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Quasi-Rational and Rational Solutions to the Defocusing Nonlinear Schrödinger Equation

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Abstract

Quasi-rational solutions to the defocusing nonlinear Schrödinger equation (dNLS) in terms of wronskians and Fredholm determinants of order $2N$ depending on $2N-2$ real parameters are given. We get families of quasi-rational solutions to the dNLS equation expressed as a quotient of two polynomials of degree $N(N+1)$ in the variables x and t. We present also rational solutions as a quotient of determinants involving certain particular polynomials.

1. Introduction

We consider the one dimensional defocusing nonlinear Schrödinger equation (dNLS) which can be written in the form

$$iv_t + v_{xx} - 2|v|^2 v = 0. \tag{1.1}$$

Nakemura and Hirota presented solutions to this equation in terms of wronskians in 1985 [1] using bilinear method. They constructed rational solutions by using a connection with a Bäcklund transformation for the classical Boussinesq system (BS)

$$\begin{cases} u_t = ((1+u)v + a^2 v_{xx})_x, \\ v_t = (u + \frac{1}{2}v^2)_x. \end{cases} \tag{1.2}$$

Hone [2] constructed rational solutions in terms of determinant by using Crum dressing method in 1997. In 1999, Barran and Kovalyov presented slowly oscillatory decaying solutions in terms of determinants [3].
Clarkson presented rational solutions and rational-oscillatory solutions expressed in terms of special polynomials associated with rational solutions of the fourth Painlevé equation in [4]. Lenells considered in 2015 solutions of the dNLS equation on the halfline [5] whose Dirichlet and Neumann boundary values become periodic for sufficiently large t. In the same year, Prinari et al. [6] derived novel dark-bright soliton solutions with nonzero boundary conditions obtained within the framework of the inverse scattering transform.

Here we present solutions to the defocusing nonlinear Schrödinger equation (dNLS) of order N depending on $2N-2$ real parameters in terms of wronskians and Fredholm determinants. Families of quasi-rational solutions to the dNLS equation are obtained. These quasi rational solutions can be expressed as a quotient of two polynomials of degree $N(N+1)$ in the variables x and t.
We present also rational solutions as a quotient of determinants using certain particular polynomials.

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2. Different representations of quasi-rational solutions to the dNLS equation

2.1. Quasi-rational solutions of the dNLS equation in terms of Fredholm determinant

We have to define the following notations.
The terms $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ are functions of the parameters λ_ν, $1 \leq \nu \leq 2N$; they are defined by the formulas:

$$\begin{aligned} &\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \qquad \delta_\nu = \kappa_\nu \lambda_\nu, \qquad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}, \\ &x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}, \qquad r = 1,3. \end{aligned} \tag{2.1}$$

The parameters $-1 < \lambda_\nu < 1$, $\nu = 1,\ldots,2N$, are real numbers such that

$$\begin{aligned} &-1 < \lambda_{N+1} < \lambda_{N+2} < \ldots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \ldots < \lambda_1 < 1, \\ &\lambda_{N+j} = -\lambda_j, \quad j = 1,\ldots,N. \end{aligned} \tag{2.2}$$

The condition (2.2) implies that

$$\kappa_{j+N} = \kappa_j, \quad \delta_{j+N} = -\delta_{j+N}, \quad \gamma_{j+N} = \gamma_j^{-1}, \quad x_{r,j+N} = x_{r,j}, \quad j = 1,\ldots,N. \tag{2.3}$$

Complex numbers e_ν $1 \leq \nu \leq 2N$ are defined in the following way:

$$\begin{aligned} e_j &= i\sum_{l=1}^{N-1} a_l (j\varepsilon)^{2l+1} - \sum_{l=1}^{N-1} b_l (j\varepsilon)^{2l+1}, \\ e_{j+N} &= i\sum_{l=1}^{N-1} a_l (j\varepsilon)^{2l+1} + \sum_{l=1}^{N-1} b_l (j\varepsilon)^{2l+1}, \\ &\quad 1 \leq j \leq N-1. \end{aligned} \tag{2.4}$$

ε, a_ν, b_ν, $\nu = 1\ldots 2N$ are arbitrary real numbers.
Let I be the unit matrix, and

$$\varepsilon_j = j \quad 1 \leq j \leq N, \quad \varepsilon_j = N + j, \quad N+1 \leq j \leq 2N. \tag{2.5}$$

Let's consider the matrix $D_r = (d_{jk}^{(r)})_{1\leq j,k\leq 2N}$ defined by:

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu} \prod_{\eta \neq \mu} \left|\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \tag{2.6}$$

Using all the previous notations, the solution to the dNLS equation can be written as

Theorem 2.1. *The function v defined by*

$$v(\tilde{x},\tilde{t}) = \frac{\det(I + D_3(x,t))}{\det(I + D_1(x,t))} e^{2it - i\varphi}{}_{\{x = i\tilde{x}, t = -\tilde{t}\}}, \tag{2.7}$$

is a solution to the defocusing dNLS equation depending on $2N-1$ real parameters a_j, b_j, ε, $1 \leq j \leq N-1$ with the matrix $D_r = (d_{jk}^{(r)})_{1\leq j,k\leq 2N}$ defined by

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu} \prod_{\eta \neq \mu} \left|\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu).$$

where κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν being defined in (2.1), (2.2) and (2.4).

