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ECONOMIC ORDER QUANTITY MODEL WITH SHORTAGES BACKORDERED UNDER INFLATION AND TIME VALUE OF MONEY

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ÖZET

Bu makalede, enflasyon ve paranın zaman değerinin dikkate alındığı durumda, stoksuzluğa izin veren bir ekonomik sipariş miktarı modeli geliştirilmiştir. Türetilen iki değişkenli toplam maliyet fonksiyonu, değişkenlere göre kısmi türevleri alınarak, tek değişkenli maliyet fonksiyonuna indirgenmiştir. Tek boyutlu bir arama metodu kullanılarak, farklı reel faiz değerlerinin, optimal parti hacimleri ve toplam maliyetler üzerine etkileri sayısal örneklerle incelenmiştir.

In this paper, an economic order quantity (EOQ) model with shortages backordered is developed under inflation and time value of money. Derived total cost function which has two variable has been reduced to a function with one variable by partially differentiating the total cost with respect to its variables. By using a one-dimensional search method, effects of some real interest values on optimal lot sizes and total costs are investigated by illustrative examples.

Ekonomik sipariş miktarı, Ardısmarlama, Enflasyon, Paranın zaman değeri.
economic order quantity (EOQ), backorder, inflation, time value of money.

1. INTRODUCTION

Most of the classical inventory models did not take into account the effect of inflation and time value of money[1]. Reason of this may be that the inflation would not influence the inventory policy to any significant degree[2].

Buzacott is pioneer researcher in discussing the first economic order quantity (EOQ) model taking inflation into account[3]. In the same year, Misra

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1 RAY J., CHAUDHURI K.S., An EOQ model with stock-dependent demand, shortage, inflation and time discounting, International Journal of Production Economics 53 (2) (1997) 27-38.

2 SARKER B.R., PAN H., Effects of inflation and the time value of money on order quantity and allowable shortages, International Journal of Production Economics 34 (1) (1994) 65-72.

3 BUZACOTT J.A., Economic order quantities with inflation, Operational Research Quarterly 26 (1975) 553-558.

also developed an EOQ model incorporating inflationary conditions. Since then a vast amount of literature has developed[4]. The next paragraph summarizes some of this literature.

Chandra and Bahner extended the result in Misra to allow for shortages[5]. Park and Son examined various conditions under which the discounting effects on EOQ and on total costs can be critical[6]. In their model, the objective is to minimize the total costs per unit time. Datta and Pal considered effects of inflation and time-value of money on an inventory model with a linear time-dependent demand rate and shortages[7]. Sarker and Pan developed a finite replenishment model when shortage is allowed[8]. They obtained optimal production quantity and shortage level by using two-dimensional search method. Bose, Goswami and Chaudhuri developed an economic order quantity (EOQ) inventory model with a linear, positive trend in demand allowing inventory shortages and backlogging for deteriorating goods[9]. Chung developed an algorithm with finite replenishment and infinite planning horizon[10]. Wee and Law developed a deterministic inventory model for deteriorating items taking account of the time-value of money, with price-dependent demand and shortages allowed[11].

In this paper, an economic order quantity (EOQ) model with shortages backordered is developed under inflation and time value of money. Total cost function with two variables is reduced to a one variable function by partially differentiating the total cost with respect to its variables. By using a one-dimensional search method, effects of the real interest on optimal lot sizes and present value of total cost are investigated by illustrative examples.

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4 MISRA R.B., A study of inflation effects on inventory system. Logistics Spectrum 9 (1975) 260-268.

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8 See, SARKER B.R., PAN H.

9 BOSE S., GOSWAMI A., CHAUDHURI A., CHAUDHURI K.S., An EOQ model for deteriorating items with linear time-dependent demand rate and shortages under inflation and time discounting, Journal of the Operational Research Society, 46 (1995) 771-782.

