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Orders of Solutions of Fractional Differential Equation in Complex Domain

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Abstract

We consider the fractional differential equation ${}^cD_z^{\alpha} f'(z)+A(z){}^cD_z^{\alpha} f(z)+B(z)f(z)=0$, where ${}^cD_z^{\alpha}$ be the Caputo fractional derivative of orders $0<\alpha\leq 1$, and z is complex number, $A(z),B(z)$ be entire functions. We will find conditions on $A(z),B(z)$ which will guarantee that every solution $f\not\equiv 0$ of the equation will have infinite order.

1. Introduction

Many researchers have been interested in the study of the order and hyper-order of solutions of linear ordinary differential equations with entire functions, meromorphic functions and analytic functions coefficients in the complex domain or in the unit disk, has significant applications in various scientific fields of research especially in physics, we cite some of them for example ([1–7]). In this new work, we study the order of solving a fractional differential equation withentire functions coefficients, we consider the following equation

$$ {}^cD_z^{\alpha} f'(z)+A(z){}^cD_z^{\alpha} f(z)+B(z)f(z)=0, \tag{1.1}$$

where ${}^cD_z^{\alpha}$ be the Caputo fractional derivative of order $0<\alpha\leq 1$, and z is complex number, $A(z),B(z)$ be entire functions.

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [8]). Let $\rho(f)$ denote the order of an entire function f, that is,

$$\rho(f)=\overline{\lim_{r\to+\infty}}\frac{\log T(r,f)}{\log r}=\overline{\lim_{r\to+\infty}}\frac{\log\log M(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f (see [8]), and

$$M(r,f)=\max_{|z|=r}|f(z)|.$$

For example, the function $f(z)=e^{z^2}$ satisfies $\rho(f)=2$.

The question which arises is: what conditions on $A(z),B(z)$ will guarantee that every solution $f\not\equiv 0$ has infinite order?

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1- In this section [9], we introduce some notations and definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows.

Definition 1.1. ([9]) The fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\alpha} d\xi, \ \ 0 \leq \alpha < 1, \tag{1.2}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z-$plane containing the origin, and the multiplicity of $(z-\xi)^{-\alpha}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

where $\Gamma(.)$ denotes the Gamma function is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \alpha > 0.$$

Definition 1.2. ([9]) The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-\xi)^{\alpha-1} f(\xi) d\xi, \ \ 0 < \alpha, \tag{1.3}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z-$plane containing the origin, and the multiplicity of $(z-\xi)^{\alpha-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Using the Caputo sense, we have

Definition 1.3. The Liouville-Caputo fractional derivative of order $n-1 < \alpha \leq n, n \in \mathbb{N}^*$, for a function $f(z)$ is defined as

$${}^cD_z^\alpha f(z) = \frac{1}{\Gamma(n-\alpha)} \int_0^z (z-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, \ \ n-1 < \alpha \leq n \tag{1.4}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z-$plane containing the origin, and the multiplicity of $(z-\xi)^{n-\alpha-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Remark 1.4. In the following we put

$${}^cD^\alpha f(z) = f^{(\alpha)}(z).$$

Lemma 1.5. ([10, 11]) Let $\alpha > 0$ then

$$I^\alpha f^{(\alpha)}(z) = f(z) + c_0 + c_1 z + \dots + c_{[\alpha]} z^{[\alpha]},$$

where are $c_0, c_1, \dots, c_{[\alpha]}$ constants in $\mathbb{C}$.

2. Main result

Lemma 2.1. ([12]) Let w be a transcendental entire function of finite order σ. Let $\Upsilon = \{(k_1, j_1), (k_2, j_2) \ , \cdots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0, i = 1, \dots, m$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 0$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, and for all $(k, j) \in \Upsilon$ we have

$$\left| \frac{w^{(k)}(z)}{w^{(j)}(z)} \right| \leq |z|^{(k-j)((\sigma-1+\varepsilon))}.$$

Lemma 2.2. Let w be entire, $0<\alpha\leq 1$, and suppose that $\left|w^{(\alpha)}(z)\right|$ is unbounded on some ray $\arg z=\theta$. Then there exists an infinites equence of points $z_n=r_ne^{i\theta}$ where $r_n\to\theta$, such that $w^{(\alpha)}(z)\to\infty$ and

$$\left|\frac{w(z_n)}{w^{(\alpha)}(z_n)}\right|\leq(1+o(1))\left|z_n\right|^{\alpha}, \tag{2.1}$$

as $z_n\to\infty$.