Proof. It is a consequence of the previous works of the author [7, 8, 9] with the change of variables defined by $\{x = i\tilde{x}, t = -\tilde{t}\}$. □

2.2. Wronskian representation

For this, we need to define the following notations :

$$\phi_{r,\nu} = \sin\Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos\Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1,3, \tag{2.8}$$

with the arguments

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2, \quad 1 \leq \nu \leq 2N. \tag{2.9}$$

The functions $\phi_{r,\nu}$ are defined by

$$\phi_{r,\nu} = \sin\Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos\Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1,3. \tag{2.10}$$

We denote $W_r(y)$ the wronskian of the functions $\phi_{r,1},\ldots,\phi_{r,2N}$ defined by

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_{r,\nu})_{\nu,\mu\in[1,\ldots,2N]}]. \tag{2.11}$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\mu\in[1,\ldots,2N]}$ defined in (2.6). Then we have the following statement [8]:

Theorem 2.2.

$$\det(I+D_r) = k_r(0)\times W_r(\phi_{r,1},\dots,\phi_{r,2N})(0), \tag{2.12}$$

where

$$k_r(y) = \frac{2^{2N}\exp(i\sum_{\nu=1}^{2N}\Theta_{r,\nu})}{\prod_{\nu=2}^{2N}\prod_{\mu=1}^{\nu-1}(\gamma_\nu-\gamma_\mu)}.$$

With these notations, we have the following result

Theorem 2.3. *The function v defined by*

$$v(\tilde{x},\tilde{t}) = \frac{W_3(\phi_{3,1},\dots,\phi_{3,2N})(0)}{W_1(\phi_{1,1},\dots,\phi_{1,2N})(0)} e^{2it-i\varphi}{}_{\{x=i\tilde{x},t=-\tilde{t}\}}.$$

is a solution to the defocusing dNLS equation depending on $2N-1$ real parameters a_j, b_j, ε, $1\le j\le N-1$ with ϕ_ν^r defined in (2.10)

$$\begin{aligned}
&\phi_{r,\nu} = \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), \quad 1\le\nu\le N,\\
&\phi_{r,\nu} = \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), \quad N+1\le\nu\le 2N, \quad r=1,3.
\end{aligned}$$

κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν being defined in (2.1), (2.2) and (2.4).

Proof. It is a consequence of [8] with the change of variables defined by $\{x=i\tilde{x}, t=-\tilde{t}\}$. □

We can give another representation of the solutions to the dNLS equation depending only on terms γ_ν, $1\le\nu\le 2N$. From the relations (2.1), we can express the terms κ_ν, δ_ν and $x_{r,\nu}$ in function of γ_ν, for $1\le\nu\le 2N$ and we obtain:

$$\begin{aligned}
&\kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j-i}{\gamma_j+i}, \quad 1\le j\le N,\\
&\kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = -\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j+i}{\gamma_j-i}, \quad N+1\le j\le 2N.
\end{aligned} \tag{2.13}$$

We have the following new representation

Theorem 2.4. *The function v defined by*

$$v(\tilde{x},\tilde{t}) = \frac{\det[(\partial_y^{\mu-1}\tilde{\phi}_{3,\nu}(0))_{\nu,\mu\in[1,\dots,2N]}]}{\det[(\partial_y^{\mu-1}\tilde{\phi}_{1,\nu}(0))_{\nu,\mu\in[1,\dots,2N]}]} e^{2it-i\varphi}{}_{\{x=i\tilde{x},t=-\tilde{t}\}} \tag{2.14}$$

is a solution to the defocusing dNLS equation (1.1) depending on $2N-1$ real parameters a_j, b_j, ε, $1\le j\le N-1$. The functions $\tilde{\phi}_{r,\nu}$ are defined by

$$\begin{aligned}
&\tilde{\phi}_{r,j}(y) = \sin\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - i\frac{(r-1)}{2}\ln\frac{\gamma_j-i}{\gamma_j+i} + \gamma_j y - ie_j\right),\\
&\tilde{\phi}_{r,N+j}(y) = \cos\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t + i\frac{(r-1)}{2}\ln\frac{\gamma_j-i}{\gamma_j+i} + \frac{1}{\gamma_j}y - ie_{N+j}\right),\\
&\textit{where } \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, 1\le j\le N.\\
&\lambda_j \textit{ is an arbitrary real parameter such that } 0<\lambda_j<1, \lambda_{N+j}=-\lambda_j, 1\le j\le N.\\
&\textit{The terms } e_\nu \textit{ are defined by } (2.4), \textit{where } a_j \textit{ and } b_j \textit{ are arbitrary real numbers}, 1\le j\le N-1.
\end{aligned} \tag{2.15}$$

Proof. We have to make the following change of variables defined by $\{x=i\tilde{x}, t=-\tilde{t}\}$ in the previous works [8, 10, 11, 12]. □

Remark 2.5. *In the formula (2.14), the determinants $\det[(\partial_y^{\mu-1}f_\nu(0))_{\nu,\mu\in[1,\dots,2N]}]$ are the wronskians of the functions $f_1,\dots,f_{2N}$ evaluated in $y=0$. In particular $\partial_y^0 f_\nu$ means f_ν.*

