383)	(0.00709)	(0.00722)	(0.00512)	(0.00409)	(0.00496)	(0.00314)
200	2.04076	3.90855	0.43746	2.18128	3.88349	0.44341	2.05285	3.67134	0.44936
	(0.00299)	(0.00328)	(0.00299)	(0.00533)	(0.00563)	(0.00400)	(0.00308)	(0.00387)	(0.00245)
20	2.44111	3.41807	0.51256	2.60920	3.39615	0.51953	2.45557	3.21062	0.52651
	(0.00548)	(0.00728)	(0.00737)	(0.00977)	(0.01248)	(0.00984)	(0.00564)	(0.00858)	(0.00603)
50	2.09534	3.58682	0.51576	2.23963	3.56382	0.52278	2.10775	3.36913	0.52979
	(0.00492)	(0.00644)	(0.00645)	(0.00876)	(0.01107)	(0.00860)	(0.00505)	(0.00761)	(0.00527)
100	2.08486	3.71430	0.51816	2.22842	3.69048	0.52520	2.09721	3.48888	0.53225
	(0.00442)	(0.00553)	(0.00552)	(0.00787)	(0.00947)	(0.00737)	(0.00454)	(0.00652)	(0.00452)
150	2.02580	3.75129	0.52068	2.16529	3.72724	0.52776	2.03780	3.52362	0.53485
	(0.00360)	(0.00439)	(0.00440)	(0.00642)	(0.00754)	(0.00587)	(0.00370)	(0.00519)	(0.00360)
200	2.05819	3.84741	0.52496	2.19992	3.82274	0.53210	2.07038	3.61391	0.53923
	(0.00271)	(0.00343)	(0.00343)	(0.00483)	(0.00589)	(0.00458)	(0.00279)	(0.00405)	(0.00281)

Table 4. The Bayes estimates and posterior risks based on IILP, IEP and ICSP under SELF using PS1, PS2, PS3 and PS4 respectively.