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2. ASSUMPTIONS AND NOTATIONS

The notations are:

$D =$ demand rate, units/year

$A =$ initial replenishment cost, dollars/order or setup

$h =$ initial inventory cost, dollars/unit/year

$\pi =$ initial shortage cost, dollars/unit/year

$b =$ maximum allowable shortage in units

$Q =$ order quantity in units

$C =$ initial purchase cost, in dollars/unit

$i =$ inflation rate

$r =$ discount rate representing time value of money

$R =$ real interest rate (net discount rate), $R = i - r$

$L =$ planning horizon length, in years.

$N =$ number of cycles in L years, $N = DL/Q$

$T =$ Cycle time in years

The assumptions are:

a) lead time is zero,

b) the inventory system has the same cycle time, $T = Q/D$,

c) inflation and time value develop continuously with time, and both rates are constant,

3. MATHEMATICAL FORMULATION

Inventory level variation with time is given in Figure From Figure 1 the following expressions can be written:

$$T_1 = (Q - b)/D$$
$$T = Q/D$$

Define $S(t)$ and $I(t)$ as respectively the shortage level and the inventory level in cycle k ; they can be expressed as

$$S(t) = \begin{cases} 0, & 0 \le t \le T_1 \\ D(t - T_1), & T_1 \le t \le T \end{cases}$$

and

$$I(t) = \begin{cases} (Q - b) - Dt, & 0 \le t \le T_1 \\ 0, & T_1 \le t \le T \end{cases}$$

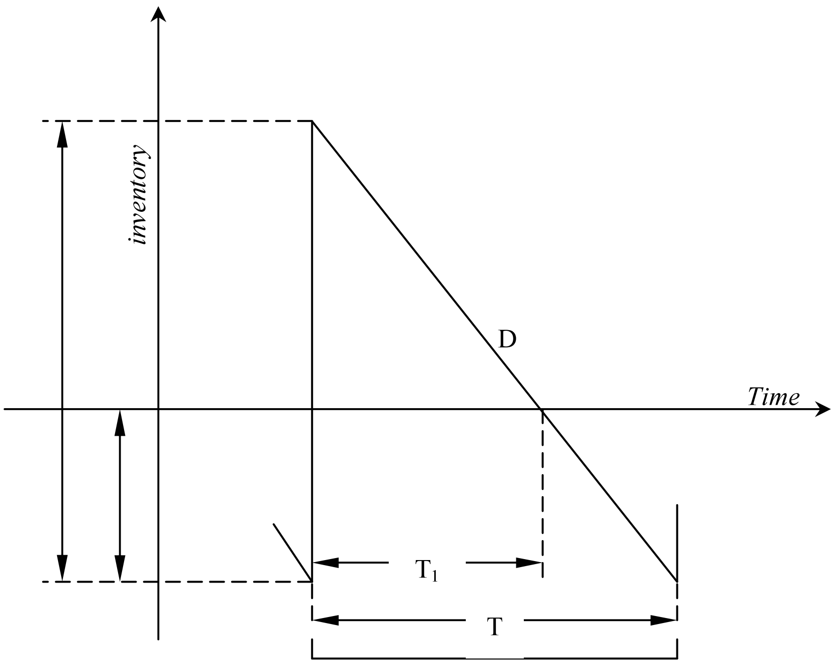

Figure 1: Inventory and shortage level as a function of time

Thus, present value of shortage cost in cycle k (SC_k) is (see the Appendix)

$$SC_k = \left\{ \int_0^T \left[\pi \, S(t)\, e^{Rt} \right] dt \right\} e^{R(k-1)T}$$

$$= \frac{\pi}{R}\left[\left(b - \frac{D}{R} \right) e^{RQ/D} + \frac{D}{R} e^{R(Q-b)/D} \right] e^{R(k-1)Q/D}$$

and present value of inventory holding cost in cycle k (HC_k) is (see the Appendix)

$$HC_k = \left\{ \int_0^T \left[h\, I(t)\, e^{Rt} \right] dt \right\} e^{R(k-1)T}$$

$$= \frac{h}{R}\left[-(Q-b) - \frac{D}{R} + \frac{D}{R} e^{R(Q-b)/D} \right] e^{R(k-1)Q/D}$$