2.3. Families of quasi-rational solutions of dNLS equation in terms of a quotient of two determinants

The following notations are used:

$$X_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2,$$

$$Y_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2,$$

for $1\le\nu\le 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in (2.1).
Parameters e_ν are defined by (2.4).
Below the following functions are used :

$$\begin{aligned}
&\varphi_{4j+1,k} = \gamma_k^{4j-1}\sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j}\cos X_k,\\
&\varphi_{4j+3,k} = -\gamma_k^{4j+1}\sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2}\cos X_k,
\end{aligned} \tag{2.16}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll} \varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin X_{N+k}, \end{array} \tag{2.17}$$

for $1 \leq k \leq N$.
We define the functions $\psi_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, the term X_k is only replaced by Y_k.

$$\begin{array}{ll} \psi_{4j+1,k} = \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} = \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} = -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} = -\gamma_k^{4j+2} \cos Y_k, \end{array} \tag{2.18}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll} \psi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin Y_{N+k}, \end{array} \tag{2.19}$$

for $1 \leq k \leq N$.
Then we get the following result

Theorem 2.6. *The function v defined by*

$$v(\tilde{x},\tilde{t}) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} e^{2it-i\varphi}{}_{\{x=i\tilde{x},t=-\tilde{t}\}} \tag{2.20}$$

is a quasi-rational solution of the defocusing dNLS equation (1.1) depending on $2N-2$ real parameters a_j, b_j, $1 \leq j \leq N-1$, where

$$\begin{array}{l} n_{j1} = \varphi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jk} = \dfrac{\partial^{2k-2}\varphi_{j,1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ n_{jN+1} = \varphi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jN+k} = \dfrac{\partial^{2k-2}\varphi_{j,N+1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ d_{j1} = \psi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jk} = \dfrac{\partial^{2k-2}\psi_{j,1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ d_{jN+1} = \psi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jN+k} = \dfrac{\partial^{2k-2}\psi_{j,N+1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ 2 \leq k \leq N,\ 1 \leq j \leq 2N. \end{array}$$

The functions φ and ψ are defined in (2.16), (2.17), (2.18), (2.19).

Proof: It is also a consequence of the previous work [10] with the following change of variables defined by $\{x = i\tilde{x}, t = -\tilde{t}\}$. □

We don't give examples of solutions in terms of Fredholm determinants, wronskians or quasi-rational solutions because these types of solutions have been already explicitly constructed by the author until order 13 in the case of the focusing equation and it is easy to deduce these in the defocusing case. These results can be found from the previous published works. We do not give all the references; for the first orders in [13], until last orders (13) in [14].

3. Structure of the multi-parametric quasi-rational solutions to the dNLS equation

Here we present a result which states the structure of the quasi-rational solutions of the dNLS equation. In this section we use the notations defined in the previous sections. The functions φ and ψ are defined in (2.16), (2.17), (2.18), (2.19).
The structure of the quasi rational solutions to the dNLS equation is given by the following theorem

Theorem 3.1. *The function v defined by*

$$v(\tilde{x},\tilde{t}) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} e^{2it-i\varphi}\{x = i\tilde{x}, t = -\tilde{t}\} \tag{3.1}$$

is a quasi-rational solution of the defocusing dNLS equation (1.1) quotient of two polynomials of degrees $N(N+1)$ in x and t depending on $2N-2$ real parameters a_j and b_j, $1 \leq j \leq N-1$.

Proof. It is sufficient to realize the following change of variables defined by $\{x = i\tilde{x}, t = -\tilde{t}\}$ in [11, 12]. □

4. Rational solutions of order k to the dNLS equation

4.1. Expression of the rational solutions of order k

We consider the polynomials $p_n(x,t)$ defined by

$$\left\{ \begin{array}{l} p_n(x,t) = \sum_{k=0}^{n} \dfrac{(-x)^k}{k!} \dfrac{t^{\left(\frac{n-k}{2}\right)}}{\left(\frac{n-k}{2}\right)!} \left(1-(n-k)+2\left[\dfrac{n-k}{2}\right]\right), \quad n \geq 0, \\ p_n(x,t) = 0, \quad n < 0, \end{array} \right. \tag{4.1}$$

where $[x]$ is the greater integer less or equal to x.
We denote $W_{n,k}(x,t)$ the following determinants

$$W_{n,k}(x,t)=\begin{vmatrix} p_n & p_{n-1} & \dots & p_k \\ -p_{n-1} & -p_{n-2} & \dots & -p_{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{n-k}p_k & (-1)^{n-k}p_{k-1} & \dots & (-1)^{n-k}p_{2k-n} \end{vmatrix} \tag{4.2}$$

We define the function v_k by

$$v_k(x,t)=\frac{W_{2k+1,k}(x,t)}{W_{2k+1,k+1}(x,t)}.$$

We will call this function a function of order k and with these notations we have the following result

