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Research Article

On The Maximum Spherical Inversions

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Abstract

In this study, the maximum spherical inversion in the maximum space, the Cartesian 3-space endowed with the maximum metric is defined and introduced. Firstly, the formula for computing the inverse of a point P with respect to the maximum sphere is given. Then, some basic properties of the maximum spherical inversion map are studied. The obtained results related to invariant line, the plane and the maximum sphere under the maximum spherical inversion are presented. In addition, the cross ratio and the harmonic conjugate in maximum space are given. Afterwars, the maximum spherical inverses of the cross ratio and the harmonic conjugate are examined.

Keywords: Maximum metric, spherical inversion, maximum space, cross ratio.

Maksimum Küresel İnversiyonlar Üzerine

Öz

Bu çalışmada, maksimum uzayda maksimum küresel inversiyon tanımlanıp, sunulmuştur. İlk olarak, maksimum küresine göre bir P noktasının inversi için formül verilmiştir. Daha sonra maksimum küresel inversiyon dönüşümünün bazı temel özellikleri incelenmiştir. Maksimum küresel inversiyon altında invaryant kalan doğru, düzlem ve maksimum küre ile ilgili elde edilen sonuçlar sunulmuştur. Bunlara ek olarak, maksimum uzayda çifte oran ve harmonik eşlenik kavramları verilmiştir. Sonrasında, çifte oran ve harmonik eşleniğin maksimum küresel inversleri incelenmiştir.

Anahtar Kelimeler: Maksimum metrik, küresel inversiyon, maksimum uzay, çifte oran.

1. Introduction

The circle inversion is one of the interesting maps in Euclidean plane, (Blair, 2000). The circle inversion was presented in the work with the title "Plane Loci" by Apollonoius of Perga. It was systematically developed by Steiner in the 1830s. It can be used to examine some theorems and problems in geometry, for example the Pappus chain theorem, Feuerbach's theorem, Ptolemi theorem, Steiner Porism, Apollonoius's problem, (Ramirez et al, 2015). There are many generalizations of the inversion map in the literature. For example, different inversion maps have been obtained by using other objects such as parallel lines, central cones, star shape sets instead of the inversion circle, (Childress, 1965; Nickel, J. 1995; Ramirez, 2013; Gdawiec, 2014; Ramirez, 2014; Ramirez and Rubian, 2014). The new inversion maps have been defined

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by taking different distance function such as taxicab distance, p-distance, Chinese Checkers distance, α-distance, (Bayar ve Ekmekçi, 2014; Gelişgen ve Ermiş, 2019; Nickel, 1995; Pekzorlu, 2019; Ramirez et al, 2015; Yüca ve Can, 2020). The spherical inversion has been defined as the generalization of the circle inversion in the three dimensional Euclidean space and its properties have been given, in (Ramirez and Rubian, 2016). An inversion with respect to an ellipsoid has been defined as the generalization of the spherical inversion, (Ramirez and Rubian, 2016). The new inversion maps with respect to a taxicab sphere and a Chinese Checkers sphere have been defined in the taxicab space and the Chinese Checkers space, respectively, (Pekzorlu ve Bayar, 2020 (a); Pekzorlu ve Bayar, 2020 (b)). There have been many studies that contributed to the literature in non-Euclidean planes and spaces, (Akça ve Kaya, 1997; Akça ve Kaya, 2004; Kaya et al, 2000; Bayar ve Kaya, 2011; Bayar ve Ekmekçi , 2015).

In this study, the inversion map with respect to the maximum sphere is defined in the three dimensional space with the maximum metric and some properties of it are given. Then the maximum spherical inverses of the cross ratio, the harmonic conjugate, a line, a plane and a maximum sphere are examined in the maximum space.

The Euclidean distance between points X and Y is

$$d_E(X,Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ are two points $\mathbb{R}^3$. The maximum distance between these points is

$$d_M(X,Y) = \max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\}.$$

The maximum space is constructed by simply replacing the Euclidean distance d_E by the maximum distance d_M and is denoted by $\mathbb{R}^{\mathbf{3}}_{\mathbf{M}}$. Linear structure of the maximum space except for the distance function is the same as the Euclidean space. Points, lines, planes of maximum space are the same as Euclidean space, angles are measured in the same way, (Ermiş ve Kaya, 2013).