The present value of production cost in cycle k (PC_k) is expressed as

$$PC_k = (A + CQ)\, e^{R(k-1)Q/D}$$

Therefore, the present value of total cost for N cycle (TC) can be given by (see the Appendix)

$$TC = \sum_{k=1}^{N} (SC_k + HC_k + PC_k)$$

$$= \left[-\frac{h}{R}\left[Q - b + \frac{D}{R} \right] + \frac{(h+\pi)D}{R^2} e^{R(Q-b)/D} + \frac{\pi}{R}\left(b - \frac{D}{R} \right) e^{RQ/D} + A + CQ \right] \left(\frac{1 - e^{RL}}{1 - e^{RQ/D}} \right) \quad (1)$$

If R is negative and L is infinite, then the present value of total cost, TC_∞, will become

$$TC_\infty = \left[-\frac{h}{R}\left[Q - b + \frac{D}{R} \right] + \frac{(h+\pi)D}{R^2} e^{R(Q-b)/D} + \frac{\pi}{R}\left(b - \frac{D}{R} \right) e^{RQ/D} + A + CQ \right] \left(\frac{1}{1 - e^{RQ/D}} \right) \quad (2)$$

Our objective is to find the optimal values of Q and b for minimizing the total cost, TC.

By partially differentiating the equation (1) with respect to Q and b, the following first order optimality conditions are obtained:

$$\left[C - \frac{h}{R} + \frac{(h+\pi)}{R} e^{R(Q-b)/D} + \frac{\pi}{D}\left(b - \frac{D}{R} \right) e^{RQ/D} \right] \left[\frac{1 - e^{RL}}{1 - e^{RQ/D}} \right] + \left[-\frac{h}{R}\left(Q - b + \frac{D}{R} \right) + \frac{(h+\pi)D}{R^2} e^{R(Q-b)/D} + A + CQ + \frac{\pi}{R}\left(b - \frac{D}{R} \right) e^{RQ/D} \right] \left[\frac{R\, e^{RQ/D}(1 - e^{RL})}{D(1 - e^{RQ/D})^2} \right] = 0 \quad (3)$$

and

$$\left[\frac{h}{R}-\frac{(h+\pi)}{R}e^{R(Q-b)/D}+\frac{\pi}{R}e^{RQ/D}\right]\left[\frac{1-e^{RL}}{1-e^{RQ/D}}\right]=0$$

or

$$b=-\frac{D}{R}\ln G \tag{4}$$

where (see the Appendix): $G=\dfrac{h+\pi\ e^{RQ/D}}{(h+\pi)e^{RQ/D}}$

After substituting (4) into (3), the two-dimensional problem of determining the optimal production quantity and maximum allowable shortage can be reduced to a one-dimensional problem of determining the optimal production quantity as following:

$$\left\{C-\frac{h}{R}+\left[\frac{(h+\pi)}{R}G-\frac{\pi}{R}(1+\ln G)\right]e^{RQ/D}\right\}\left[\frac{1-e^{RL}}{1-e^{RQ/D}}\right]$$
$$+\left\{-\frac{h}{R}\left[Q+\frac{D}{R}(1+\ln G)\right]+A+CQ+\left[\frac{(h+\pi)D}{R^2}G\right.\right.$$
$$\left.\left.-\frac{\pi D}{R^2}(1+\ln G)\right]e^{RQ/D}\right\}\left[\frac{R\,e^{RQ/D}\left(1-e^{RL}\right)}{D\left(1-e^{RQ/D}\right)^2}\right]=0 \tag{5}$$

Therefore, we can solve the equation (5) by using any one-dimensional search method to find the optimal production quantity, Q, which minimize the present value of the total cost, TC. Then, corresponding values of the maximum allowable shortage, b, and total cost, TC, can be obtained expressions (4) and (1) respectively.

Theorem : There is at least one solution to equations (3) and (4) (for proof of the Theorem , see the Appendix).

Remark: Let $f(Q)=$ the left side of (5). If $f'(Q)=df(Q)/dQ>0$, the solution to equations (3) and (4) is unique (see the Appendix for $f'(Q)$). Yet, since $f(Q)$ is a complex exponential function, it seems proving its first derivative to be positive would be a difficult task.

