e-ISSN: 2564-7954 CUJSE 19(01): 029-039 (2022) Research Article

Çankaya University

Journal of Science and Engineering

https://dergipark.org.tr/cankujse

CUJSE

A Precise Analytical Method to Solve the Nonlinear System of Partial Differential Equations with the Caputo Fractional Operator

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Keywords

Homotopy perturbation method, Caputo fractional derivative, ZZ integral transform, Nonlinear fractional differential equations.

Abstract

In this paper, we present a new technique by combination the homotopy perturbation method with ZZ transform method, we get the homotopy perturbation ZZ transform method to solve systems of nonlinear fractional partial differential equations. The fractional derivative is described in the Caputo sense. The results show that this method is appropriate and effective to solve the nonlinear system of nonlinear fractional differential equations and other nonlinear problems.

1. Introduction

For several decades, the analytical approximation methods have experienced considerable enthusiasm for differential equations which model natural phenomena affecting our environment or scientific problems of societies. These mathematical models draw their representations from many scientific fields such as physics and chemistry in all their generality, engineering sciences...

Differential equations flourished considerably with the development of mathematical analysis at the beginning of the 17th century. With the emergence of nonlinear sciences, the search for analytical solutions of differential equations became a central interest for mathematicians of the time. As a result, we see the emergence of several methods that are proposed to respond to such concerns. Among these methods, we mention, for example, the Adomian decomposition method which has been applied to solve linear and nonlinear boundary problems [1–4]. This method resulted in the development of several variants of analytical resolution methods such as the variational iteration method [5, 6], the homotopy perturbation method (HPM).

This last method developed by Ji-Huan He [7,8] allowed to solve a great variety of problems modeled by linear and nonlinear partial differential equations. Subsequently, the homotopy perturbation method was generalized to the fractional differential equations, to the nonlinear partial differential equations of fractional order according to the time variable in [9]. Several researches have been done to apply and extend this method to the nonlinear partial differential equations of fractional order according to the time variable or the dimensional variable or even according to both, for example, it was applied to the fractional biological population equation in [10], the fractional Cahn-Hilliard equation in [11], the fractional Fisher's equation in [12] and the fractional nonlinear dispersive K(2,2) equations in [13].

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Received: September 2, 2021, Accepted: January 26, 2022

Fractional differential equations are of great interest in many physical problems. The interest of the fractional derivative is linked to the mechanical modeling of materials which conserve and which memorize past deformations. The fractional derivation is ideally suited for studying this problem. The fractional calculus approach is a very attractive tool for studying the properties and characteristics of viscoelastic objects compared to known and already used methods. Consequently, we find that many researchers have been interested in solving this kind of differential equations, whether ODEs or PDEs with fractional derivative [14–17].

In order to facilitate the solution of this type of equations, especially nonlinear ones, we find that many researchers benefit from the combined the homotopy perturbation method with some known transforms, such as: Laplace transform [18, 19], Sumudu transform [20, 21], Elzaki transform [22], and that ZZ transform [23].

In our paper, we will extend the homotopy perturbation method combined with the ZZ transform which gives the homotopy perturbation ZZ transform method (HPZZTM) to solve the nonlinear system of partial differential equations of fractional order. This method will be applied to different types of system of nonlinear fractional partial differential equations.

2. Basic theory of fractional calculus

In this section, we will present the basics of fractional local calculus, these concepts include: Fractional derivative, fractional integral, some important results and fractional ZZ transform.

2.1. Fractional calculus

We present some basic definitions and properties of the fractional calculus theory as the Riemann-Liouville fractional integrals and Caputo fractional derivative (see [24, 25]).

Theorem 1. *[24, 25] Let $\sigma \geq 0$ and let $n = [\sigma] + 1$. If $\psi(\zeta) \in AC^n\,[a, b]$, then the Caputo fractional derivative $({}^cD^{\sigma}_{0+}\psi)(\zeta)$ exist almost everywhere on $[a, b]$. If $\sigma \notin \mathbb{N}$, $({}^cD^{\sigma}_{0+}\psi)(\zeta)$ is represented by*

$$({}^cD^{\sigma}_{0+}\psi)(\zeta) = \frac{1}{\Gamma(n-\sigma)} \int_0^{\zeta} \frac{\psi^{(n)}(\tau)d\tau}{(\zeta-\tau)^{\sigma-n+1}}, \tag{1}$$

where $D = \frac{d}{d\zeta}$ and $n = [\sigma] + 1$.

Remark 1. *[24] We consider the time-fractional derivative in the Caputo's sense. When $\sigma \in \mathbb{R}_+$, the time-fractional derivative is defined as*

$$\begin{aligned} ({}^cD^{\sigma}_{\zeta}\upsilon)(\varkappa,\zeta) &= \frac{\partial^{\sigma}\upsilon(\varkappa,\zeta)}{\partial\zeta^{\sigma}} \\ &= \begin{cases} \frac{1}{\Gamma(m-\sigma)}\int_0^{\zeta}(\zeta-\tau)^{m-\sigma-1}\frac{\partial^m\upsilon(\varkappa,\tau)}{\partial\tau^m}, & m-1<\sigma<m, \\ \frac{\partial^m\upsilon(\varkappa,\zeta)}{\partial\zeta^m}, & \sigma = m, \end{cases} \end{aligned}$$

where $m \in \mathbb{N}^*$.

Definition 1. *[24] Let $\sigma \in \mathbb{R}_+$; the operator I^{σ}_a defined on $L_1[a, b]$ by*

$$(I^{\sigma}_a\psi)(\zeta) = \frac{1}{\Gamma(\sigma)} \int_a^{\zeta} (\zeta-\tau)^{\sigma-1}\psi(\tau)d\tau;\ \sigma > 0, \tag{2}$$

is called the Riemann-Liouville fractional integral operator of order σ. Here $\Gamma(\cdot)$ is the gamma function.