In $\mathbb{R}^{\mathbf{3}}_{\mathbf{M}}$, the maximum sphere centered at the point $M = (m_1, m_2, m_3)$ and radius r is

$$\mathcal{K} = \{X : d_M(M,X) = r, r > 0\},$$

that is

$$\{(x,y,z) : \max\{|x - m_1|, |y - m_2|, |z - m_3| = r\}.$$

The maximum unit sphere is

$$\mathcal{K} = \{(x,y,z) : \max\{|x|, |y|, |z| = 1\},$$

that is the cube.

2. Material and Methods

2.1. The Maximum Spherical Inversion

Let $\mathcal{K}$ be the maximum sphere centered at the point O and radius r in $\mathbb{R}^3_{\mathrm{M}}$. The maximum spherical inversion in $\mathcal{K}$ is

$$I_{\mathcal{K}}: \mathbb{R}^{\mathbf{3}}_{\mathbf{M}} \backslash \{O\} \to \mathbb{R}^{\mathbf{3}}_{\mathbf{M}} \backslash \{O\}$$

$$P \to I_{\mathcal{K}}(P) = P'$$

where the point P' is on the ray $\overrightarrow{OP}$ and $d_M(O,P).d_M(O,P') = r^2$. $\mathcal{K}$ is called the sphere of the maximum spherical inversion; O is called the center of the maximum spherical inversion; r is called the radius of the maximum spherical inversion; and the point P' is said the maximum spherical inverse of the point P with respect to $I_{\mathcal{K}}$. Since $I_{\mathcal{K}}\big(I_{\mathcal{K}}(P)\big) = P$ for every point $P \neq O$ in $\mathbb{R}^3_{\mathrm{M}}$, the maximum spherical inversion $I_{\mathcal{K}}$ is an involutive map.

In Euclidean space, the points inside the sphere of the maximum spherical inversion are mapped to the points outside the sphere of the maximum inversion, and conversely. In the following theorem, it is stated that this property is valid in the maximum space.

Theorem 2.1. The maximum spherical inversion moves each point outside of the (except for the center of inversion) sphere of inversion to the point inside of the sphere, and conversely.

Proof. Let $\mathcal{K}$ and $I_{\mathcal{K}}$ be the maximum sphere centered at the point O and radius r and the maximum spherical inversion in $\mathbb{R}^3_{\mathrm{M}}$, respectively. For every point P outside $\mathcal{K}$ $d_M(O,P)> r$ is valid. If the point P' is the maximum spherical inverse of P, then $I_{\mathcal{K}}(P) = P'$ and $d_M(O,P).d_M(O,P') = r^2$. It can be seen easily that $r > d_M(O,P')$ and the point P' is inside $\mathcal{K}$.

Theorem 2.2. The maximum spherical inversion leaves every point on the inversion sphere fixed.

Theorem 2.3. Let $I_{\mathcal{K}}$ be the maximum spherical inversion centered at the point O and radius r in $\mathbb{R}^3_{\mathrm{M}}$. If the point P' is the maximum spherical inverse of $\neq O$, then

$$P' - O = \frac{r^2}{\big(d_M(O,P)\big)^2}(P - O).$$

Proof. If the point P' is the maximum spherical inverse of $P \neq O$, then $d_M(O,P).d_M(O,P') = r^2$ and P' is on the ray $\overrightarrow{OP}$. Therefore, $\overrightarrow{OP'} = \lambda\overrightarrow{OP}$, $\lambda \epsilon \mathbb{R}^+$ and $P' - O = \lambda(P - O)$. From the following equality

$$d_M(O,P') = |P' - O|_M = |\lambda(P - O)|_M$$

$$= \lambda|(P - O)|_M = \lambda d_M(O,P),$$

one gets $\lambda = \frac{r^2}{d_M(O,P)^2}$. And then

$$P' - O = \left(\frac{r}{d_M(O,P)}\right)^2 (P - O)$$

is obtained.