4. NUMERICAL EXAMPLE

Assume an infinite replenishment inventory system with D = 500 units/year, A=\$1000/setup, h = \$10/unit/year, C = \$5/unit and π = \$50/unit/year.

Using expression (5), optimal order quantity, Q, can be first obtained by applying the Newton-Raphson method. Then, we can derive the maximum allowable shortage, b, and total cost, TC, by using expressions (4) and (1) respectively.

The optimal solutions for the differences, which are varied from 0.1% to 175%, between the inflation rate and time value of money for $L = 1$ are given in Table 1. From Table 1, when R increases optimal ordering quantity, Q, and present value of total cost, TC, increases, but maximum allowable shortage, b, increases approximately up until $R = 1$. After $R > 1$, b gradually declines.

The optimal solutions for $L = 1$ and ∞ and for the differences, which are varied from -0.1% to -175%, between the inflation rate and time value of money are given in table 2. From table 2, when $|R|$ increases, optimal ordering quantity, Q, maximum allowable shortage, b, and present value of total costs, TC and TC_{∞}, decrease.

For all the values of R in the numerical example, since principals minors of order 1 and 2 of Hessian matrix (Δ_1 and Δ_2) are positive, for optimal Q and b values, each of the total cost function has a minimum.

Table 1: Optimal ordering quantities and maximum allowed shortages under different $R, (R > 0)$

$R = i - r$	Q	b	TC	Δ_1	Δ_2
0.001	347	57.82	5388.0	0.0287	0.0041
0.01	348	57.83	5398.9	0.0286	0.0041
0.05	353	57.97	5447.8	0.0278	0.0041
0.10	360	58.23	5509.3	0.0268	0.0040
0.15	367	58.43	5571.1	0.0258	0.0039
0.25	383	58.95	5695.7	0.0236	0.0038
0.35	401	59.49	5820.8	0.0213	0.0035
0.50	431	60.13	6008.3	0.0181	0.0032
0.75	496	61.02	6312.2	0.0126	0.0025
1.00	590	61.34	6588.9	0.0076	0.0018
1.25	740	60.54	6814.4	0.0035	0.0010
1.50	1032	57.77	6967.2	0.0008	0.0003
1.75	1899	52.02	7075.2	0.00002	0.000007

$D = 500, A = 1000, C = 5, L = 1, h = 10, \pi = 50$

Table 2: Optimal ordering quantities and maximum allowed shortages under different R, $(R < 0)$

$R = i - r$	Q	b	$L = 1$			$L = \infty$		
			TC	Δ_1	Δ_2	TC	Δ_1	Δ_2
-0.001	346	57.68	5385.5	0.0290	0.0042	5388229.1	28.965	4185.8
-0.01	345	57.67	5374.6	0.0291	0.0042	540151.7	2.9250	42.287
-0.05	340	57.48	5326.2	0.0299	0.0042	109209.0	0.6133	1.7824
-0.10	334	57.24	5266.2	0.0309	0.0043	55338.4	0.3249	0.4750
-0.15	328	56.96	5206.7	0.0320	0.0044	37379.5	0.2295	0.2254
-0.25	317	56.45	5089.6	0.0340	0.0045	23009.0	0.1535	0.0917
-0.35	307	55.97	4975.1	0.0359	0.0046	16846.9	0.1215	0.0525
-0.50	293	55.19	4808.8	0.0387	0.0047	12221.5	0.0984	0.0305
-0.75	273	53.98	4546.9	0.0432	0.0049	8617.4	0.0818	0.0175
-1.00	256	52.83	4304.7	0.0473	0.0050	6810.0	0.0748	0.0124
-1.25	241	51.63	4082.3	0.0513	0.0051	5721.6	0.0719	0.0099
-1.50	228	50.52	3878.9	0.0550	0.0051	4993.0	0.0708	0.0084
-1.75	217	49.59	3693.6	0.0582	0.0051	4470.4	0.0705	0.0074