Definition 2. *[24] The Mittag-Leffler function plays an important role in the solution of differential equations of fractional order, it's defined by*

$$E_{\sigma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\sigma k+1)}, \qquad \Re(\sigma) > 0; z \in \mathbb{C}. \tag{3}$$

For $\sigma = 1$, we get $E_{\sigma}(z) = e^z$.

2.2. Main result of the ZZ integral transform

In this part, we will give some basic definitions and properties of the ZZ transform (see [26], [27]).

Definition 3. *[26] Let $\upsilon(\zeta)$ be a function defined for all $\zeta \geq 0$. The ZZ transform of $\upsilon(\zeta)$ is the function $T(\upsilon, s)$ defined by*

$$Z\left[\upsilon(\zeta)\right] = T(\upsilon, s) = s\int_0^\infty \psi(\upsilon\zeta)e^{-s\zeta}d\zeta. \tag{4}$$

Theorem 2. *[27] If $\psi(\zeta)$ is piecewise continuous in every finite interval $0 \leq \zeta \leq K$ and of exponential order μ for $\zeta > K$, then its ZZ transform T(v,s) exists for all $s > \mu,\ \upsilon > \mu$.*

Proof. see [27] □

2.2.1. Some properties of the ZZ integral transform

1. The ZZ transform of the n^{th} derivative of $\upsilon(\zeta)$ is given by

$$Z[\upsilon^{(n)}(\zeta)] = \frac{s^n}{\upsilon^n}Z[\upsilon(\zeta)] - \sum_{k=0}^{n-1}\frac{s^{n-k}}{\upsilon^{n-k}}\upsilon^{(k)}(0) \tag{5}$$

2. ZZ transform of some elementary functions

$\upsilon(\zeta)$	$Z\left[\upsilon(\zeta)\right]$
1	1
ζ	$\frac{\upsilon}{s}$
ζ^n	$n!\frac{\upsilon^n}{s^n}$, $n = 0, 1, 2, \ldots$
ζ^σ	$\Gamma(\sigma+1)\frac{\upsilon^\sigma}{s^\sigma}, \sigma \geq 0.$

Proposition 1. *The ZZ transform of the time-fractional derivative in the Caputo's sense is defined as*

$$Z\left[({}^cD^\sigma_{0+}\upsilon)(\zeta); (\upsilon, s)\right] = \frac{s^\sigma}{\upsilon^\sigma}Z[\upsilon(\zeta)] - \sum_{k=0}^{n-1}\frac{s^{\sigma-k}}{\upsilon^{\sigma-k}}\upsilon^{(k)}(0)\ ,\ \ n-1 < \sigma \leq n,\ \ n = 1, 2, \ldots \tag{6}$$

Proof. See [26] □

3. Analysis of the homotopy perturbation ZZ transform method (HPZZTM)

We consider the general nonlinear system of fractional partial differential equations of the form

$$\begin{cases} {}^cD^\sigma_\zeta\upsilon(\varkappa,\zeta) + R\omega(\varkappa,\zeta) + N\upsilon(\varkappa,\zeta) = h_1(\varkappa,\zeta), \\ {}^cD^\delta_\zeta\omega(\varkappa,\zeta) + R\upsilon(\varkappa,\zeta) + N\omega(\varkappa,\zeta) = h_2(\varkappa,\zeta), \end{cases} \tag{7}$$

where $n-1 < \sigma, \delta \leq n, n = 1, 2, ...$
and the initial conditions

$$\begin{cases} \left[\frac{\partial^{n-1}\upsilon(\varkappa,\zeta)}{\partial\zeta^{n-1}}\right]_{\zeta=0} = \psi_{n-1}(\varkappa), \quad n = 1, 2, ... \\ \left[\frac{\partial^{n-1}\omega(\varkappa,\zeta)}{\partial\zeta^{n-1}}\right]_{\zeta=0} = \varphi_{n-1}(\varkappa), \quad n = 1, 2, ... \end{cases} \tag{8}$$

${}^cD^\sigma_\zeta\upsilon(\varkappa,\zeta)$, ${}^cD^\delta_\zeta\omega(\varkappa,\zeta)$ are the Caputo fractional derivatives of the functions $\upsilon(\varkappa,\zeta)$ and $\omega(\varkappa,\zeta)$ respectively, R is the linear differential operator, N represent the general nonlinear differential operator, and $h_1(\varkappa,\zeta)$, $h_2(\varkappa,\zeta)$ are the source terms.

Theorem 3. *The solutions of nonlinear system of partial differential equations with Caputo time-fractional derivative (7)-(8) by HPZZTM are given in the form of an infinite series which converges rapidly to the exact solution as follows*

$$\upsilon(\varkappa,\zeta) = \lim_{N\to\infty}\sum_{n=0}^{N}\upsilon_n(\varkappa,\zeta) \quad \omega(\varkappa,\zeta) = \lim_{N\to\infty}\sum_{n=0}^{N}\omega_n(\varkappa,\zeta).$$

Proof. By applying the ZZ transform to both sides of (7) and using the differentiation property, we get

$$\begin{cases} Z\left[\upsilon(\varkappa,\zeta)\right] = \frac{v^{\sigma}}{s^{\sigma}} \sum_{k=0}^{n-1} \frac{s^{\sigma-k}}{v^{\sigma-k}} \upsilon^{(k)}(\varkappa,0) + \frac{v^{\sigma}}{s^{\sigma}} Z\left[h_1(\varkappa,\zeta)\right] - \frac{v^{\sigma}}{s^{\sigma}} Z\left[R\omega(\varkappa,\zeta) + Nu(\varkappa,\zeta)\right], \\ Z\left[\omega(\varkappa,\zeta)\right] = \frac{v^{\delta}}{s^{\delta}} \sum_{k=0}^{n-1} \frac{s^{\delta-k}}{v^{\delta-k}} \omega^{(k)}(\varkappa,0) + \frac{v^{\delta}}{s^{\delta}} Z\left[h_2(\varkappa,\zeta)\right] - \frac{v^{\delta}}{s^{\delta}} Z\left[R\upsilon(\varkappa,\zeta) + N\omega(\varkappa,\zeta)\right]. \end{cases} \tag{9}$$