Theorem 4.1. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=\frac{W_{2k+1,k}(x,t)}{W_{2k+1,k+1}(x,t)} \tag{4.3}$$

is a rational to the (dNLS) equation

$$iv_t+v_{xx}-2|v|^2v=0.$$

Proof. It is well known that $v=\dfrac{G}{F}$, where F and G are polynomials, is a solution to the dNLS equation if G and F verify the two following equations:

$$(iD_t+D_x^2)G\cdot F=0 \tag{4.4}$$

$$D_x^2F\cdot F+2\overline{G}G=0, \tag{4.5}$$

where D is the bilinear differential Hirota operator.
We have to verify (4.4) for $G=W_{2k+1,k}(x,t)$ and $F=W_{2k+1,k+1}(x,t)$. We denote C_l and $\tilde{C}_l$ the following columns :

$$C_l=\begin{pmatrix} p_l \\ -p_{l-1} \\ \vdots \\ (-1)^{k+1}p_{l-k-1}\end{pmatrix}, \quad \tilde{C}_l=\begin{pmatrix} p_l \\ -p_{l-1} \\ \vdots \\ (-1)^{k}p_{l-k}\end{pmatrix}. \tag{4.6}$$

With these notations, $W_{2k+1,k}(x,t)$ and $W_{2k+1,k+1}(x,t)$ can be written as

$$W_{2k+1,k}(x,t)=|C_{2k+1},\dots,C_k| \quad \text{and} \quad W_{2k+1,k+1}(x,t)=|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|. \tag{4.7}$$

We denote A the expression $A=(iD_t+D_x^2)W_{2k+1,k}(x,t)\cdot W_{2k+1,k+1}(x,t)$. We have to evaluate A.
The polynomials p_k verify $\partial_x(p_k)=-p_{k-1}$ and $\partial_t(p_k)=ip_{k-2}$.
So A can be written as

$$\begin{aligned} A \;=\;& |C_{2k+1},\dots,C_{k+2},C_k,C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|-|C_{2k+1},\dots,C_{k+1},C_{k-2}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}| \\ & -|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k|+|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_{k-1}| \\ & +|C_{2k+1},\dots,C_{k+2},C_k,C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|+|C_{2k+1},\dots,C_{k+1},C_{k-2}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}| \\ & -2|C_{2k+1},\dots,C_{k+1},C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_k|+|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k| \\ & +|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_{k-1}|. \end{aligned}$$

A can be reduced to

$$\begin{aligned} A \;=\;& 2(|C_{2k+1},\dots,C_{k+2},C_k,C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|+|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_{k-1}| \\ & -|C_{2k+1},\dots,C_{k+1},C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|). \end{aligned}$$

A can be rewritten as the following determinant of order $2k+3$

$$A=\begin{vmatrix} C_{2k+1} & \dots & C_{k+2} & C_{k+1} & C_k & 0 & \dots & 0 & -C_{k-1} \\ 0 & \dots & 0 & -\tilde{C}_{k+1} & -\tilde{C}_k & \tilde{C}_{2k+1} & \dots & \tilde{C}_{k+2} & \tilde{C}_{k-1} \end{vmatrix} \tag{4.8}$$

We denote by $\mathscr{L}$ the rows and by $\mathscr{C}$ the columns of this determinant of order $2k+3$.

We combine the lines of the previous determinant in the following way:
We replace $\mathscr{L}_{k+2+j}$ by $\mathscr{L}_{k+2+j}+\mathscr{L}_j$ for $1\leq j\leq k+1$, then we obtain the following determinant

$$A=\begin{vmatrix} p_{2k+1} & p_{2k} & \dots & p_{k+1} & p_k & 0 & \dots & 0 & -p_{k-1}\\ -p_{2k} & -p_{2k-1} & \dots & -p_k & -p_{k-1} & 0 & \dots & 0 & p_{k-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (-1)^{k+1}p_k & (-1)^{k+1}p_{k-1} & \dots & \dots & 0 & 0 & \dots & 0 & 0\\ p_{2k+1} & p_{2k} & \dots & 0 & 0 & p_{2k+1} & \dots & p_{k+2} & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (-1)^k p_{k+1} & (-1)^k p_k & \dots & 0 & 0 & (-1)^k p_{k+1} & \dots & (-1)^k p_2 & 0 \end{vmatrix} \tag{4.9}$$

Then replacing $\mathscr{C}_j$ by $\mathscr{C}_j-\mathscr{C}_{k+2+j}$ for $1\leq j\leq k+1$, when we obtain the following determinant

$$A=\begin{vmatrix} p_{2k+1} & p_{2k} & \dots & p_{k+1} & p_k & 0 & \dots & 0 & -p_{k-1}\\ -p_{2k} & -p_{2k-1} & \dots & -p_k & -p_{k-1} & 0 & \dots & 0 & p_{k-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (-1)^{k+1}p_k & (-1)^{k+1}p_{k-1} & \dots & \dots & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 0 & p_{2k+1} & \dots & p_{k+2} & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0 & p_k & \dots & p_2 & 0 \end{vmatrix} \tag{4.10}$$

This last determinant is clearly equal to 0, which proves that:

$$A=(iD_t+D_x^2)W_{2k+1,k}(x,t)\cdot W_{2k+1,k+1}(x,t)=0.$$

The relation (4.5) can be proven with the same type of arguments.
We give a sketch of the proof.
We denote B the expression $B=D_x^2F\cdot F+2\overline{G}G$. We have to evaluate B.
The polynomials p_k verify $\partial_x(p_k)=-p_{k-1}$.
So B can be written as

$$\begin{aligned} B \;=\; & 2(|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|+|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|\\ & -|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|+|C_{2k+1},\dots,C_k|\times|\overline{C_{2k+1}},\dots,\overline{C_k}|). \end{aligned}$$