Corollary 2.4. Let $I_{\mathcal{K}}$ be the maximum spherical inversion centered at the point $O = (a, b, c)$ and radius r in $\mathbb{R}^3_{\mathrm{M}}$. If the point $P' = (x', y', z')$ is the maximum spherical inverse of the point $= (x, y, z) \neq O$, then

$$x' = \left(\frac{r}{d_M(O,P)}\right)^2 (x - a) + a,$$

$$y' = \left(\frac{r}{d_M(O,P)}\right)^2 (y - b) + b,$$

$$z' = \left(\frac{r}{d_M(O,P)}\right)^2 (z - c) + c,$$

where $d_M(O,P) = \max\{|x - a|, |y - b|, |z - c|\}$.

Corollary 2.5. Let $I_{\mathcal{K}}$ be the maximum spherical inversion centered at the point $O = (0,0,0)$ and radius r in $\mathbb{R}^3_{\mathrm{M}}$. If the point $P' = (x', y', z')$ is the maximum spherical inverse of the point $= (x, y, z) \neq O$, then

$$P' = \frac{r^2}{(d_M(O,P))^2} P$$

and

$$(x', y', z') = \left(\frac{r}{\max\{|x|,|y|,|z|\}}\right)^2 (x, y, z).$$

Theorem 2.6. Let O, P, Q be three different collinear points in $\mathbb{R}^3_{\mathrm{M}}$. According to the maximum spherical centered at the point $O = (0,0,0)$ and radius r, if the maximum spherical inverses of the points P and Q are the points P' and Q', then

$$d_M(P', Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P) d_M(O,Q)}.$$

Proof. Let $I_{\mathcal{K}}$ be the maximum spherical inversion centered at the point $O = (0,0,0)$ and radius r in $\mathbb{R}^3_{\mathrm{M}}$. If P and P'; Q and Q' are pairs of inverse points with respect to maximum spherical inversion $I_{\mathcal{K}}$, then

$$d_M(O,P).\, d_M(O,P') = r^2,$$

$$d_M(O,Q).\, d_M(O,Q') = r^2.$$

Since O, P, Q are collinear points,

$$d_M(P',Q') = |d_M(O,P') - d_M(O,Q')|$$

$$= \left| \frac{r^2}{d_M(O,P)} - \frac{r^2}{d_M(O,Q)} \right|$$

$$= \frac{r^2 d_M(P,Q)}{d_M(O,P)d_M(O,Q)}$$

is obtained.

The equality given in theorem 2.6 is not always valid for three noncollinear points in the maximum space. Now, in the following theorem, the cases where this equality is valid for three noncollinear points are given.

Theorem 2.7. Let O, P, Q be three different noncollinear points in $\mathbb{R}^3_{\mathrm{M}}$ and the points P' and Q' be the inverse points of P and Q with respect to the maximum spherical inversion centered at the point O and radius r. If the directions of the rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ belong to one of the sets D_i i=1,2,3, then

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P)d_M(O,Q)}$$

where

D_1={(1,0,0),(0,1,0),(0,0,1)},

D_2={(1,1,1),(-1,1,1),(1,-1,1),(1,1,-1)},

D_3={(1,$\pm$1,0),(0,1, $\pm$1),(1,0, $\pm$1)}.

Proof. Since all translations are an isometry of the maximum space, without loss of generality, it can be taken the center of maximum spherical inversion as origin. Let $I_{\mathcal{K}}$ be the maximum spherical inversion centered at the point $O = (0,0,0)$ and radius r in $\mathbb{R}^3_{\mathrm{M}}$. We suppose that the directions of the rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ belong to the set D_1. In this situation, if the points $P = (0, p, 0)$ and $Q = (0,0,q)$ are taken, then $d_M(O,P) = |p|$, $d_M(O,Q) = |q|$. The inverse points of P and Q with respect to $I_{\mathcal{K}}$ are $P' = (0, \frac{r^2}{p}, 0)$ ve $Q' = (0,0, \frac{r^2}{q})$, respectively. We get

$$d_M(P',Q') = max\left\{\left|\frac{r^2}{p}\right|, \left|\frac{r^2}{q}\right|\right\}.$$