The inverse ZZ transform of both sides of the equations (9) with the initial conditions (8) gives

$$\begin{cases} \upsilon(\varkappa,\zeta) = G(\varkappa,\zeta) - Z^{-1}\left(\frac{v^{\sigma}}{s^{\sigma}} Z[R\omega(\varkappa,\zeta) + N\upsilon(\varkappa,\zeta)]\right), \\ \omega(\varkappa,\zeta) = H(\varkappa,\zeta) - Z^{-1}\left(\frac{v^{\delta}}{s^{\delta}} Z[R\upsilon(\varkappa,\zeta) + N\omega(\varkappa,\zeta)]\right), \end{cases} \tag{10}$$

where $G(\varkappa,\zeta)$ and $H(\varkappa,\zeta)$ are representing the terms arising from the non homogeneous terms and the prescribed initial conditions. Then, the solutions represent as follows

$$\upsilon(\varkappa,\zeta) = \sum_{n=0}^{\infty} p^n \upsilon_n(\varkappa,\zeta) \ , \omega(\varkappa,\zeta) = \sum_{n=0}^{\infty} p^n \omega_n(\varkappa,\zeta), \tag{11}$$

and the nonlinear terms can be decomposed as

$$N\upsilon(\varkappa,\zeta) = \sum_{n=0}^{\infty} H_n, \ \ Nw(\varkappa,\zeta) = \sum_{n=0}^{\infty} D_n \tag{12}$$

where H_n and D_n are the He polynomials [28], and they can be calculated by

$$H_n = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i \upsilon_i\right)\right]_{p=0}, \ \ D_n = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i \omega_i\right)\right]_{p=0}, \ i = 0,1,2,\cdots \tag{13}$$

By using (11) and (12), we can rewrite (10) as

$$\begin{cases} \sum_{n=0}^{\infty} p^n \upsilon_n(\varkappa,\zeta) = G(\varkappa,\zeta) - p\left(Z^{-1}\left[\frac{v^{\sigma}}{s^{\sigma}} Z\left[R\sum_{n=0}^{\infty} p^n \omega_n + \sum_{n=0}^{\infty} p^n H_n\right]\right]\right) \\ \sum_{n=0}^{\infty} p^n \omega_n(\varkappa,\zeta) = H(\varkappa,\zeta) - p\left(Z^{-1}\left[\frac{v^{\delta}}{s^{\delta}} Z\left[R\sum_{n=0}^{\infty} p^n \upsilon_n + \sum_{n=0}^{\infty} D_n\right]\right]\right) \end{cases}. \tag{14}$$

We compare the both sides of (14), then we obtain the first terms of the solution

$$\begin{aligned} &\upsilon_0(\varkappa,\zeta) = G(\varkappa,\zeta), \\ &\upsilon_1(\varkappa,\zeta) = -Z^{-1}\left[\frac{v^{\sigma}}{s^{\sigma}} Z\left[R\omega_0(\varkappa,\zeta) + H_0\right]\right], \\ &\upsilon_2(\varkappa,\zeta) = -Z^{-1}\left[\frac{v^{\sigma}}{s^{\sigma}} Z\left[R\omega_1(\varkappa,\zeta) + H_1\right]\right]. \\ &\vdots \end{aligned} \tag{15}$$

And

$$\begin{aligned} &\omega_0(\varkappa,\zeta) = H(\varkappa,\zeta), \\ &\omega_1(\varkappa,\zeta) = -Z^{-1}\left[\frac{v^{\delta}}{s^{\delta}} Z\left[R\upsilon_0(\varkappa,\zeta) + D_0\right]\right], \\ &\omega_2(\varkappa,\zeta) = -Z^{-1}\left[\frac{v^{\delta}}{s^{\delta}} Z\left[R\upsilon_1(\varkappa,\zeta) + D_1\right]\right], \\ &\vdots \end{aligned} \tag{16}$$

by continuing in the same way, we find the general recursive relations

$$\begin{cases} \upsilon_{n+1}(\varkappa,\zeta) = -Z^{-1}\left[\frac{v^{\sigma}}{s^{\sigma}} Z\left[R\omega_n(\varkappa,\zeta) + H_n\right]\right], \ \ n \geq 1 \\ \omega_{n+1}(\varkappa,\zeta) = -Z^{-1}\left[\frac{v^{\delta}}{s^{\delta}} Z\left[R\upsilon_n(\varkappa,\zeta) + D_n\right]\right], \ \ n \geq 1. \end{cases}. \tag{17}$$

At last, the approximate solution is calculated by

$$\upsilon(\varkappa,\zeta) = \lim_{N\to\infty} \sum_{n=0}^{N} \upsilon_n(\varkappa,\zeta) \quad \omega(\varkappa,\zeta) = \lim_{N\to\infty} \sum_{n=0}^{N} w_n(\varkappa,\zeta). \tag{18}$$

□

4. Illustrative examples and results

In this part, we apply the HPZZTM method for solving some nolinear systems of fractional partial differential equations.