n	Non-informative Prior			Improved Gamma Prior			Improved Jeffreys Gamma Prior		
	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$
20	1.35991 (0.00828)	1.69928 (0.00981)	0.43263 (0.00641)	1.37428 (0.01036)	1.60333 (0.01256)	0.43088 (0.00851)	1.37121 (0.00922)	1.59927 (0.01232)	0.42912 (0.00916)
50	1.16728 (0.00742)	1.78317 (0.00872)	0.43533 (0.00561)	1.17962 (0.00929)	1.68248 (0.01114)	0.43356 (0.00745)	1.17699 (0.00827)	1.67822 (0.01092)	0.43179 (0.00801)
100	1.07542 (0.00667)	1.84655 (0.00744)	0.43735 (0.00480)	1.08678 (0.00835)	1.74228 (0.00954)	0.43558 (0.00638)	1.08436 (0.00743)	1.73787 (0.00935)	0.43380 (0.00686)
150	1.04494 (0.00544)	1.86495 (0.00593)	0.43948 (0.00382)	1.05599 (0.00681)	1.75963 (0.00759)	0.43770 (0.00508)	1.05364 (0.00606)	1.75518 (0.00744)	0.43591 (0.00546)
200	1.03733 (0.00409)	1.91273 (0.00462)	0.44310 (0.00299)	1.04829 (0.00512)	1.80472 (0.00593)	0.44129 (0.00397)	1.04596 (0.00456)	1.80015 (0.00581)	0.43949 (0.00427)
20	1.34009 (0.00751)	1.67270 (0.01026)	0.51916 (0.00734)	1.35426 (0.00938)	1.57824 (0.01313)	0.51705 (0.00976)	1.35124 (0.00835)	1.57424 (0.01288)	0.51494 (0.01050)
50	1.15028 (0.00673)	1.75528 (0.00910)	0.52240 (0.00643)	1.16244 (0.00841)	1.65616 (0.01165)	0.52027 (0.00853)	1.15984 (0.00749)	1.65197 (0.01142)	0.51815 (0.00918)
100	1.05975 (0.00605)	1.81766 (0.00780)	0.52483 (0.00550)	1.07095 (0.00756)	1.71502 (0.00997)	0.52270 (0.00731)	1.06857 (0.00673)	1.71068 (0.00978)	0.52056 (0.00786)
150	1.02973 (0.00493)	1.83577 (0.00620)	0.52739 (0.00438)	1.04061 (0.00616)	1.73211 (0.00794)	0.52524 (0.00582)	1.03829 (0.00549)	1.72772 (0.00778)	0.52309 (0.00626)
200	1.04619 (0.00370)	1.88281 (0.00483)	0.53172 (0.00342)	1.05725 (0.00464)	1.77648 (0.00620)	0.52955 (0.00454)	1.05489 (0.00413)	1.77198 (0.00608)	0.52739 (0.00489)
20	2.52673 (0.01003)	3.40295 (0.01212)	0.42158 (0.01053)	2.70071 (0.01894)	3.38112 (0.02201)	0.42732 (0.01492)	2.54171 (0.01093)	3.19642 (0.01514)	0.43305 (0.00914)
50	2.16883 (0.00898)	3.57095 (0.01076)	0.42421 (0.00922)	2.31818 (0.01698)	3.54805 (0.01952)	0.42999 (0.01305)	2.18169 (0.00979)	3.35423 (0.01343)	0.43575 (0.00800)
100	2.15799 (0.00808)	3.69787 (0.00922)	0.42619 (0.00789)	2.30659 (0.01526)	3.67415 (0.01671)	0.43198 (0.01117)	2.17078 (0.00880)	3.47344 (0.01150)	0.43778 (0.00685)
150	2.09685 (0.00658)	3.73470 (0.00732)	0.42826 (0.00628)	2.24124 (0.01244)	3.71075 (0.01330)	0.43408 (0.00890)	2.10928 (0.00718)	3.50803 (0.00915)	0.43991 (0.00545)
200	2.08157 (0.00495)	3.83038 (0.00572)	0.43177 (0.00490)	2.22491 (0.00936)	3.80582 (0.01039)	0.43765 (0.00695)	2.09391 (0.00540)	3.59792 (0.00714)	0.44352 (0.00426)
20	2.48993 (0.00907)	3.34971 (0.01269)	0.50590 (0.01207)	2.66138 (0.01715)	3.32822 (0.02301)	0.51278 (0.01709)	2.50468 (0.00989)	3.14641 (0.01583)	0.51967 (0.01047)
50	2.13724 (0.00814)	3.51508 (0.01123)	0.50906 (0.01056)	2.28442 (0.01537)	3.49254 (0.02040)	0.51598 (0.01495)	2.14991 (0.00887)	3.30175 (0.01404)	0.52290 (0.00916)
100	2.12656 (0.00731)	3.64001 (0.00963)	0.51142 (0.00904)	2.27299 (0.01382)	3.61667 (0.01747)	0.51838 (0.01280)	2.13916 (0.00797)	3.41910 (0.01202)	0.52533 (0.00785)
150	2.06631 (0.00596)	3.67627 (0.00765)	0.51391 (0.00720)	2.20860 (0.01127)	3.65269 (0.01391)	0.52090 (0.01019)	2.07856 (0.00650)	3.45315 (0.00957)	0.52789 (0.00625)
200	2.09935 (0.00449)	3.77046 (0.00599)	0.51813 (0.00562)	2.24391 (0.00847)	3.74628 (0.01086)	0.52518 (0.00796)	2.11179 (0.00489)	3.54163 (0.00747)	0.53222 (0.00488)

7. Real life example

This section covers the analysis of real life data set regarding the breaking strengths of 64 single carbon fibers of length 10, presented by [21]. The idea has been to see whether the results and properties of the Bayes estimators, explored by simulation study, are applicable to a real life situation. We have taken $n = 64$, $r = 7$ and $s = 58$ in order to have censoring rate close to 20% (that has been used in simulation study). The results of the analysis have been reported in the following tables. The amounts of posterior risks associated with each estimate have been presented in the parenthesis in the tables.

Table 5 contain the Bayes estimates and posterior risks for the mixture distribution using real life data. In case of point estimation, the performance of improved informative priors is better than non-informative prior. The minimum amounts of posterior risks have been observed under the assumption of improved inverse levy prior using SLLF. Therefore, the findings from the analysis of real life data are in accordance with the simulation study.

Table 5. Bayes estimates and posterior risks under real life data.