The determinant $\overline{G}=|\overline{C_{2k+1}},\dots,\overline{C_k}|$ is equal to $|C^*_{2k+1},\dots,C^*_{k+2}|$, where C^*_l is defined by:

$$C^*_l=\begin{pmatrix} p_l\\ -p_{1-l}\\ \vdots\\ (-1)^{k-1}p_{1-k+1} \end{pmatrix}. \tag{4.11}$$

The product $G\times\overline{G}$ can be written as $G\times(G[k+1,k+1])\,[k+2,k+2]$, where $G[i,j]$ means that $G[i,j]$ is obtained from G by deleting the row i and the column j.
We denote $\widehat{C}_l$

$$\widehat{C}_l=\begin{pmatrix} p_l\\ -p_{1-l}\\ \vdots\\ (-1)^{k-1}p_{1-k+1}\\ (-1)^{k+1}p_{1-k-1} \end{pmatrix}. \tag{4.12}$$

Using the Jacobi identity, we can write $G\times\overline{G}$ as

$$\begin{aligned} G\times(G[k+1,k+1])G[k+2,k+2]-G[k+1,k+2]G[k+2,k+1] \;=\; & |\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|\\ & -|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|. \end{aligned}$$

So, B can be rewritten as the sum

$$\begin{aligned} B \;=\; & 2|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|\times(|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k|+|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+2},\tilde{C}_{k-1}|+|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_k|)\\ & -2|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|\times(|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|+|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_{k+1}|). \end{aligned}$$

But the sums

$$|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k|+|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+2},\tilde{C}_{k-1}|+|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_k|)$$

and

$$(|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|+|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_{k+1}|)$$

are equal to 0 which proves that $B=0$.
Then we get the relation (4.5).
So we get the result. □

4.2. Some examples of rational solutions to the dNLS equation

In this section we will give some explicit examples of rational solutions to the dNLS equation. We recall that k means the order of the solution defined by

$$v_k(x,t)=\frac{W_{2k+1,k}(x,t)}{W_{2k+1,k+1}(x,t)}.$$

4.2.1. Rational solutions of order 1 to the dNLS equation

Example 4.2. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=-2\,\frac{x\left(-x^2+6\,it\right)}{-x^4+12t^2} \tag{4.13}$$

is a rational solution to the (dNLS) equation.

4.2.2. Rational solutions of order 2 to the dNLS equation

Example 4.3. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=-3\,\frac{-x^8+16\,ix^6t+120\,x^4t^2-720\,t^4}{x\left(-x^8+72\,x^4t^2+2160\,t^4\right)} \tag{4.14}$$

is a rational solution to the (dNLS) equation.

4.2.3. Rational solutions of order 3 to the dNLS equation

Example 4.4. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=\frac{n(x,t)}{d(x,t)} \tag{4.15}$$

with

$$n(x,t)=4\,(-x^{14}+30\,itx^{12}+540\,t^2x^{10}-4200\,it^3x^8-10800\,t^4x^6+151200\,it^5x^4-504000\,t^6x^2+3024000\,it^7)x$$

and

$$d(x,t)=x^{16}-240\,t^2x^{12}-7200\,t^4x^8-2016000\,t^6x^4+6048000\,t^8$$

is a rational solution to the (dNLS) equation.

4.2.4. Rational solutions of order 4 to the dNLS equation

Example 4.5. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=\frac{n(x,t)}{d(x,t)} \tag{4.16}$$

with

$$\begin{aligned} n(x,t) \;=\;& -5x^{24}+240\,itx^{22}+7560\,t^2x^{20}-134400\,it^3x^{18}-1436400\,t^4x^{16}+12096000\,it^5x^{14}+98784000\,t^6x^{12}\\ &+677376000\,it^7x^{10}-1905120000\,t^8x^8+71124480000\,it^9x^6+533433600000\,t^{10}x^4-1066867200000\,t^{12}\end{aligned}$$

and

$$d(x,t)=(x^{24}-600\,t^2x^{20}+25200\,t^4x^{16}-14112000\,t^6x^{12}-4021920000\,t^8x^8+106686720000\,t^{10}x^4+1066867200000\,t^{12})x$$

is a rational solution to the (dNLS) equation.

4.2.5. Rational solutions of order 5 to the dNLS equation

Example 4.6. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=\frac{n(x,t)}{d(x,t)} \tag{4.17}$$

with

$$\begin{aligned} n(x,t) \;=\;& -6\,(-x^{34}+70\,itx^{32}+3360\,t^2x^{30}-100800\,ix^{28}t^3-2116800\,t^4x^{26}+33022080\,it^5x^{24}+423360000\,t^6x^{22}\\ &-3217536000\,ix^{20}t^7-1778112000\,t^8x^{18}+522764928000\,it^9x^{16}+2782389657600\,t^{10}x^{14}\\ &+39431411712000\,it^{11}x^{12}+1552163751936000\,t^{12}x^{10}-11435109396480000\,ix^8t^{13}\\ &-141953082163200000\,t^{14}x^6+198734315028480000\,it^{15}x^4\\ &-2484178937856000000\,t^{16}x^2+14905073627136000000\,it^{17})x\end{aligned}$$

and

$$\begin{aligned} d(x,t) &= -x^{36}+1260t^2x^{32}-302400t^4x^{28}+76204800t^6x^{24}+30939148800t^8x^{20}+12943232870400t^{10}x^{16} \\ &\quad -1623857227776000t^{12}x^{12}-21292962324480000t^{14}x^8-2235761044070400000t^{16}x^4 \\ &\quad +2981014725427200000t^{18} \end{aligned}$$

is a rational solution to the (dNLS) equation.