Example 1. *First, we consider the nonlinear system of partial differential equations with time-fractional derivatives*

$$\begin{cases} {}^cD_\zeta^\sigma \upsilon(\varkappa,\zeta)+\omega(\varkappa,\zeta)\upsilon_\varkappa(\varkappa,\zeta)+\upsilon(\varkappa,\zeta)=1, & 0\leq\sigma<1 \\ {}^cD_\zeta^\delta \omega(\varkappa,\zeta)-\upsilon(\varkappa,\zeta)w_\varkappa(\varkappa,\zeta)-\omega(\varkappa,\zeta)=1, & 0\leq\delta<1 \end{cases} \tag{19}$$

subject the initial conditions

$$\begin{cases} \upsilon(\varkappa,0)=e^{\varkappa} \\ \omega(\varkappa,0)=e^{-\varkappa}. \end{cases} \tag{20}$$

By taking the ZZ transform on both sides of (19) and using its differentiation property, we obtain

$$\begin{cases} Z\left[\upsilon(\varkappa,\zeta)\right]=e^{\varkappa}+\frac{\upsilon^\sigma}{s^\sigma}Z\left[1-\omega(\varkappa,\zeta)\upsilon_\varkappa(\varkappa,\zeta)-\upsilon(\varkappa,\zeta)\right], \\ Z\left[\omega(\varkappa,\zeta)\right]=e^{-\varkappa}+\frac{\upsilon^\delta}{s^\delta}Z\left[1+\upsilon(\varkappa,\zeta)\omega_\varkappa(\varkappa,\zeta)+\omega(\varkappa,\zeta)\right]. \end{cases} \tag{21}$$

The inverse ZZ transform on both sides of (21) gives

$$\begin{cases} \upsilon(\varkappa,\zeta)=e^{\varkappa}+\frac{\zeta^\sigma}{\Gamma(\sigma+1)}-Z^{-1}\left(\frac{\upsilon^\sigma}{s^\sigma}Z\left[\omega(\varkappa,\zeta)\upsilon_\varkappa(\varkappa,\zeta)+\upsilon(\varkappa,\zeta)\right]\right), \\ \omega(\varkappa,\zeta)=e^{-\varkappa}+\frac{\zeta^\delta}{\Gamma(\delta+1)}+Z^{-1}\left(\frac{\upsilon^\delta}{s^\delta}Z\left[\upsilon(\varkappa,\zeta)\omega_\varkappa(\varkappa,\zeta)+\omega(\varkappa,\zeta)\right]\right). \end{cases} \tag{22}$$

The approximate solution represent as

$$\upsilon(\varkappa,\zeta)=\sum_{n=0}^{\infty}p^n\upsilon_n(\varkappa,\zeta),\quad \omega(\varkappa,\zeta)=\sum_{m=0}^{\infty}p^n\omega_n(\varkappa,\zeta) \tag{23}$$

Note that these nonlinear terms

$$\omega\upsilon_\varkappa=\sum_{n=0}^{\infty}p^nH_n\ ,\ \ \upsilon\omega_\varkappa=\sum_{n=0}^{\infty}p^nD_n \tag{24}$$

are the He polynomials [28]. The first few components of these polynomials are given by

$$\begin{aligned} H_0 &= \omega_0\upsilon_{0\varkappa}, \\ H_1 &= \omega_0\upsilon_{1\varkappa}+\omega_1\upsilon_{0\varkappa}, \\ H_2 &= \omega_0\upsilon_{2\varkappa}+\omega_2\upsilon_{0\varkappa}+\omega_1\upsilon_{1\varkappa}, \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} D_0 &= \upsilon_0\omega_{0\varkappa}, \\ D_1 &= \upsilon_0\omega_{1\varkappa}+\upsilon_1\omega_{0\varkappa}, \\ D_2 &= \upsilon_0\omega_{2\varkappa}+\upsilon_2\omega_{0\varkappa}+\upsilon_1\omega_{1\varkappa}, \\ &\vdots \end{aligned}$$

Substituting (23) and (24) in (22), we get

$$\begin{cases} \sum_{n=0}^{\infty}p^n\upsilon_n(\varkappa,\zeta)=e^{\varkappa}+\frac{\zeta^\sigma}{\Gamma(\sigma+1)}-p\left(Z^{-1}\left(\frac{\upsilon^\sigma}{s^\sigma}Z\left[\sum_{n=0}^{\infty}p^nH_n+\sum_{n=0}^{\infty}p^n\upsilon_n(\varkappa,\zeta)\right]\right)\right) \\ \sum_{n=0}^{\infty}p^n\omega_n(\varkappa,\zeta)=e^{-\varkappa}+\frac{\zeta^\delta}{\Gamma(\delta+1)}+p\left(Z^{-1}\left(\frac{\upsilon^\delta}{s^\delta}Z\left[\sum_{n=0}^{\infty}p^nD_n+\sum_{m=0}^{\infty}p^n\omega_n(\varkappa,\zeta)\right]\right)\right) \end{cases}. \tag{25}$$

By comparing the both sides of (25), the recursive relations are given by

$$\begin{array}{l} p^0: \upsilon_0(\varkappa,\zeta) = e^{\varkappa} + \frac{\zeta^\sigma}{\Gamma(\sigma+1)} \\ p^1: \upsilon_1(\varkappa,\zeta) = -Z^{-1}\left(\frac{v^\sigma}{s^\sigma} Z\left[H_0 + \upsilon_0(\varkappa,\zeta)\right]\right) \\ p^2: \upsilon_2(\varkappa,\zeta) = -Z^{-1}\left(\frac{v^\sigma}{s^\sigma} Z\left[H_1 + \upsilon_1(\varkappa,\zeta)\right]\right) \\ p^3: \upsilon_3(\varkappa,\zeta) = -Z^{-1}\left(\frac{v^\sigma}{s^\sigma} Z\left[H_2 + \upsilon_2(\varkappa,\zeta)\right]\right) \\ \vdots \\ p^{n+1}: \upsilon_{n+1}(\varkappa,\zeta) = -Z^{-1}\left(\frac{v^\sigma}{s^\sigma} Z\left[H_n + \upsilon_n(\varkappa,\zeta)\right]\right), \ \ n \geq 0, \end{array} \tag{26}$$