If $|p| \leq |q|$, then

$$d_M(P',Q') = \frac{r^2}{|p|} = \frac{r^2|q|}{|p||q|} = \frac{r^2 d_M(P,Q)}{d_M(O,P)d_M(O,Q)}.$$

If $|p| > |q|$, then

$$d_M(P', Q') = \frac{r^2}{|q|} = \frac{r^2|p|}{|q||p|} = \frac{r^2 d_M(P,Q)}{d_M(O,P) d_M(O,Q)}.$$

Suppose that the directions of the rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ belong to the set D_2. If $P = (p, -p, p)$ and $Q = (q, q, -q)$ are taken, then $d_M(O, P) = |p|$, $d_M(O, Q) = |q|$. The inverse points of P and Q with respect to $I_{\mathcal{K}}$ are

$P' = (\frac{r^2}{p}, -\frac{r^2}{p}, \frac{r^2}{p})$ and $Q' = (\frac{r^2}{q}, \frac{r^2}{q}, -\frac{r^2}{q})$, respectively. We get

$$d_M(P', Q') = max\left\{\left|\frac{r^2}{p} - \frac{r^2}{q}\right|, \left|\frac{r^2}{p} + \frac{r^2}{q}\right|\right\}.$$

If $|q - p| \leq |q + p|$, then

$$d_M(P', Q') = \frac{r^2|q + p|}{|p||q|} = \frac{r^2 d_M(P, Q)}{d_M(O, P) d_M(O, Q)}.$$

If $|q - p| > |q + p|$, then

$$d_M(P', Q') = \frac{r^2|q-p|}{|p||q|} = \frac{r^2 d_M(P,Q)}{d_M(O,P) d_M(O,Q)}.$$

Suppose that the directions of the rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ belong to the set D_3 . If $P = (p, -p, 0)$ and $Q = (0, q, -q)$ are taken, then $d_M(O, P) = |p|$, $d_M(O, Q) = |q|$. The inverse points of P and Q with respect to $I_{\mathcal{K}}$ are $P' = (\frac{r^2}{p}, -\frac{r^2}{p}, 0)$ and $Q' = (0, \frac{r^2}{q}, -\frac{r^2}{q})$. We get that

$$d_M(P, Q) = max\{|p|, |q|, |p + q|\}$$

and

$$d_M(P', Q') = max\left\{\left|\frac{r^2}{p}\right| \left|\frac{r^2}{p} + \frac{r^2}{q}\right|, \left|\frac{r^2}{q}\right|\right\}$$

$$= \frac{r^2}{|p||q|} max\{|p|, |q|, |p + q|\}$$

$$= \frac{r^2 d_M(P, Q)}{d_M(O, P) d_M(O, Q)}.$$

Similarly, it can be easily seen that the equality is valid for the other cases concerning the sets D_i, i=1,2,3.

Theorem 2.8. The maximum inverse of a plane through the center of the maximum spherical inversion is the plane itself.

Proof. Since translations preserve maximum distance, without loss of generality, it is enough to take the center of maximum spherical inversion as origin. Let $I_{\mathcal{K}}$ be the maximum spherical inversion centered at the point $O=(0,0,0)$ and radius r in $\mathbb{R}^3_{\mathrm{M}}$. $I_{\mathcal{K}}$ transforms the plane with equation $Ax+By+Cz=0$ passing through origin to the plane with equation $Ax'+By'+Cz'=0$. The proof is completed.

Corollary 2.9. The maximum inverse of a line through the center of the maximum spherical inversion is the line itself.

Theorem 2.10. The inverse of a maximum sphere with the inversion center is again a maximum sphere with inversion center.

Proof. Let $I_{\mathcal{K}}$ be the maximum spherical inversion centered at the point $O=(0,0,0)$ and radius r in $\mathbb{R}^3_{\mathrm{M}}$. The maximum sphere centered at the point $O=(0,0,0)$ and radius r' is

$$\mathcal{K}'=\{(x,y,z): \max\{|\mathrm{x}|,|\mathrm{y}|,|\mathrm{z}|\}=\mathrm{r}'\}.$$

The inverse of $\mathcal{K}'$ with respect to $I_{\mathcal{K}}$ is

$$\max\{|\mathrm{x}'|,|\mathrm{y}'|,|\mathrm{z}'|\}=\mathrm{r}''$$

where $r''=\frac{r^2}{r'}$. This is a maximum sphere centered at the point $O=(0,0,0)$ and radius r''.