4.2.6. Rational solutions of order 6 to the dNLS equation

Example 4.7. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=\frac{n(x,t)}{d(x,t)}$$

with

$$\begin{aligned} n(x,t) &= 7x^{48}-672itx^{46}-45360t^2x^{44}+2016000it^3x^{42}+67102560t^4x^{40}-1717148160it^5x^{38}-35611349760t^6x^{36} \\ &\quad +580375756800it^7x^{34}+6847687123200t^8x^{32}-82242658713600it^9x^{30}-1292786998272000t^{10}x^{28} \\ &\quad -2839061643264000it^{11}x^{26}-158869157787648000t^{12}x^{24}+2004377520144384000it^{13}x^{22} \\ &\quad -10427589509544345600 0t^{14}x^{20}+251136579215189606400 0it^{15}x^{18}+229593878833259888640 00t^{16}x^{16} \\ &\quad -142130012484808212480000it^{17}x^{14}-1281924569969568645120000t^{18}x^{12} \\ &\quad -3781319401921409187840000it^{19}x^{10}-4253984327161585336320000t^{20}x^8 \\ &\quad -15881541488069918588928000 0it^{21}x^6-119111561160524389416960000 0t^{22}x^4 \\ &\quad +119111561160524389416960000 0t^{24} \end{aligned}$$

and

$$\begin{aligned} d(x,t) &= (-x^{48}+2352t^2x^{44}-1481760t^4x^{40}+516499200t^6x^{36}+79481606400t^8x^{32}+125617211596800t^{10}x^{28} \\ &\quad +52451663859302400t^{12}x^{24}-2576448441262080000 0t^{14}x^{20}+427924663835074560000t^{16}x^{16} \\ &\quad -15480051747376398336000 0t^{18}x^{12}-17488602233886517493760000t^{20}x^8 \\ &\quad +2382231223210487788339200000t^{22}x^4+119111561160524389416960000 0t^{24})x \end{aligned}$$

is a rational solution to the (dNLS) equation.

4.2.7. Rational solutions of order 7 to the dNLS equation

Example 4.8. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=\frac{n(x,t)}{d(x,t)} \tag{4.18}$$

with

$$\begin{aligned} n(x,t) &= 8(-x^{62}+126itx^{60}+11340t^2x^{58}-693000ix^{56}t^3-32916240t^4x^{54}+1236422880it^5x^{52}+38294182080t^6x^{50} \\ &\quad -981414403200it^7x^{48}-20719094457600t^8x^{46}+373342708569600it^9x^{44}+6234317431372800t^{10}x^{42} \\ &\quad -78116020651468800ix^{40}t^{11}-380937010696704000t^{12}x^{38}+7441864983641088000it^{13}x^{36} \\ &\quad +234509627737921536000t^{14}x^{34}-1049152892936782233600 0it^{15}x^{32}+28872638199346765824000t^{16}x^{30} \\ &\quad -674072893110830653440000 0it^{17}x^{28}-16947489318197019918336000 0t^{18}x^{26} \\ &\quad +2400831552640985128304640000it^{19}x^{24}+29802589444030637579304960000t^{20}x^{22} \\ &\quad -228707538154566157550223360000it^{21}x^{20}+2302924801111261090283520000 00t^{22}x^{18} \\ &\quad +1171623855418189510158581760000 0it^{23}x^{16}+9113325209167385962862346240000 0t^{24}x^{14} \\ &\quad -2352007515362870156841713664000 00it^{25}x^{12}+1301619354591317976182905896960000 0t^{26}x^{10} \\ &\quad -93948903546617439225800294400000000it^{27}x^8-67643210553564556242576211968000000t^{28}x^6 \\ &\quad +947004947749903787396066967552000000it^{29}x^4-6313366318332691915973779783680000000t^{30}x^2 \\ &\quad +3788019790999615149584267870208000000it^{31})x \end{aligned}$$

and

$$\begin{aligned} d(x,t) &= x^{64}-4032t^2x^{60}+5201280t^4x^{56}-3353011200t^6x^{52}+613601049600t^8x^{48}-737286926745600t^{10}x^{44} \\ &\quad -6096479298674688 00t^{12}x^{40}-16400169840505651200 0t^{14}x^{36}+43623320815260426240000 0t^{16}x^{32} \\ &\quad -57117582890101985771520000t^{18}x^{28}+10315477913333460605337600000t^{20}x^{24} \\ &\quad +196052767973849246025646080000 0t^{22}x^{20}+32724126909677602873855180800000 0t^{24}x^{16} \\ &\quad -2459753111038711136093680435200000 0t^{26}x^{12}-1803818948095054833135365652480000 00t^{28}x^8 \\ &\quad -1010138610933230706555804765388800000 0t^{30}x^4+757603958199923029916853574041600000 0t^{32} \end{aligned}$$

is a rational solution to the (dNLS) equation.