and

$$\begin{array}{l} p^0: \omega_0(\varkappa,\zeta) = e^{-\varkappa} + \frac{\zeta^\delta}{\Gamma(\delta+1)} \\ p^1: \omega_1(\varkappa,\zeta) = Z^{-1}\left(\frac{v^\delta}{s^\delta} Z\left[D_0 + \omega_0(\varkappa,\zeta)\right]\right) \\ p^2: \omega_2(\varkappa,\zeta) = Z^{-1}\left(\frac{v^\delta}{s^\delta} Z\left[D_1 + \omega_1(\varkappa,\zeta)\right]\right) \\ p^3: \omega_3(\varkappa,\zeta) = Z^{-1}\left(\frac{v^\delta}{s^\delta} Z\left[D_2 + \omega_2(\varkappa,\zeta)\right]\right) \\ \vdots \\ p^{n+1}: \omega_{n+1}(\varkappa,\zeta) = Z^{-1}\left(\frac{v^\delta}{s^\delta} Z\left[D_n + \omega_n(\varkappa,\zeta)\right]\right), \ \ n \geq 0. \end{array} \tag{27}$$

The first few components of $u_n(\varkappa,\zeta)$ and $w_n(\varkappa,\zeta)$ are

$$\left\{ \begin{array}{l} \upsilon_1(\varkappa,\zeta) = -\dfrac{1+e^{\varkappa}}{\Gamma(\sigma+1)}\zeta^\sigma - \dfrac{e^{\varkappa}}{\Gamma(\sigma+\delta+1)}\zeta^{\sigma+\delta} - \dfrac{\zeta^{2\sigma}}{\Gamma(2\sigma+1)} \\ w_1(\varkappa,\zeta) = \dfrac{-1+e^{-\varkappa}}{\Gamma(\delta+1)}\zeta^\delta - \dfrac{e^{-\varkappa}}{\Gamma(\sigma+\delta+1)}\zeta^{\sigma+\delta} + \dfrac{\zeta^{2\delta}}{\Gamma(2\delta+1)} \end{array} \right. ,$$

and

$$\begin{array}{rl} u_2(\varkappa,\zeta) & = \frac{2+e^{\varkappa}}{\Gamma(2\sigma+1)}\zeta^{2\sigma} + \frac{e^{\varkappa}-1}{\Gamma(\sigma+\delta+1)}\zeta^{\sigma+\delta} \\ & + \left(1 + 2e^{\varkappa} + \frac{\Gamma(\sigma+\delta+1)e^{\varkappa}}{\Gamma(\sigma+1)\Gamma(\delta+1)}\right)\frac{\zeta^{2\sigma+\delta}}{\Gamma(2\sigma+\delta+1)} \\ & + \frac{\Gamma(\sigma+2\delta+1)e^{\varkappa}}{\Gamma(\delta+1)\Gamma(\sigma+\delta+1)\Gamma(2\sigma+2\delta+1)}\zeta^{2\sigma+2\delta} \\ & - \frac{e^{\varkappa}}{\Gamma(\sigma+2\delta+1)}\zeta^{\sigma+2\delta} + \frac{\zeta^{3\sigma}}{\Gamma(3\sigma+1)} \\ w_2(\varkappa,\zeta) & = \frac{-2+e^{-\varkappa}}{\Gamma(2\delta+1)}\zeta^{2\delta} + \frac{1+e^{-\varkappa}}{\Gamma(\sigma+\delta+1)}\zeta^{\sigma+\delta} \\ & + \left(2 - e^{-\varkappa} + \frac{\Gamma(\sigma+\delta+1)e^{-\varkappa}}{\Gamma(\sigma+1)\Gamma(\delta+1)}\right)\frac{\zeta^{\sigma+2\delta}}{\Gamma(\sigma+2\delta+1)} \\ & + \frac{\Gamma(2\sigma+\delta+1)e^{-\varkappa}}{\Gamma(\sigma+1)\Gamma(\sigma+\delta+1)\Gamma(2\sigma+2\delta+1)}\zeta^{2\sigma+2\delta} \\ & + \frac{e^{-\varkappa}}{\Gamma(2\sigma+\delta+1)}\zeta^{2\sigma+\delta} + \frac{\zeta^{3\delta}}{\Gamma(3\delta+1)}. \end{array}$$

By continuing in the same way, we find the other components.
At last, the series solution $\upsilon(\varkappa,\zeta)$ and $\omega(\varkappa,\zeta)$ of (19) are given by

$$\begin{array}{rl} \upsilon(\varkappa,\zeta) & = e^{\varkappa} - \dfrac{e^{\varkappa}}{\Gamma(\sigma+1)}\zeta^\sigma + \dfrac{1+e^{\varkappa}}{\Gamma(2\sigma+1)}\zeta^{2\sigma} \\ & -\dfrac{1}{\Gamma(\sigma+\delta+1)}\zeta^{\sigma+\delta} + \left(1 + 2e^{\varkappa} + \dfrac{\Gamma(\sigma+\delta+1)e^{\varkappa}}{\Gamma(\sigma+1)\Gamma(\delta+1)}\right)\dfrac{\zeta^{2\sigma+\delta}}{\Gamma(2\sigma+\delta+1)} \\ & +\dfrac{\Gamma(\sigma+2\delta+1)e^{\varkappa}}{\Gamma(\delta+1)\Gamma(\sigma+\delta+1)\Gamma(2\sigma+2\delta+1)}\zeta^{2\sigma+2\delta} \\ & -\dfrac{e^{\varkappa}}{\Gamma(\sigma+2\delta+1)}\zeta^{\sigma+2\delta} + \dfrac{\zeta^{3\sigma}}{\Gamma(3\sigma+1)} + ... \end{array}$$