$D = 500,\ A = 1000,\ C = 5,\ h = 10,\ \pi = 50$

5. CONCLUSION

In this paper, an economic order quantity (EOQ) model with shortages backordered is developed under inflation and time value of money. Derived total cost function which has two variable has been reduced to a function with one variable by partially differentiating the total cost with respect to its variables. By using Newton-Raphson search method, effects of some real interest values on optimal lot sizes and total costs are investigated by illustrative examples. When positive real interest rate increases, optimal ordering quantity, Q, and present value of total cost, TC, increases, but maximum allowable shortage, b, increases approximately up until $R = 1$. After $R > 1$, b gradually declines. For negative real interest rate, when their absolute values, $|R|$, increases, optimal ordering quantity, Q, maximum allowable shortage, b, and present value of total costs, TC and TC_∞, decrease.

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APPENDIX

Derivation of present value of shortage cost:

$$SC_k = \left\{ \int_0^T \left[\pi\ S(t)\ e^{Rt}\right] dt \right\} e^{R(k-1)T}$$

$$= \left\{ \int_{T_1}^T \pi\left[D(t-T_1)\right] e^{Rt}\, dt \right\} e^{R(k-1)T}$$

$$= \pi\left[\frac{D}{R}\left(T-\frac{1}{R}\right)e^{RT} - \frac{D}{R}\left(T_1-\frac{1}{R}\right)e^{RT_1} - \frac{DT_1}{R}e^{RT} + \frac{DT_1}{R}e^{RT_1}\right]e^{R(k-1)T}$$

$$= \pi\left[\left(\frac{Q}{R}-\frac{D}{R^2}\right)e^{RQ/D} + \left(\frac{D}{R^2}-\frac{(Q-b)}{R}\right)e^{R(Q-b)/D} - \frac{(Q-b)}{R}e^{RQ/D} + \frac{(Q-b)}{R}e^{R(Q-b)/D}\right]e^{R(k-1)T}$$

$$=\frac{\pi}{R}\left[\left(b-\frac{D}{R}\right)e^{R\,Q/D}+\frac{D}{R}\,e^{R\,(Q-b)/D}\right]e^{R\,(k-1)\,T}$$

Derivation of present value of inventory holding cost:

$$HC_k=\left\{\int_0^T\left[h\,I(t)e^{R\,t}\right]dt\right\}e^{R\,(k-1)\,T}$$

$$=\left\{\int_0^{T_1}\left[h(Q-b-D\,t)\,e^{R\,t}\right]dt\right\}e^{R\,(k-1)\,T}$$

$$=h\left[\frac{(Q-b)}{R}\left(e^{R\,(Q-b)/D}-1\right)-\frac{D}{R}\left(T_1-\frac{1}{R}\right)e^{R\,(Q-b)/D}-\frac{D}{R^2}\right]e^{R\,(k-1)T}$$

$$=\frac{h}{R}\left[-\left(Q-b+\frac{D}{R}\right)+\frac{D}{R}\,e^{R\,(Q-b)/D}\right]e^{R\,(k-1)\,T}$$

Derivation of expression (4):

$$\frac{\partial TC}{\partial b}=\left\{\frac{h}{R}-\frac{(h+\pi)}{R}\,e^{R\,(Q-b)/D}+\frac{\pi}{R}\,e^{R\,Q/D}\right\}\left[\frac{1-e^{R\,L}}{1-e^{R\,Q/D}}\right]=0$$

then, we get;

$$h-(h+\pi)\,e^{R\,Q/D}\,e^{-R\,b/D}+\pi\,e^{R\,Q/D}=0$$

or

$$b=-\frac{D}{R}ln\left[\frac{h+\pi\,e^{R\,Q/D}}{(h+\pi)e^{R\,Q/D}}\right]$$

Derivation of Hessian Matrix:

$$\frac{\partial^2 TC}{\partial Q^2}=\left\{\frac{(h+\pi)}{D}\,e^{R\,(Q-b)/D}+\frac{\pi\,R}{D^2}\left(b-\frac{D}{R}\right)e^{R\,Q/D}\right\}\left[\frac{1-e^{R\,L}}{1-e^{R\,Q/D}}\right]$$