Proof. Let $M\left(r,w^{(\alpha)},\theta\right)=\max\left|w^{(\alpha)}(z)\right|$ over all satisfying $0\leq|z|\leq r$ and $\arg z=\theta$. It follows that there exists an infinite sequence of points $z_n=r_ne^{i\theta}$ where $r_n\to 0$, such that $M\left(r_n,w^{(\alpha)},\theta\right)=\left|w^{(\alpha)}(r_ne^{i\theta})\right|$ for all Then for each n, we have

$$\begin{aligned}
|w(z_n)| &= \left|I^{\alpha}w^{(\alpha)}(z_n)-c_0\right| \\
&\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{z_n}(z_n-\xi)^{\alpha-1}w^{(\alpha)}(\xi)d\xi\right|+|c_0| \\
&\leq \frac{\left|w^{(\alpha)}(z_n)\right|}{\Gamma(1+\alpha)}\left|z_n\right|^{\alpha}+|c_0| \\
&\leq \left|w^{(\alpha)}(z_n)\right|\left(\frac{1}{\Gamma(1+\alpha)}+\frac{|c_0|}{\left|w^{(\alpha)}(z_n)\right|\left|z_n\right|^{\alpha}}\right)\left|z_n\right|^{\alpha}.
\end{aligned}$$

Since $w^{(\alpha)}(z_n)\to\infty$, we obtain

$$\left|\frac{w(z_n)}{w^{(\alpha)}(z_n)}\right|\leq(1+o(1))\left|z_n\right|^{\alpha}.$$

□

Lemma 2.3. Let w be analytic on a ray $\arg z=\theta$ and $0<\alpha\leq 1$, and suppose that for some constant $\kappa>1$ we have

$$\left|\frac{w^{(\alpha)}(z)}{w(z)}\right|=O\left(|z|^{-\kappa}\right)|z|^{1-\alpha}, \tag{2.2}$$

as $z\to\infty$ along $\arg z=\theta$. Then there exists a constant $c\neq 0$ such that $w(z)\to c$ as $z\to\infty$ along $\arg z=\theta$.

Proof. From (2.2) it follows that there exists an $R_0>0$ and a simply connected domain D such that $\frac{w^{(\alpha)}}{w}$ is analytic on D and where if z satisfies $\arg z=\theta$ and $|z|\geq R_0$ then $z\in D$. Hence there exists an analytic function $F(z)$ on D such that $F'=\frac{w^{(\alpha)}}{w}$ on D. If $z_1=r_1e^{i\theta}$ and $z_2=r_2e^{i\theta}$ are large, where $R_0<r_1<r_2$, then by consideration of

$$F(z_2)-F(z_1)=\int_{z_1}^{z_2}\frac{w^{(\alpha)}(t)}{w(t)}dt,$$

and (2.2), it can be deduced that there exists a constant b such that $F(z)\to b$ as $z\to\infty$ along $\arg z=\theta$. It follows that there exists a constant $c\neq 0$ such that $w(z)\to c$ as $z\to\infty$ along $\arg z=\theta$. □

Theorem 2.4. Let $A(z),B(z)\not\equiv 0$ be entire functions such that for real constants $\lambda,\eta,\theta_1,\theta_2$ where $\lambda>0,\eta>0$, and $\theta_1<\theta_2$, we have

$$|A(z)|\geq\exp\left\{(1+o(1))\lambda|z|^{\eta\alpha}\right\} \tag{2.3}$$

and

$$|B(z)|\leq\frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)}\exp\left\{o(1)|z|^{\eta\alpha}\right\}, \tag{2.4}$$

as $z \to \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Let $\varepsilon > 0$ be a given small constant, and let $S(\varepsilon)$ denote the angle $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$. If $f \not\equiv 0$, $\max\limits_{\zeta \in [0,z]} |f'(\xi)| = |f'(z)|$ and $\max\limits_{\zeta \in [0,z]} |f''(\xi)| = |f''(z)|$ is a solution of equation (1.1) where $\rho(f) < \infty$, then the following conclusions hold:

(i) There exists a constant $b \neq 0$ such that $f(z) \to b$ as $z \to \infty$ in $S(\varepsilon)$. Furthermore,

$$|f(z) - b| \leq \exp\left\{(1 + o(1))\,\lambda\,|z|^{\eta\alpha}\right\}, \tag{2.5}$$

as $z \to \infty$ in $S(\varepsilon)$.

(ii) For each $k \geq \alpha$, as $z \to \infty$ in $S(\varepsilon)$

$$\left|f^{(k)}(z)\right| \leq \exp\left\{-(1 + o(1))\lambda\,|z|^{\eta\alpha}\right\}. \tag{2.6}$$

Theorem 2.5. Let $A(z)$ and $B(z) \not\equiv 0$ be entire functions such that for real constants $\lambda, \eta, \theta_1, \theta_2$ where $\lambda > 0, \eta > 0$, and $\theta_1 < \theta_2$, we have

$$|B(z)| \geq \frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)} \exp\left\{(1 + o(1))\,\lambda\,|z|^{\alpha\eta}\right\} \tag{2.7}$$

and

$$|A(z)| \leq \exp\left\{o(1)\,|z|^{\alpha\eta}\right\}, \tag{2.8}$$

as $z \to \infty$ in $\theta_1 \leq \arg z \leq \theta_2$, if $\max\limits_{\zeta \in [0,z]} |f'(\xi)| = |f'(z)|$ and $\max\limits_{\zeta \in [0,z]} |f''(\xi)| = |f''(z)|$. Then every solution $f \not\equiv 0$, of equation (1.1) has infinite order.

3. Proof of Theorem 2.4

Suppose that $f \not\equiv 0$ and $\sup\limits_{\zeta \in [0,z]} |f''(\xi)| = |f''(z)|$, is a solution of (1.1) with $\rho(f) < \infty$. Set $\sigma = \rho(f)$. Then from Lemma 2.1 there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then

$$\begin{aligned}
\left|\frac{(f')^{(\alpha)}(z)}{f^{(\alpha)}(z)}\right| &= \frac{\frac{1}{\Gamma(1-\alpha)}\left|\int\limits_0^z (z-\xi)^{1-\alpha-1} f''(\xi)d\xi\right|}{\frac{1}{\Gamma(1-\alpha)}\left|\int\limits_0^z (z-\xi)^{1-\alpha-1} f'(\xi)d\xi\right|} \\
&\leq \frac{\max\limits_{\zeta \in [0,z]} |f''(\xi)| \left|\int\limits_0^z (z-\xi)^{-\alpha} d\xi\right|}{\max\limits_{\zeta \in [0,z]} \left|\int\limits_0^z (z-\xi)^{-\alpha} d\xi\right|} \\
&\leq \frac{\max\limits_{\zeta \in [0,z]} |f''(\xi)|}{\max\limits_{\zeta \in [0,z]} |f'(\xi)|} \\
&\leq \frac{\max\limits_{\zeta \in [0,z]} |f''(\xi)|}{|f'(z)|} \\
&\leq \left|\frac{f''(z)}{f'(z)}\right| \\
&= o(1)\,|z|^{\sigma},
\end{aligned} \tag{3.1}$$

as z$\to \infty$ along arg ψ_0.