Theorem 2.11. Every vertex, edge and face of the maximum inversion sphere is invariant with respect to maximum spherical inversion.

Proof. Since the maximum inversion sphere is invariant with respect to maximum spherical inversion, the proof is clear.

3. Main Theorem and Proof

3.1. Cross Ratio

Definition 3.1.1. Let A and B be any two points on a directed line d. The directed maximum length of the line segment $\overline{AB}$ is denoted by $d_M[AB]$. If the line segment $\overline{AB}$ and d have the same direction, then $d_M[AB]=d_M(A,B)$ and if in the opposite direction then $d_M[AB]=-d_M(A,B)$.

If the points A, B, C are on the same directed line and the point C is between the points A and B, then these points are stated in the array ACB , (Salihova, 2006; Özcan and Kaya,2002).

Definition 3.1.2. Let A, B, C and D be four distinct points on the directed line in the maximum space. The maximum cross ratio of these four points is defined by

$$(AB,CD)_M=\frac{d_M[AC]}{d_M[AD]}\cdot\frac{d_M[BD]}{d_M[BC]}.$$

Corollary 3.1.3. Let A, B, C and D be four distinct points on the directed line in the maximum space. The maximum cross ratio $(AB, CD)_M$ is positive when both C and D points are between A and B points or not.

Proof. Suppose that both C and D points be between A and B points. For the array $ACDB$, the maximum cross ratio is

$$(AB, CD)_M = \frac{d_M[AC]}{d_M[AD]} \cdot \frac{d_M[BD]}{d_M[BC]} = \frac{d_M(A, C)}{d_M(A, D)} \cdot \frac{\left(-d_M(B, D)\right)}{\left(-d_M(B, C)\right)} > 0.$$

Now, suppose that neither C nor D points be between A and B points. For the array $CABD$, the maximum cross ratio is

$$(\mathrm{AB}, \mathrm{CD})_\mathrm{M} = \frac{\mathrm{d_M[AC]}}{\mathrm{d_M[AD]}} \cdot \frac{\mathrm{d_M[BD]}}{\mathrm{d_M[BC]}} = \frac{(-\mathrm{d_M(A, C)})}{\mathrm{d_M(A, D)}} \cdot \frac{\mathrm{d_M(B, D)}}{(-\mathrm{d_M(B, C)})} > 0.$$

Similarly, it can be seen that the maximum cross ratio is positive for other arrays of points.

Corollary 3.1.4. Let A, B, C and D be four distinct points on the directed line in the maximum space. If the pairs $\{A, B\}$ and $\{C, D\}$ separate each other, the maximum cross ratio $(AB, CD)_M$ is negative.

Proof. Suppose that the pairs $\{A, B\}$ and $\{C, D\}$ separate each other. For the array ACBD, the maximum cross ratio is

$$(\mathrm{AB}, \mathrm{CD})_\mathrm{M} = \frac{\mathrm{d_M[AC]}}{\mathrm{d_M[AD]}} \cdot \frac{\mathrm{d_M[BD]}}{\mathrm{d_M[BC]}} = \frac{\mathrm{d_M(A, C)}}{\mathrm{d_M(A, D)}} \cdot \frac{\left(-\mathrm{d_M(B, C)}\right)}{\mathrm{d_M(B, D)}} < 0.$$

Similarly, it can be seen that the maximum cross ratio is negative for other arrays of points with the same property.

In Euclidean space, the cross ratio is invariant under spherical inversion such that the points A, B, C and D are different from the center of the inversion sphere. The version in maximum space of this property is given in the following theorem.

Theorem 3.1.5. The maximum cross ratio is invariant under maximum spherical inversions.