$$
\begin{aligned}
\omega(\varkappa,\zeta) &= e^{-\varkappa} + \frac{e^{-\varkappa}}{\Gamma(\delta+1)}\zeta^{\delta} + \frac{-1+e^{-\varkappa}}{\Gamma(2\delta+1)}\zeta^{2\delta} + \frac{e^{-\varkappa}}{\Gamma(\sigma+\delta+1)}\zeta^{\sigma+\delta} \\
&+ \left(2 - e^{-\varkappa} + \frac{\Gamma(\sigma+\delta+1)e^{-\varkappa}}{\Gamma(\sigma+1)\Gamma(\delta+1)}\right)\frac{\zeta^{\sigma+2\delta}}{\Gamma(\sigma+2\delta+1)} \\
&+ \frac{\Gamma(2\sigma+\delta+1)e^{-\varkappa}}{\Gamma(\sigma+1)\Gamma(\sigma+\delta+1)\Gamma(2\sigma+2\delta+1)}\zeta^{2\sigma+2\delta} \\
&+ \frac{e^{-\varkappa}}{\Gamma(2\sigma+\delta+1)}\zeta^{2\sigma+\delta} + \frac{\zeta^{3\delta}}{\Gamma(3\delta+1)} + \ldots
\end{aligned}
$$

When $\sigma = 1$ and $\delta = 1$, the series solutions of (19) are

$$
\left\{
\begin{array}{l}
\upsilon(\varkappa,\zeta) = u_0(\varkappa,\zeta) + u_1(\varkappa,\zeta) + u_2(\varkappa,\zeta) + \ldots = e^{\varkappa}\left(1 - \zeta + \frac{\zeta^2}{2!} - \frac{\zeta^3}{3!} + \ldots\right) = e^{\varkappa-\zeta} \\
\omega(\varkappa,\zeta) = \omega_0(\varkappa,\zeta) + \omega_1(\varkappa,\zeta) + \omega_2(\varkappa,\zeta) + \ldots = e^{-\varkappa}\left(1 + \zeta + \frac{\zeta^2}{2!} + \frac{\zeta^3}{3!} + \ldots\right) = e^{-\varkappa+\zeta},
\end{array}
\right.
$$

they represent the solutions exact solution of (19)-(20) given in [29].

Example 2. *Now, we consider the following nonlinear system of partial differential equations with time-fractional derivatives*

$$
\left\{
\begin{array}{ll}
{}^{c}D_{\zeta}^{\sigma}\upsilon = -\upsilon - h_{\varkappa}w_y + h_y\omega_{\varkappa}, & 0 \le \sigma < 1 \\
{}^{c}D_{\zeta}^{\delta}h = h, & 0 \le \delta < 1 \\
{}^{c}D_{\zeta}^{\mu}\omega = \omega - \upsilon_{\varkappa}w_{\varkappa} - u_y\omega_y, & 0 \le \mu < 1
\end{array}
\right. \tag{28}
$$

subject the initial conditions

$$
\upsilon(\varkappa,y,0) = \varkappa + y\,; \quad h(\varkappa,y,0) = 1 + \varkappa - y; \quad \omega(\varkappa,y,0) = -\varkappa + y. \tag{29}
$$

By applying the ZZ transform with its differentiation property on (28) and using the initial conditions (29), we get

$$
\left\{
\begin{array}{l}
Z[\upsilon] = \varkappa + y + \frac{\upsilon^{\sigma}}{s^{\sigma}}Z[-\upsilon - h_{\varkappa}w_y + h_y\omega_{\varkappa}], \\
Z[h] = 1 + \varkappa - y + \frac{\upsilon^{\delta}}{s^{\delta}}Z[h], \\
Z[\omega] = -\varkappa + y + \frac{\upsilon^{\mu}}{s^{\mu}}Z[\omega - \upsilon_{\varkappa}\omega_{\varkappa} - \upsilon_y\omega_y].
\end{array}
\right. \tag{30}
$$

The inverse ZZ transform on both sides of (30) gives

$$
\left\{
\begin{array}{l}
\upsilon(\varkappa,y,\zeta) = \varkappa + y + Z^{-1}\left(\frac{\upsilon^{\sigma}}{s^{\sigma}}Z[-\upsilon - h_{\varkappa}\omega_y + h_y\omega_{\varkappa}]\right), \\
h(\varkappa,y,\zeta) = 1 + \varkappa - y + Z^{-1}\left(\frac{\upsilon^{\delta}}{s^{\delta}}Z[h]\right), \\
\omega(\varkappa,y,\zeta) = -\varkappa + y + Z^{-1}\left(\frac{\upsilon^{\mu}}{s^{\mu}}Z[\omega - \upsilon_{\varkappa}\omega_{\varkappa} - \upsilon_y\omega_y]\right).
\end{array}
\right. \tag{31}
$$

The solution forms an infinite series represent as

$$
\upsilon(\varkappa,y,\zeta) = \sum_{n=0}^{\infty} p^n\upsilon_n(\varkappa,y,\zeta), \quad h(\varkappa,y,\zeta) = \sum_{n=0}^{\infty} p^n h_n(\varkappa,y,\zeta), \omega(\varkappa,y,\zeta) = \sum_{n=0}^{\infty} p^n\omega_n(\varkappa,y,\zeta). \tag{32}
$$