4.2.8. Rational solutions of order 8 to the dNLS equation

Example 4.9. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{4.19}$$

with

$$\begin{aligned} n(x,t) \quad = \quad & -9x^{80} + 1440\,itx^{78} + 166320t^2x^{76} - 13305600\,it^3x^{74} - 846568800t^4x^{72} + 43445445120\,it^5x^{70} \\ & +1870141996800t^6x^{68} - 68163843686400\,it^7x^{66} - 2125554631200000t^8x^{64} \\ & +57307521503232000\,it^9x^{62} + 1363297604917248000t^{10}x^{60} - 28253298553159680000\,it^{11}x^{58} \\ & -484086300466728960000t^{12}x^{56} + 7566415631881666560000\,it^{13}x^{54} + 133152327700676444160000t^{14}x^{52} \\ & -2058963324486458277888000\,it^{15}x^{50} - 22021666720077485260800000t^{16}x^{48} \\ & -139857830692567689461760000\,it^{17}x^{46} - 3066455043451203128524800000t^{18}x^{44} \\ & -43732810958030201621053440000\,it^{19}x^{42} - 465799874464596133601633894400000t^{20}x^{40} \\ & +14255045283077121923887398912000000\,it^{21}x^{38} + 312639496448818711727126347776000000t^{22}x^{36} \\ & -5090182968557659371426514206720000000\,it^{23}x^{34} - 49325155597355070655188691845120000000t^{24}x^{32} \\ & +393967965258117624505590752477184000000\,it^{25}x^{30} \\ & +6848848992934365727781791035555840000000t^{26}x^{28} \\ & -22946638527138348968719720088862720000000\,it^{27}x^{26} \\ & +147787806733217686894511699420774400000000t^{28}x^{24} \\ & -9472010854510354292766985028213145600000000\,it^{29}x^{22} \\ & +33214909842533672004456976034939535360000000t^{30}x^{20} \\ & -198711070178640618136097905352890122240000000\,it^{31}x^{18} \\ & -1648749560903000743664085988421633310720000000t^{32}x^{16} \\ & +7477294657146652804799178630899328614400000000\,it^{33}x^{14} \\ & +69816498810133160472172135327273372876800000000t^{34}x^{12} \\ & +117025750386508916600974245881905844060160000000\,it^{35}x^{10} \\ & +268184011302416267210565980146034225971200000000t^{36}x^{8} \\ & +273060084235187472068939907057780302807040000000\,it^{37}x^{6} \\ & +204795063176390604051704930293335227105280000000t^{38}x^{4} \\ & -122877037905834362431022958176001136263168000000t^{40} \end{aligned}$$

and

$$\begin{aligned} d(x,t) \quad = \quad & (x^{80} - 6480t^2x^{76} + 14968800t^4x^{72} - 17603308800t^6x^{68} + 10318053715200t^8x^{64} - 6006932976844800t^{10}x^{60} \\ & -2425558108925952000t^{12}x^{56} - 356811818824581120000t^{14}x^{52} + 17711277416544699187200000t^{16}x^{48} \\ & +75989182484107429163827200000t^{18}x^{44} - 3390923311730068095298437120000t^{20}x^{40} + \\ & 916481140720063998978215116800000t^{22}x^{36} + 10661196076862440946653200384000000t^{24}x^{32} \\ & +642484723767584547487877840240640000000t^{26}x^{28} \\ & +912282169251705857390796131952230400000t^{28}x^{24} \\ & -3638429895511987315358195445179351040000000t^{30}x^{20} \\ & +32250817833265306075499427215808921600000000t^{32}x^{16} \\ & -792788081726786803896460144741932072960000000t^{34}x^{12} \\ & -4957340814984058272680159026941844783104000000000t^{36}x^{8} \\ & +409590126352781208103409860586670454210560000000t^{38}x^{4} \\ & +122877037905834362431022958176001136263168000000t^{40})x \end{aligned}$$

is a rational solution to the (dNLS) equation.