Proof. Let $I_{\mathcal{K}}$ be the maximum spherical inversion centered at the point O with the radius r and A, B, C, D be any four collinear points different from O in $\mathbb{R}^3_\mathrm{M}$. Suppose that the points A', B', C', D' are the maximum spherical inverse of these four points with respect to $\mathrm{I}_{\mathcal{K}}$, respectively. The points close to O map to points far from O under $I_{\mathcal{K}}$. So, the directions of the line segment $\overline{AB}$ and its maximum spherical inverse $\overline{A'B'}$ are opposite. Also, the maximum spherical inversion preserves the separation and non separation of the pairs

$\{A, B\}$ and $\{C, D\}$. Therefore, it is enough to show that $|(\mathrm{A'B'}, \mathrm{C'D'})_{\mathrm{M}}| = |(\mathrm{AB}, \mathrm{CD})_{\mathrm{M}}|$.

$$|(\mathrm{A'B'}, \mathrm{C'D'})_{\mathrm{M}}| = \left|\frac{\mathrm{d_M}[\mathrm{A'C'}]}{\mathrm{d_M}[\mathrm{A'D'}]} \cdot \frac{\mathrm{d_M}[\mathrm{B'D'}]}{\mathrm{d_M}[\mathrm{B'C'}]}\right| = \frac{\mathrm{d_M}(\mathrm{A'}, \mathrm{C'})}{\mathrm{d_M}(\mathrm{A'}, \mathrm{D'})} \cdot \frac{\mathrm{d_M}(\mathrm{B'}, \mathrm{D'})}{\mathrm{d_M}(\mathrm{B'}, \mathrm{C'})}$$

$$= \frac{\dfrac{\mathrm{r}^2 \mathrm{d_M}(\mathrm{A}, \mathrm{C})}{\mathrm{d_M}(\mathrm{O}, \mathrm{A}) \cdot \mathrm{d_M}(\mathrm{O}, \mathrm{C})}}{\dfrac{\mathrm{r}^2 \mathrm{d_M}(\mathrm{A}, \mathrm{D})}{\mathrm{d_M}(\mathrm{O}, \mathrm{A}) \cdot \mathrm{d_M}(\mathrm{O}, \mathrm{D})}} \cdot \frac{\dfrac{\mathrm{r}^2 \mathrm{d_M}(\mathrm{B}, \mathrm{D})}{\mathrm{d_M}(\mathrm{O}, \mathrm{B}) \cdot \mathrm{d_M}(\mathrm{O}, \mathrm{D})}}{\dfrac{\mathrm{r}^2 \mathrm{d_M}(\mathrm{B}, \mathrm{C})}{\mathrm{d_M}(\mathrm{O}, \mathrm{B}) \cdot \mathrm{d_M}(\mathrm{O}, \mathrm{C})}} = \frac{\mathrm{d_M}[\mathrm{AC}]}{\mathrm{d_M}[\mathrm{AD}]} \cdot \frac{\mathrm{d_M}[\mathrm{BD}]}{\mathrm{d_M}[\mathrm{BC}]} = |(\mathrm{AB}, \mathrm{CD})_{\mathrm{M}}|$$

is obtained.

3.2. Harmonic Conjugates

Definition 3.2.1. Let A and B be any two points on a line l in $\mathbb{R}^3_{\mathrm{M}}$, any pair C and D on l for which

$$\frac{\mathrm{d_M}[\mathrm{AC}]}{\mathrm{d_M}[\mathrm{CB}]} = \frac{\mathrm{d_M}[\mathrm{AD}]}{\mathrm{d_M}[\mathrm{DB}]}$$

is said to divide A and B harmonically. The points C and D are called maximum harmonic conjugates with respect to A and B, the maximum harmonic set of points is denoted by $H(AB, CD)_M$.

Corollary 3.2.2. Two distinct points C and D are the maximum harmonic conjugates with respect to A and B in $\mathbb{R}^3_{\mathrm{M}}$ if and only if $(AB, CD)_M = -1$.

Theorem 3.2.3. Let $\mathcal{K}$ be the maximum inversion sphere centered at the point $O = (0,0,0)$ and with diameter $\overline{\mathrm{AB}}$ line segment in $\mathbb{R}^3_{\mathrm{M}}$. Let the points C and C' on the ray $\overrightarrow{\mathrm{OB}}$ divide the line segment $\overline{\mathrm{AB}}$ internally and externally, respectively. Then the points C and C' are the maximum harmonic conjugates with respect to the points A and B in $\mathbb{R}^3_{\mathrm{M}}$ if and only if the points C and C' are the maximum spherical inverse points with respect to $\mathcal{K}$.