The nonlinear terms

$$
h_{\varkappa}\omega_y = \sum_{n=0}^{\infty} p^n A_n\,; \quad h_y\omega_{\varkappa} = \sum_{n=0}^{\infty} p^n B_n\,; \quad \upsilon_{\varkappa}\omega_{\varkappa} = \sum_{n=0}^{\infty} p^n C_n\,; \quad \upsilon_y\omega_y = \sum_{n=0}^{\infty} p^n D_n \tag{33}
$$

are the He polynomials. Substituting (32) and (33) in (31), we get

$$\begin{cases} \sum_{n=0}^{\infty} p^n \upsilon_n(\varkappa, y, \zeta) = \varkappa + y + pZ^{-1}\left(\frac{v^{\sigma}}{s^{\sigma}} Z\left[-\sum_{n=0}^{\infty} p^n \upsilon_n - \sum_{n=0}^{\infty} p^n A_n + \sum_{n=0}^{\infty} p^n B_n\right]\right), \\ \sum_{n=0}^{\infty} p^n h_n(\varkappa, y, \zeta) = 1 + \varkappa - y + pZ^{-1}\left(\frac{v^{\delta}}{s^{\delta}} Z\left[\sum_{n=0}^{\infty} p^n h_n\right]\right), \\ \sum_{n=0}^{\infty} p^n \omega_n(\varkappa, y, \zeta) = -\varkappa + y + pZ^{-1}\left(\frac{v^{\mu}}{s^{\mu}} Z\left[\sum_{n=0}^{\infty} p^n \omega_n - \sum_{n=0}^{\infty} p^n C_n - \sum_{n=0}^{\infty} p^n D_n\right]\right). \end{cases} \tag{34}$$

We compare the both sides of (34), then we get the recursive relations

$$\begin{array}{l} \upsilon_0(\varkappa, y, \zeta) = \varkappa + y \\ \upsilon_1(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\sigma}}{s^{\sigma}} Z\left[-\upsilon_0 - A_0 + B_0\right]\right) \\ \upsilon_2(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\sigma}}{s^{\sigma}} Z\left[-\upsilon_1 - A_1 + B_1\right]\right) \\ \upsilon_3(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\sigma}}{s^{\sigma}} Z\left[-\upsilon_2 - A_2 + B_2\right]\right) \\ \vdots \\ \upsilon_{n+1}(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\sigma}}{s^{\sigma}} Z\left[-\upsilon_n - A_n + B_n\right]\right), \ \ n \geq 0, \end{array} \tag{35}$$

and

$$\begin{array}{l} h_0(\varkappa, y, \zeta) = 1 + \varkappa - y \\ h_1(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\delta}}{s^{\delta}} Z\left[h_0\right]\right) \\ h_2(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\delta}}{s^{\delta}} Z\left[h_1\right]\right) \\ h_3(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\delta}}{s^{\delta}} Z\left[h_2\right]\right) \\ \vdots \\ h_{n+1}(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\delta}}{s^{\delta}} Z\left[h_n\right]\right), \ \ n \geq 0. \end{array} \tag{36}$$

Finally

$$\begin{array}{l} \omega_0(\varkappa, y, \zeta) = -\varkappa + y \\ \omega_1(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\mu}}{s^{\mu}} Z\left[\omega_0 - C_0 - D_0\right]\right) \\ \omega_2(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\mu}}{s^{\mu}} Z\left[\omega_1 - C_1 - D_1\right]\right) \\ \omega_3(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\mu}}{s^{\mu}} Z\left[\omega_2 - C_2 - D_2\right]\right) \\ \vdots \\ \omega_{n+1}(\varkappa, y, \zeta) = Z^{-1}\left(\frac{v^{\mu}}{s^{\mu}} Z\left[\omega_n - C_n - D_n\right]\right), \ \ n \geq 0. \end{array} \tag{37}$$

The first few components of $\upsilon_n(\varkappa, y, \zeta), h_n(\varkappa, y, \zeta)$ and $\omega_n(\varkappa, y, \zeta)$ are

$$\begin{cases} \upsilon_1(\varkappa, y, \zeta) & = Z^{-1}\left(\frac{v^{\sigma}}{s^{\sigma}} Z\left[-\upsilon_0 - h_{0\varkappa}\omega_{0y} + h_{0y}\omega_{0x}\right]\right) \\ & = -(\varkappa + y)\frac{\zeta^{\sigma}}{\Gamma(\sigma+1)} \\ \\ h_1(\varkappa, y, \zeta) & = Z^{-1}\left(\frac{v^{\delta}}{s^{\delta}} Z\left[h_0\right]\right) \\ & = (1 + \varkappa - y)\frac{\zeta^{\delta}}{\Gamma(\delta+1)} \\ \\ \omega_1(\varkappa, y, \zeta) & = Z^{-1}\left(\frac{v^{\mu}}{s^{\mu}} Z\left[\omega_0 - \upsilon_{0\varkappa}\omega_{0\varkappa} - \upsilon_{0y}\omega_{0y}\right]\right) \\ & = (-\varkappa + y)\frac{\zeta^{\mu}}{\Gamma(\mu+1)} \end{cases}$$

And the second component of the solutions are given by the formulas

$$\begin{cases} \upsilon_2(\varkappa,y,\zeta) & = Z^{-1}\left(\frac{\upsilon^\sigma}{s^\sigma} Z\left[-\upsilon_1 - h_{0\varkappa}\omega_{1y} - h_{1\varkappa}\omega_{0y} + h_{0y}\omega_{1\varkappa} + h_{1y}\omega_{0\varkappa}\right]\right) \\ & = (\varkappa+y)\frac{\zeta^{2\sigma}}{\Gamma(2\sigma+1)} \\ h_2(\varkappa,y,\zeta) & = Z^{-1}\left(\frac{\upsilon^\delta}{s^\delta} Z\left[h_1\right]\right) \\ & = (1+\varkappa-y)\frac{\zeta^{2\delta}}{\Gamma(2\delta+1)} \\ \omega_2(\varkappa,y,\zeta) & = Z^{-1}\left(\frac{\upsilon^\mu}{s^\mu} Z\left[w_1 - \upsilon_{0\varkappa}\omega_{1\varkappa} - \upsilon_{1\varkappa}\omega_{0\varkappa} - \upsilon_{0y}\omega_{1y} - \upsilon_{1y}\omega_{0y}\right]\right) \\ & = (-\varkappa+y)\frac{\zeta^{2\mu}}{\Gamma(2\mu+1)}. \end{cases}$$