Now suppose that $\left|f^{(\alpha)}(z)\right|$ is unbounded on some ray $\arg z = \phi_0$ where $\phi_0 \in [\theta_1, \theta_2] - E$. Then from Lemma 2.2, there exists an infinite sequence of points $z_n = r_n \exp(i\phi_0)$ where $r_n \to \infty$, such that $f^{(\alpha)}(z_n) \to \infty$ and

$$\left|\frac{f(z_n)}{f^{(\alpha)}(z_n)}\right| \leq (1+o(1))\,|z_n|^{\alpha}, \tag{3.2}$$

as $z_n \to \infty$. From (1.1),

$$|A(z)| \leq \left|\frac{(f')^{(\alpha)}}{f^{(\alpha)}}\right| + |B(z)|\left|\frac{f}{f^{(\alpha)}}\right|. \tag{3.3}$$

By using (2.3), (2.4), (3.1), and (3.2), , we will obtain a contradiction in (3.3) as $z_n \to \infty$. Therefore, $\left|f^{(\alpha)}(z)\right|$ is bounded on any ray $\arg z = \phi$ where $\phi \in [\theta_1, \theta_2] - E$. It then follows from the classical Phragmen-Lindelof theorem [8] that there exists a constant $M > 0$ such that

$$\begin{aligned} \left|f^{(\alpha)}(z)\right| &\leq \frac{|z|^{1-\alpha}\,|f'(z)|}{\Gamma(2-\alpha)} \\ &\leq M, \end{aligned} \tag{3.4}$$

for all $z \in S(\varepsilon)$.

If $\theta_0 \in [\theta_1+\varepsilon, \theta_2+\varepsilon] - E$, then when $\arg z = \theta_0$, we obtain from (3.4) that

$$\begin{aligned} |f(z)| &= \left|I^{\alpha} f^{(\alpha)}(z) - c_0\right| \\ &\leq \left|I^{\alpha} f^{(\alpha)}(z)\right| + |c_0| \\ &\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^z (z-\xi)^{\alpha-1} f^{(\alpha)}(\xi)\,d\xi\right| + |c_0| \\ &\leq M\frac{|z|^{\alpha}}{\Gamma(1+\alpha)} + |c_0| \\ &\leq N|z|^{\alpha} + |c_0|, \end{aligned} \tag{3.5}$$

where $N = \frac{M}{\Gamma(1+\alpha)}$. From (3.1), (3.5), and (1.1), we obtain that

$$|A(z)|\left|f^{(\alpha)}(z)\right| \leq o(1)\,|z|^{\sigma}\left|f^{(\alpha)}(z)\right| + |B(z)|\left(N|z|^{\alpha} + |c_0|\right). \tag{3.6}$$

as $z \to \infty$ along $\arg z = \theta_0$. From (2.3), (2.4), and (3.6), we can deduce that

$$\left|f^{(\alpha)}(z)\right| \leq \left[\frac{|B(z)|\left(N|z|^{\alpha}+|c_0|\right)}{|A(z)| - o(1)\,|z|^{\sigma}}\right] \leq \frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)}\exp\left\{-(1+o(1))\lambda\,|z|^{\eta\alpha}\right\}, \tag{3.7}$$

as $z \to \infty$ along $\arg z = \infty$. By using an application of the Phragmén-Lindelöf theorem on (3.7), it can be deduced that

$$\left|f^{(\alpha)}(z)\right| \leq \frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)}\exp\left\{-(1+o(1))\lambda\,|z|^{\eta\alpha}\right\}, \tag{3.8}$$

as $z \to \infty$ in $S(2\varepsilon)$. This gives $k=1$ in (2.6)

Now let $z \in S(3\varepsilon)$ where $|z| > 1$, let Ω be a circle of radius one with center at z, and let $k \geq 1$ be an integer. Then from the Cauchy integral formula and (3.8), we obtain as $z \to \infty$ in $S(3\varepsilon)$,

$$\begin{aligned} \left|f^{(k)}(z)\right| &\leq \frac{(k-1)!}{2\pi}\int_{\Omega}\left|(z-\xi)^{-k} f'(\xi)\,d\xi\right| \\ &\leq \frac{(k-2)!}{2\pi}\left|f'(z)\right| \end{aligned}$$

and we have

$$\left|f'(z)\right| \geq \Gamma(2-\alpha)\left|f^{(\alpha)}(z)\right| |z|^{\alpha-1}.$$