4.2.9. Rational solutions of order 9 to the dNLS equation

Example 4.10. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{4.20}$$

with

$$
\begin{aligned}
n(x,t) \quad = \quad & -10\,(-x^{98}+198\,itx^{96}+28512\,t^2x^{94}-2882880\,ix^{92}t^3-235414080\,t^4x^{90}+15723227520\,it^5x^{88}+892747215360\,t^6x^{86}\\
& -43550538071040\,it^7x^{84}-1851377348620800\,t^8x^{82}+69126330181708800\,it^9x^{80}+2290609252566835200\,t^{10}x^{78}\\
& -67587198729187737600\,it^{11}x^{76}-1769295226634333798400\,t^{12}x^{74}+41464108043171573760000\,it^{13}x^{72}\\
& +896769265742927216640000\,t^{14}x^{70}-17596890565184340393984000\,it^{15}x^{68}\\
& -294155011471335642980352000\,t^{16}x^{66}+3342933346282600754331648000\,it^{17}x^{64}\\
& -24509684290061469981081600000\,t^{18}x^{62}-243328718022336010567286784000\,it^{19}x^{60}\\
& -13093765110025787377929682944000\,t^{20}x^{58}+30755168429120881573902365491200000\,it^{21}x^{56}\\
& -627749361947142353377618506547200000\,t^{22}x^{54}+6426139169515222305039698834227200000\,it^{23}x^{52}\\
& +2711537814255492085577714208079872000000\,t^{24}x^{50}-78014081861464809452654641620713472000000\,it^{25}x^{48}\\
& -1593441818517645683457723068376416256000000\,t^{26}x^{46}+24885149560584639176620849366107684864000000\,it^{27}x^{44}\\
& +386625555598655080628036924868824924160000000\,t^{28}x^{42}\\
& -5038144976069943692864084422006941941760000000\,it^{29}x^{40}\\
& -28927990798127747807146732998684384952320000000\,t^{30}x^{38}\\
& +114495548645121617338289433808403913768960000000\,it^{31}x^{36}\\
& -196667249543515120711633358856754871402496000000\,t^{32}x^{34}\\
& -1900734076500974034241033934729985883373568000000000\,it^{33}x^{32}\\
& -24381387177210004003082439955396709767446528000000000\,t^{34}x^{30}\\
& -203985702147802569257556825474715381206614016000000000\,it^{35}x^{28}\\
& -923547534927361745814123173374918057435909324800000000\,t^{36}x^{26}\\
& +11785088381343870446323453595723655998357058355200000000\,it^{37}x^{24}\\
& +13978640714927931017608180739336927908177379328000000000\,t^{38}x^{22}\\
& -109578921093144586384883140031773893192623063040000000000\,it^{39}x^{20}\\
& +105103418022380273009575830246200385511960766054400000000\,t^{40}x^{18}\\
& +2409791909103468872738751319695802585131816635596800000000\,it^{41}x^{16}\\
& +18657374134884620337030400509550060264339394474803200000000\,t^{42}x^{14}\\
& -101005642479635415063356068952410491738385847156736000000000\,it^{43}x^{12}\\
& +135894491039195071532838449101474552421819561174630400000000\,t^{44}x^{10}\\
& -971330243967917515513199716362438362036128002657484800000000\,it^{45}x^{8}\\
& -454128425751234163097080386870750403029878027216486400000000\,t^{46}x^{6}\\
& +6357797960517278283359125416190505642418292381030809600000000\,it^{47}x^{4}\\
& -26490824835488659513996355900793773510076218254295040000000000\,t^{48}x^{2}\\
& +15894494901293195708397813540476264106045730952577024000000000\,it^{49})\,x
\end{aligned}
$$

and

$$
\begin{aligned}
d(x,t) \quad = \quad & -x^{100}+9900\,t^2x^{96}-37540800\,t^4x^{92}+75243168000\,t^6x^{88}-86477751360000\,t^8x^{84}+69030822212352000\,t^{10}x^{80}\\
& -18841861512714240000\,t^{12}x^{76}+3213319037194506240000\,t^{14}x^{72}+80012828841888983040000000\,t^{16}x^{68}\\
& -7169606358805218392064000000\,t^{18}x^{64}-1223508256241822655062999040000\,t^{20}x^{60}\\
& +16323950793276454578355961856000000\,t^{22}x^{56}-88634453383495458565418231267328000000\,t^{24}x^{52}\\
& +59769291166864434105376230604800000000000\,t^{26}x^{48}\\
& -9037618519244139622561816267613798400000000\,t^{28}x^{44}\\
& -297859518059409014875858745094586761216000000000\,t^{30}x^{40}\\
& +18265383607659591222210729071928001167360000000000\,t^{32}x^{36}\\
& +732385622391101433187551102767316006548275200000000000\,t^{34}x^{32}\\
& -597279198999947252378234412245692079501475840000000000\,t^{36}x^{28}\\
& +7743361476932166250721550026156249805131612160000000000000\,t^{38}x^{24}\\
& +87292314466005855930727753040515199535829273804800000000000\,t^{40}x^{20}\\
& +815065546574204996007977012015903440428932372889600000000000\,t^{42}x^{16}\\
& -4042431062558334406356586777069179723940202136207360000000000000\,t^{44}x^{12}\\
& -18922017739634756795711682786281266792911584467353600000000000000\,t^{46}x^{8}\\
& -66227062088721648784990889751984433775190545635737600000000000000\,t^{48}x^{4}\\
& +3178898980258639141679562708095252821209146190515404800000000000000\,t^{50}
\end{aligned}
$$

is a rational solution to the (dNLS) equation.

We could go on and present more explicit rational solutions, but they become very complicated. For example, in the case of order 10 the numerator includes 60 terms and the denominator 31 terms with big coefficients. It will be relevant to study in detail the structure of these solutions.

5. Conclusion

Different representations of quasi-rational solutions to the defocusing nonlinear Schrödinger equation have been given. First quasi rational solutions in terms of wronskians of order $2N$ depending on $2N-2$ real parameters have been presented. Another representation in terms of Fredholm determinants are given depending on $2N-2$ real parameters. These solutions give families of quasi-rational solutions to the dNLS equation expressed as a quotient of two polynomials of degree $N(N+1)$ in the variables x and t depending on $2N-2$ real parameters. Rational solutions as a quotient of determinants have been also given using certain particular polynomials and some explicit expressions are given for some orders. It will be relevant to study the structure of these last solutions.

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