LF	$\pi = 0.45$								
	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\pi}$
	Non-informative Prior			Improved Gamma Prior			Improved Jeffreys Gamma Prior		
SLLF	254.532	280.913	0.390	256.355	280.383	0.392	259.469	283.789	0.393
	(0.51382)	(0.27110)	(0.20301)	(0.46501)	(0.24650)	(0.18934)	(0.41666)	(0.22110)	(0.16351)
SELF	260.936	268.699	0.383	262.806	268.192	0.385	265.998	271.449	0.386
	(0.64342)	(0.33944)	(0.25420)	(0.58222)	(0.30864)	(0.23707)	(0.52205)	(0.27651)	(0.20480)
	Improved Inverse Levy Prior			Improved Exponential Prior			Improved Chi Square Prier		
SLLF	255.690	279.349	0.411	250.825	276.518	0.468	252.622	275.997	0.470
	(0.37755)	(0.18838)	(0.15292)	(0.46529)	(0.28342)	(0.23256)	(0.42104)	(0.21789)	(0.21692)
SELF	262.124	267.203	0.404	257.136	264.495	0.460	258.978	263.997	0.462
	(0.49625)	(0.24759)	(0.20105)	(0.61170)	(0.37267)	(0.30581)	(0.55353)	(0.28646)	(0.28517)
	$\pi = 0.55$								
	Non-informative Prior			Improved Gamma Prior			Improved Jeffreys Gamma Prior		
SLLF	251.106	277.192	0.475	252.905	276.670	0.478	255.977	280.030	0.479
	(0.48973)	(0.38929)	(0.25093)	(0.44321)	(0.35396)	(0.23402)	(0.39713)	(0.31749)	(0.20210)
SELF	257.424	265.140	0.467	259.269	264.639	0.469	262.418	267.854	0.471
	(0.61939)	(0.48742)	(0.31420)	(0.56048)	(0.44319)	(0.29302)	(0.50255)	(0.39705)	(0.25313)
	Improved Inverse Levy Prior			Improved Exponential prior			Improved Chi Square Prior		
SLLF	252.249	275.649	0.501	247.449	272.856	0.571	249.222	272.341	0.573
	(0.35985)	(0.27050)	(0.18900)	(0.44348)	(0.40698)	(0.28745)	(0.40130)	(0.31289)	(0.26811)
SELF	258.596	263.663	0.492	253.675	260.991	0.561	255.493	260.500	0.563
	(0.45497)	(0.33860)	(0.23667)	(0.56082)	(0.50965)	(0.35998)	(0.50748)	(0.39175)	(0.33569)

8. Hazard rate for the mixture of Burr type VII distribution

The hazard rate is a useful way of describing the distribution of time to event because it has a natural interpretation that relates to the aging of a population. The hazard function is the risk of failure in a small time interval, given survival at the beginning of the time interval. As a function of time, a hazard function may be increasing; meaning as time increases the rate for failure increases, for example, when a patient is untreated for a disease such as cancer or the medication do not work properly; may be decreasing, for example, as a person is recovering from severe trauma like a surgery, or may be constant, meaning the rate of failure is the same regardless of how much time has passed. The constant hazard rate is mostly unrealistic. The hazard rate for the mixture of Burr type VII distribution has been compared under a range of parametric values.

Hazard rate function for mixture of Burr Type VII distribution is

$$H\left(t\right) = \frac{\pi\theta_1 2^{-\theta_1} sech^2(t)\{1+\tanh(t)\}^{\theta_1-1} + (1-\pi)\theta_2 2^{-\theta_2} sech^2(t)\{1+\tanh(t)\}^{\theta_2-1}}{1-\left\{\pi 2^{-\theta_1}\{1+\tanh(t)\}^{\theta_1} + (1-\pi)2^{-\theta_2}\{1+\tanh(t)\}^{\theta_2}\right\}}. \quad (8.1)$$