$$+2\left\{C-\frac{h}{R}+\frac{(h+\pi)}{R}\,e^{R\,(Q-b)/D}+\frac{\pi}{D}\left(b-\frac{D}{R}\right)e^{R\,Q/D}\right\}$$

$$*\left[\frac{R\,e^{R\,Q/D}\left(1-e^{R\,L}\right)}{D\left(1-e^{R\,Q/D}\right)^2}\right]+\left\{A+CQ-\frac{h}{R}\left(Q-b+\frac{D}{R}\right)\right.$$

$$+\frac{(h+\pi)D}{R^2}e^{R(Q-b)/D}+\frac{\pi}{R}\left(b-\frac{D}{R}\right)e^{RQ/D}\Bigg\}$$

$$*\left[\frac{R^2e^{RQ/D}\left(1+e^{RQ/D}\right)\left(1-e^{RL}\right)}{D^2\left(1-e^{RQ/D}\right)^3}\right]$$

$$\frac{\partial^2 TC}{\partial Q\,\partial b}=\left\{-\frac{(h+\pi)}{D}e^{R(Q-b)/D}+\frac{\pi}{D}e^{RQ/D}\right\}\left[\frac{1-e^{RL}}{1-e^{RQ/D}}\right]$$

$$+\left\{\frac{h}{R}-\frac{(h+\pi)}{R}e^{R(Q-b)/D}+\frac{\pi}{R}e^{RQ/D}\right\}\left[\frac{R\ e^{RQ/D}\left(1-e^{RL}\right)}{D\left[1-e^{RQ/D}\right]^2}\right]$$

$$\frac{\partial^2 TC}{\partial b^2}=\left\{\frac{(h+\pi)}{D}e^{R(Q-b)/D}\right\}\left[\frac{1-e^{RL}}{1-e^{RQ/D}}\right]$$

Thus,

$$\Delta_1=\frac{\partial^2 TC}{\partial Q^2}\quad\text{and}\qquad \Delta_2=\left(\frac{\partial^2 TC}{\partial Q^2}\right)\left(\frac{\partial^2 TC}{\partial b^2}\right)-\left(\frac{\partial^2 TC}{\partial Q\,\partial b}\right)^2$$

Derivation of $f'(Q)$:

$f'(Q)=$

$$\left\{\left[\frac{(h+\pi)G'}{R}-\frac{\pi G'}{RG}+\frac{(h+\pi)G-\pi(1+\ln G)}{D}\right]e^{RQ/D}\right\}\left[\frac{1-e^{RL}}{1-e^{RQ/D}}\right]$$

$$+\left\{C-\frac{h}{R}+\left[\frac{(h+\pi)G-\pi(1+\ln G)}{R}\right]e^{RQ/D}\right\}\left[\frac{R\,e^{RQ/D}\left(1-e^{RL}\right)}{D\left(1-e^{RQ/D}\right)^2}\right]$$

$$+\Bigg\{C-\frac{h}{R}\left(1+\frac{DG'}{RG}\right)$$

$$+\left[\frac{(h+\pi)DG'}{R^2}-\frac{\pi\,DG'}{R^2G}+\frac{(h+\pi)G-\pi(1+\ln G)}{R}\right]e^{RQ/D}\Bigg\}$$

$$* \left[\frac{R\, e^{RQ/D}\left(1 - e^{RL}\right)}{D\left(1 - e^{RQ/D}\right)^{2}} \right]$$

$$+ \left\{ A + CQ + \left[\frac{(h+\pi)DG - \pi\, D(1 + \ln G)}{R^{2}} \right] e^{RQ/D} \right.$$

$$\left. - \frac{h}{R}\left[Q + \frac{D}{R}(1 + \ln G) \right] \right\} \left[\frac{R^{2}\, e^{RQ/D}\left(1 + e^{RQ/D}\right)\left(1 - e^{R}\ \right)}{D^{2}\left(1 - e^{RQ/D}\right)^{3}} \right]$$

where:

$$G^{'} = \frac{dG}{dQ} = \frac{-hR}{D(h+\pi)\, e^{R\,Q/D}}$$