In case $|f'(z)| = \Gamma(2-\alpha)\left|f^{(\alpha)}(z)\right| |z|^{\alpha-1}$, we have

$$\begin{aligned} \left|f^{(k)}(z)\right| &\leq \frac{(k-2)!}{2\pi}\Gamma(2-\alpha)\left|f^{(\alpha)}(z)\right| |z|^{\alpha-1} \\ &\leq \exp\left\{-(1+o(1))\lambda |z|^{\eta\alpha}\right\}. \end{aligned} \tag{3.9}$$

This proves (2.6). Now fix θ, ψ, where $\theta_1 + \varepsilon \leq \theta, \psi \leq \theta_2 - \varepsilon$, and set

$$c = \int_0^\infty \left(e^{i\psi} - e^{i\theta}\right)^{\alpha-1} e^{i\theta} f^{(\alpha)}(te^{i\theta})dt, \tag{3.10}$$

where we note that $c \in \mathbb{C}$ from (2.6). Let $z = |z| e^{i\psi}$ where $\theta_1 + \varepsilon \leq \psi \leq \theta_2 - \varepsilon$,. Then from the Cauchy theorem and (3.10), we obtain

$$\begin{aligned} & f(z) + c_0 - c \\ = & I^\alpha f^{(\alpha)}(z) - c \\ = & \frac{1}{\Gamma(\alpha)}\int_0^z (z-\xi)^{\alpha-1} f^{(\alpha)}(\xi)d\xi - \int_0^\infty \left(te^{i\psi} - te^{i\theta}\right)^{\alpha-1} f^{(\alpha)}(te^{i\theta})e^{i\theta}dt \\ = & \frac{i}{\Gamma(\alpha)}\int_\theta^\psi |z| e^{ix}\left(|z| e^{i\psi} - |z| e^{ix}\right)^{\alpha-1} f^{(\alpha)}(|z| e^{ix})dx - \int_{|z|}^\infty \left(te^{i\psi} - te^{i\theta}\right)^{\alpha-1} e^{i\theta} f^{(\alpha)}(te^{i\theta})dt \\ = & \frac{i\,|z|^\alpha}{\Gamma(\alpha)}\int_\theta^\psi e^{ix}\left(e^{i\psi} - e^{ix}\right)^{\alpha-1} f^{(\alpha)}(|z| e^{ix})dx - \int_{|z|}^\infty \left(te^{i\psi} - te^{i\theta}\right)^{\alpha-1} e^{i\theta} f^{(\alpha)}(te^{i\theta})dt, \end{aligned} \tag{3.11}$$

From (2.6) and (3.11), it can be deduced that

$$\begin{aligned} & |f(z) - b| \\ \leq & \frac{\left|f^{(\alpha)}(z)\right| |z|^\alpha}{\Gamma(\alpha)}\left|\int_\theta^\psi \left(e^{i\psi} - e^{ix}\right)^{\alpha-1} dx\right| + \left|f^{(\alpha)}(z)\right| \left|\int_{|z|}^\infty \left(te^{i\psi} - te^{i\theta}\right)^{\alpha-1} dt\right| \\ \leq & \frac{\left|f^{(\alpha)}(z)\right| |z|^\alpha}{\Gamma(\alpha)}\int_\theta^\psi \left|\left(e^{i\psi} - e^{ix}\right)^{\alpha-1}\right| dx + \left|f^{(\alpha)}(z)\right| \left|\int_{|z|}^\infty t^{\alpha-1}\left(e^{i\psi} - e^{i\theta}\right)^{\alpha-1} dt\right| \\ \leq & \frac{2^{\alpha-1}\left|f^{(\alpha)}(z)\right| |z|^\alpha}{\Gamma(\alpha)}|\psi - \theta| + 2^{\alpha-1}\left|f^{(\alpha)}(z)\right| \frac{|z|^\alpha}{\alpha} \\ \leq & 2^{\alpha-1}|z|^\alpha\left(\frac{|\psi-\theta|}{\Gamma(\alpha)} + \frac{1}{\alpha}\right)\left|f^{(\alpha)}(z)\right| \\ \leq & \frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)}\exp\left\{-(1+o(1))\lambda |z|^{\eta\alpha}\right\}, \end{aligned} \tag{3.12}$$

as $z \to \infty$ in $S(\varepsilon)$, where $b = c - c_0$. (Note: It follows that c in (3.10) is independent of θ.) Since (3.12) is the inequality (2.5), it remains only to show that $b \neq 0$.