The graphs for the hazard rate of the mixture model, for different parametric values and for the various ranges of the variable, are presented in Figures 3 to 6. The abbreviations in the graphs are: H(t): Hazard rate; PR1: $\theta_1 = 0.50$, $\theta_2 = 0.75$; PR2: $\theta_1 = 50$, $\theta_2 = 75$; PR3: $\theta_1 = 0.50$, $\theta_2 = 75$; PR4: $\theta_1 = 50$, $\theta_2 = 0.75$; PR5: $\theta_1 = 100$, $\theta_2 = 120$; PR6: $\theta_1 = 100$, $\theta_2 = 0.75$; PR7: $\theta_1 = 0.50$, $\theta_2 = 120$.
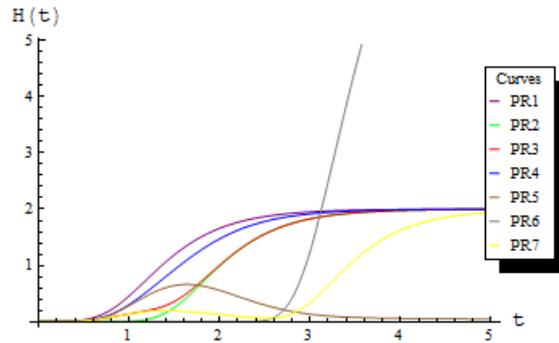
Figure 3. Graph of hazard rates for mixture of model using $\pi = 0.25$.
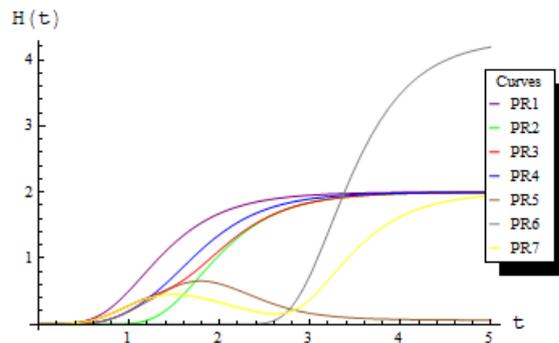
Figure 4. Graph of hazard rates for mixture of model using $\pi = 0.75$.
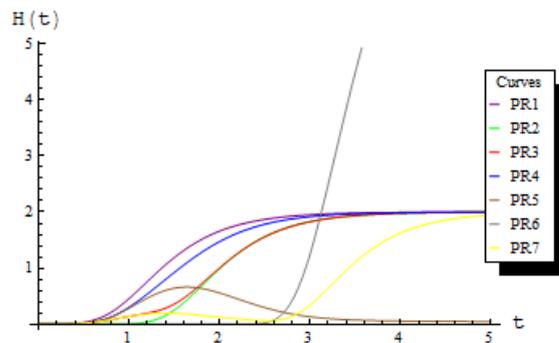
Figure 5. Graph of hazard rates for mixture of model using $\pi = 0.50$.
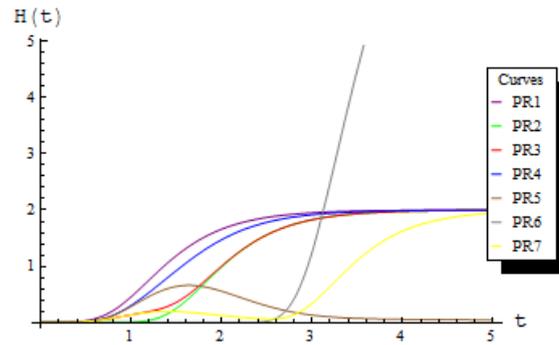
Figure 6. Graph of hazard rates for mixture of model using $\pi = 0.90$.

The graphs suggest that the hazard rate for the mixture model tend to decrease for $t < 2$ for closer choice of the values of the parameters. However, when the θ_1 and θ_2 differ significantly, the increase in the values of t shows ebbs and flows in the curves of the hazard rate. Except PR5 and PR6, the curves of hazard rate functions tend to converge to each other for $t > 4$.

9. Conclusion

The study has been carried out to introduce the methodology for Bayesian analysis of mixed lifetime models under doubly censored samples: with an application to Burr type VII distribution. A class of improved informative priors have been proposed and elicited for the analysis. A couple of loss functions have been assumed for the Bayes estimation. After a detailed analysis, it has been assessed that the improved inverse levy prior has provided the supreme results. Furthermore, the employment of the squared logarithmic loss function can produce the significantly better estimates for the parameters of the mixed Burr type VII distribution. The proposed estimators are consistent and capable of providing the stable results from moderate to large samples. The analysis of real life data further strengthened the findings of the simulation study. The study is useful for the analysts from different fields dealing with the analysis of the lifetime models when cases of failure are more than one and the duration time of the analysis is between two events.

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