Proof. Since the points C and C' divide the line segment $\overline{\mathrm{AB}}$ internally and externally, respectively, then

$$\frac{\mathrm{d_M}[\mathrm{AC}]}{\mathrm{d_M}[\mathrm{CB}]} = -\frac{\mathrm{d_M}[\mathrm{AC'}]}{\mathrm{d_M}[\mathrm{C'B}]}.$$

Suppose that the points C and C' are the maximum harmonic conjugates with respect to the points A and B. Then

$$(\mathrm{AB}, \mathrm{CC'})_{\mathrm{M}} = -1$$

and

$$\frac{\mathrm{d_M}[\mathrm{AC}]}{\mathrm{d_M}[\mathrm{AC'}]}.\frac{\mathrm{d_M}[\mathrm{BC'}]}{\mathrm{d_M}[\mathrm{BC}]}=-1$$

where $\mathrm{d_M}[\mathrm{BC}]=-\mathrm{d_M}(\mathrm{C},\mathrm{B})$. Since the point C divides the line segment $\overline{\mathrm{AB}}$ internally,

$$\mathrm{d_M}(\mathrm{C},\mathrm{B})=\mathrm{r}-\mathrm{d_M}(\mathrm{O},\mathrm{C}),$$

$$\mathrm{d_M}(\mathrm{A},\mathrm{C})=\mathrm{r}-\mathrm{d_M}(\mathrm{O},\mathrm{C}).$$

Since the point C' divides the line segment $\overline{\mathrm{AB}}$ externally,

$$\mathrm{d_M}(\mathrm{A},\mathrm{C'})=\mathrm{r}+\mathrm{d_M}(\mathrm{O},\mathrm{C'}),$$

$$\mathrm{d_M}(\mathrm{B},\mathrm{C'})=\mathrm{d_M}(\mathrm{O},\mathrm{C'})-\mathrm{r}.$$

From the equality

$$\frac{\left(\mathrm{r}+\mathrm{d_M}(\mathrm{O},\mathrm{C})\right)}{\left(\mathrm{r}+\mathrm{d_M}(\mathrm{O},\mathrm{C'})\right)}.\frac{\left(\mathrm{d_M}(\mathrm{O},\mathrm{C'})-\mathrm{r}\right)}{\left(-\mathrm{r}+\mathrm{d_M}(\mathrm{O},\mathrm{C})\right)}=-1,$$

$\mathrm{d_M}(\mathrm{O},\mathrm{C}).\mathrm{d_M}(\mathrm{O},\mathrm{C'})=\mathrm{r}^2$ is obtained. It shows that the points C and C' are the maximum spherical inverse.

Conversely, we suppose that the points C and C' are the maximum spherical inverse. Then

$$\mathrm{d_M}(\mathrm{O},\mathrm{C}).\mathrm{d_M}(\mathrm{O},\mathrm{C'})=\mathrm{r}^2.$$

From

$$(\mathrm{AB},\mathrm{CC'})_{\mathrm{M}}=\frac{\mathrm{d_M}[\mathrm{AC}]}{\mathrm{d_M}[\mathrm{AC'}]}.\frac{\mathrm{d_M}[\mathrm{BC'}]}{\mathrm{d_M}[\mathrm{BC}]},$$

$(\mathrm{AB},\mathrm{CC'})_{\mathrm{M}}=-1$ is obtained. This shows that the points C and C' are the maximum harmonic conjugates with respect to the points A and B.

4. Conclusion

In this study, we define and introduce the maximum spherical inversion in the maximum space, the Cartesian 3-space endowed with the maximum metric. Firstly, we give the formula depending on the cartesian coordinates to compute the inverse of a point P with respect to the maximum sphere. Then, we explore some well-known basic properties of the maximum spherical inversion map. We present the obtained results related to invariant line, the plane and the maximum sphere under the maximum spherical inversion. Also, we give the cross ratio and the harmonic conjugate in maximum space and study the maximum spherical inverses of cross ratio and harmonic conjugate. We think that it will contribute to the literature in non-Euclidean spaces.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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