By contination the calculations, we find

$$\begin{aligned} \upsilon_3(\varkappa,y,\zeta) & = Z^{-1}\left(\frac{\upsilon^\sigma}{s^\sigma} Z\left[-\upsilon_2 - A_2 + B_2\right]\right) \\ & = -(\varkappa+y)\frac{\zeta^{2\sigma}}{\Gamma(2\sigma+1)} \\ & \vdots \\ \upsilon_n(\varkappa,y,\zeta) & = (-1)^n(\varkappa+y)\frac{\zeta^{n\sigma}}{\Gamma(n\sigma+1)}, \\ \textit{et} & \\ h_3(\varkappa,y,\zeta) & = Z^{-1}\left(\frac{\upsilon^\delta}{s^\delta} Z\left[h_2\right]\right) \\ & = (1+\varkappa-y)\frac{\zeta^{3\delta}}{\Gamma(3\delta+1)} \\ & \vdots \\ h_n(\varkappa,y,\zeta) & = (1+\varkappa-y)\frac{\zeta^{n\delta}}{\Gamma(n\delta+1)}. \\ \textit{et} & \\ \omega_3(\varkappa,y,\zeta) & = Z^{-1}\left(\frac{\upsilon^\mu}{s^\mu} Z\left[\omega_2 - C_2 - D_2\right]\right) \\ & = (-\varkappa+y)\frac{\zeta^{3\mu}}{\Gamma(3\mu+1)} \\ & \vdots \\ \omega_n(\varkappa,y,\zeta) & = (-\varkappa+y)\frac{\zeta^{n\mu}}{\Gamma(n\mu+1)}. \end{aligned} \tag{38}$$

Finally, the series solutions of (28) are given by

$$\begin{aligned} \upsilon(\varkappa,y,\zeta) & = \sum_{n=0}^{\infty}\upsilon_n(\varkappa,y,\zeta) \\ & = (\varkappa+y)\left(1 - \frac{\zeta^\sigma}{\Gamma(\sigma+1)} + \frac{\zeta^{2\sigma}}{\Gamma(2\sigma+1)} - \frac{\zeta^{3\sigma}}{\Gamma(3\sigma+1)} + \ldots \pm \frac{\zeta^{n\sigma}}{\Gamma(n\sigma+1)} \pm \ldots\right) \\ & = (\varkappa+y)\sum_{n=0}^{\infty}\frac{(-\zeta^\sigma)^n}{\Gamma(n\sigma+1)} \\ & = (\varkappa+y)E_\sigma(-\zeta^\sigma), \end{aligned} \tag{39}$$

$$\begin{aligned} h(\varkappa,y,\zeta) & = \sum_{n=0}^{\infty}h_n(\varkappa,y,\zeta) \\ & = (1+\varkappa-y)\left(1 + \frac{\zeta^\delta}{\Gamma(\delta+1)} + \frac{\zeta^{2\delta}}{\Gamma(2\delta+1)} + \frac{\zeta^{3\delta}}{\Gamma(3\delta+1)} + \ldots + \frac{\zeta^{n\delta}}{\Gamma(n\delta+1)} + \ldots\right) \\ & = (1+\varkappa-y)\sum_{n=0}^{\infty}\frac{(\zeta^\delta)^n}{\Gamma(n\delta+1)} \\ & = (1+\varkappa-y)E_\delta(\zeta^\delta), \end{aligned} \tag{40}$$

$$\begin{aligned} \omega(\varkappa,y,\zeta) & = \sum_{n=0}^{\infty}w_n(\varkappa,y,\zeta) \\ & = (-\varkappa+y)\left(1 + \frac{\zeta^\mu}{\Gamma(\mu+1)} + \frac{(\zeta^\mu)^2}{\Gamma(2\mu+1)} + \frac{(\zeta^\mu)^3}{\Gamma(3\mu+1)} + \ldots + \frac{(\zeta^\mu)^n}{\Gamma(n\mu+1)} + \ldots\right) \\ & = (-\varkappa+y)\sum_{n=0}^{\infty}\frac{(\zeta^\mu)^n}{\Gamma(n\mu+1)} \\ & = (-\varkappa+y)E_\mu(\zeta^\mu). \end{aligned} \tag{41}$$

When $\sigma=1$, $\delta=1$ and $\mu=1$, we get

$$\begin{cases} \upsilon(\varkappa,y,\zeta) = (\varkappa+y)E_\sigma(-\zeta) = (\varkappa+y)e^{-\zeta}. \\ h(\varkappa,y,\zeta) = (1+\varkappa-y)E_\delta(\zeta) = (1+\varkappa-y)e^{\zeta}. \\ \omega(\varkappa,y,\zeta) = (-\varkappa+y)E_\mu(\zeta) = (-\varkappa+y)e^{\zeta}, \end{cases} \tag{42}$$

which are the solutions of our nonlinear system (28)-(29) presented in [29]).

5. Conclusions

In this work, we applied a precise analytical method called the homotopy perturbation ZZ transform method (HPZZTM) to solve the nonlinear system of fractional partial differential equations. This method is a combination of two methods: the homotopy perturbation method and the ZZ transform method. In our main study, we extend the study of the work presented in [23], which deals with the solution of some nonlinear fractional differential equations. Where we have extending the HPZZTM method to obtain analytical solutions of nonlinear systems of fractional partial differential equations. This algorithm is easy to apply and effective in reaching the desired results, as illustrated by the examples of coupled and triple nonlinear systems that we have solved. These results lead us to say that this algorithm is powerful and effective to apply to this type of systems, and thus can be applied to the others nonlinear system of fractional partial differential equations without or with variable coefficients of engineering or medical sciences as: Rotavirus epidemic system and Susceptible-Infected-Recovered (SIR) and other nonlinear problems [30].

Acknowledgements

This work was supported by the Algerian Ministry of Higher Education and Scientific Research under the PRFU-Project: COOL03EP310220210001.

Declaration of Competing Interest

The author declares that there is no conflict of interest.

Authorship Contribution Statement

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