There exists a ray $\arg z = \psi_1$ where $\theta_1 + \varepsilon \leq \psi_1 \leq \theta_2 - \varepsilon$, such that

$$\left|\frac{(f')^{(\alpha)}(z)}{f(z)}\right| = \frac{o(1)\,|z|^{2\sigma-\alpha+1}}{\Gamma(2-\alpha)}, \tag{3.13}$$

as $z \to \infty$ along $\arg z = \psi_1$. Then from (2.3), (2.4), (3.13), and (1.1), we obtain that

$$\left|\frac{f^{(\alpha)}(z)}{f(z)}\right| \leq \left|\frac{B(z)}{A(z)}\right| + \left|\frac{(f')^{(\alpha)}(z)}{A(z)f(z)}\right| \leq \frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)} \exp\left\{-(1+o(1))\lambda\,|z|^{\eta\alpha}\right\}, \tag{3.14}$$

as $z \to \infty$ along $\arg z = \psi_1$. By applying Lemma 2.3 to (3.14), and noting that $f(z) - b$ as $z \to \infty$ in $S(\varepsilon)$ from (3.12), we see that $b \neq 0$. Thus part (i) is proved,and the proof of Theorem 2.4 is now complete.

4. Proof of Theorem 2.5

Suppose that $f \not\equiv 0$ and $\sup\limits_{\zeta\in[0,z]} |f''(\xi)| = |f''(z)|$, is a solution of (1.1) of finite order. Set $\sigma = \rho(f)$. Then from Lemma 2.1 there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then

$$\begin{aligned}
\left|\frac{(f')^{(\alpha)}(z)}{f(z)}\right| &= \frac{\left|\int\limits_0^z (z-\xi)^{-\alpha} f''(\xi)d\xi\right|}{\Gamma(1-\alpha)\,|f(z)|} \\
&\leq \frac{\max\limits_{\zeta\in[0,z]} |f''(\xi)| \left|\int\limits_0^z (z-\xi)^{-\alpha} d\xi\right|}{\Gamma(1-\alpha\,|f(z)|} \\
&\leq \frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)} \left|\frac{f''(z)}{f(z)}\right| \\
&= o(1)\frac{|z|^{2\sigma+1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\left|\frac{f^{(\alpha)}(z)}{f(z)}\right| &= \frac{\left|\int\limits_0^z (z-\xi)^{-\alpha} f'(\xi)d\xi\right|}{\Gamma(1-\alpha)\,|f(z)|} \\
&\leq \max\limits_{\zeta\in[0,z]} |f'(\xi)| \left|\int\limits_0^z (z-\xi)^{-\alpha} d\xi\right| \\
&\leq \frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)} \left|\frac{f'(z)}{f(z)}\right| \\
&= o(1)\frac{|z|^{\sigma+1-\alpha}}{\Gamma(2-\alpha)},
\end{aligned} \tag{4.2}$$

as $z \to \infty$ along $\arg z = \psi_0$. Then from (4.1), (4.2) and (1.1), we obtain

$$\begin{aligned}
|B(z)| &\leq \left|\frac{(f')^{(\alpha)}}{f}\right| + |A(z)|\left|\frac{f^{(\alpha)}}{f}\right| \\
&\leq \frac{o(1)}{\Gamma(2-\alpha)}\,|z|^{2\sigma+1-\alpha} + |A(z)|\,\frac{o(1)}{\Gamma(2-\alpha)}\,|z|^{\sigma+1-\alpha},
\end{aligned}$$

as $z \to \infty$ along $\arg z = \psi_0$, and this contradicts (2.7) and (2.8).

5. Conclusion

In this new work, using the Nevanlinna value distribution theory of meromorphic functions, we studied the order of solving a fractional differential equation with the coefficients of the full functions, where we showed that the solution of the fractional differential equation (1.1) has an infinite order.

Declaration of Competing Interest

No conflict of interest was declared by the authors.

Authorship Contribution Statement

Hamid Beddani: Writing, Reviewing, Methodology and Supervision.

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