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EXPLICIT FORMULAS FOR EXPONENTIAL OF 2×2 SPLIT-COMPLEX MATRICES

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Abstract. Split-complex (hyperbolic) numbers are ordered pairs of real numbers, written in the form $x + jy$ with $j^2 = 1$, used to describe the geometry of the Lorentzian plane. Since a null split-complex number does not have an inverse, some methods to calculate the exponential of complex matrices are not valid for split-complex matrices. In this paper, we examined the exponential of a 2×2 split-complex matrix in three cases : i. $\Delta = 0$, ii. $\Delta \neq 0$ and Δ is not null split-complex number, iii. $\Delta \neq 0$ and Δ is a null split-complex number where $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4\det\mathbf{A}$.

1. Introduction

The exponential of a matrix could be computed in many ways: series, matrix decomposition, differential equations and, polynomial methods. The matrix exponential gives a connection between any matrix Lie algebra and the corresponding Lie group. The matrix exponential does not satisfy some properties of the number exponential since matrix multiplication is not commutative. For example, the property $e^{a+b} = e^a e^b$ is not true for matrix exponential. The equality $e^{A+B} = e^A e^B$ is only true in the case the matrices A and B commute. Detailed information on the exponential matrix can be found in many sources. In this study, especially references [2], [26], [16], [3] and, [4] were used.

The purpose of this article is to determine the exponential of split-complex number matrices and to give useful formulas by classifying them. The formulas of calculating the matrix exponential for 2×2 complex numbers can be found in Bernstein's study [4]. Standard methods can be used to calculate the exponential of a matrix defined on a field such as complex numbers. However, for a set of numbers defined

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on a ring such as split-complex numbers, some problems are encountered in using these methods. The methods used in the literature for complex numbers can be useless, since some elements do not have an inverse in the ring of split-complex numbers. Diagonalization and finding eigenvalues and eigenvectors of a matrix defined on a ring is not as easy as in matrices defined over a field. There are many studies on this subject in the literature [18], [21], [15]. In particular, exponential of a matrix defined on the ring of quaternions can be found in Casey's paper [17]. Casey constructed a transformation from quaternionic square matrices to complex square matrices to compute the exponential of the quaternionic matrix. Also, Ablamowicz computed matrix exponential of a real, complex, and quaternionic matrix, using an isomorphism between matrix algebras and orthogonal Clifford algebras [1]. The exponential of a matrix defined in the rings of split and hyperbolic split quaternions can be found in the references [7], [8] and [20]. In this paper, exponential of a matrix defined over the split-complex numbers is studied. In the first part, some basic information and definitions about split-complex numbers are given. In the second part, the cases where some methods and formulas are insufficient are determined. In the last chapter, it is examined with the help of isomorphism between split complex matrices and real matrices.

2. Preliminaries

The set of split-complex numbers is defined as follows:

$$\mathbb{P} = \{\mathbf{z} = z_1 + jz_2 : z_1, z_2 \in \mathbb{R}\}$$

where the split-complex unit j satisfies $j^2 = 1$ and $j \neq 1$. In the literature, these numbers are also called double, spacetime, hyperbolic or perplex numbers [10], [28], [27], [5], [11], [25], [24], [9], [12]. For any $\mathbf{z} = z_1 + jz_2 \in \mathbb{P}$ we define the real part of $\mathbf{z}$ as $\mathrm{Re}(\mathbf{z}) = z_1$ and the split-complex part of $\mathbf{z}$ as $\mathrm{Im}(\mathbf{z}) = z_2$. The conjugate of $\mathbf{z}$ is denoted by $\overline{\mathbf{z}}$ and it is $\overline{\mathbf{z}} = z_1 - jz_2$. The inner product of $\mathbf{z} = z_1 + jz_2$ and $\mathbf{w} = w_1 + jw_2$ is defined as

$$\langle\ ,\ \rangle : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$$
$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathrm{Re}\,(\overline{\mathbf{z}}\mathbf{w}) = w_1 z_1 - w_2 z_2.$$

This product is a nondegenerate, symmetrical bilinear form, known as the scalar product in the Lorentzian plane. Since this scalar product is not positively defined, we will need to classify the split-complex numbers, as in the Lorentzian plane [6], [23], [13], [22], [14]. We will call a split-complex number $\mathbf{z} = z_1 + jz_2$ spacelike, timelike, or null, according to $\langle \mathbf{z}, \mathbf{z} \rangle = \overline{\mathbf{z}}\mathbf{z} > 0$, < 0 or $= 0$, respectively. So, for a split-complex number $\mathbf{z} = z_1 + jz_2$, we can call $\mathbf{z}$ spacelike, timelike or null according to $|z_1| > |z_2|$, $|z_1| < |z_2|$ and $z_1 = \pm z_2$, respectively. Null split-complex numbers have no inverse. In this paper, the set of null split-complex numbers is denoted by $\mathbb{P}_0$. Norm of $\mathbf{z} = z_1 + jz_2$ is defined as

$$|\mathbf{z}| = \mathbf{z}\overline{\mathbf{z}} = \sqrt{|z_1^2 - z_2^2|}.$$

The square root of the split-complex number $\mathbf{z} = z_1 + jz_2$ is found by

$$\sqrt{z_1 + jz_2} = \frac{\sqrt{z_1+z_2}+\sqrt{z_1-z_2}}{2} + \frac{\sqrt{z_1+z_2}-\sqrt{z_1-z_2}}{2}j. \tag{1}$$

A necessary and sufficient condition for the square root of a non-null split-complex number to be defined is that this number is spacelike. Moreover, for the null split-complex number $\mathbf{z} = x + yj$, we have $x = \pm y$ and obtain

$$\text{If } x = y, \quad \sqrt{x+yj} = \frac{\sqrt{2x}}{2}(1+j)$$
$$\text{If } x = -y, \quad \sqrt{x+yj} = \frac{\sqrt{2x}}{2}(1-j).$$

In this paper, by a split-complex matrix, we mean simply a matrix with split-complex number entries. We denote the set of $m \times n$ split-complex matrices with $\mathbb{M}_{m\times n}(\mathbb{P})$. We may write $\mathbf{X} = X_1 + jX_2$ for any $\mathbf{X} \in \mathbb{M}_{n\times n}(\mathbb{P})$ where $X_1, X_2 \in \mathbb{M}_{n\times n}(\mathbb{R})$. There exists a ring isomorphism between $\mathbb{M}_{n\times n}(\mathbb{P})$ and the algebra of the matrices of the form

$$\left\{ \begin{bmatrix} X_1 & X_2 \\ X_2 & X_1 \end{bmatrix} \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) : X_1, X_2 \in \mathbb{M}_{n\times n}(\mathbb{R}) \right\}$$

for $\mathbf{X} = X_1 + jX_2 \in \mathbb{M}_{n\times n}(\mathbb{P})$. In this study, split-complex numbers and matrices will be shown in bold small and bold big letters, respectively.

According to the fundamental theorem of the set of split-complex numbers, every polynomial of the n-th degree has a split-complex root of n^2. For example, a second-order polynomial defined in the split-complex number has exactly 4 split-complex roots, and the polynomial can be factored in two different ways. Since the roots of polynomial $P(\mathbf{z}) = \mathbf{z}^2 - 4$ are $\mathbf{z}_1 = 2$, $\mathbf{z}_2 = -2$, $\mathbf{z}_3 = 2j$, $\mathbf{z}_4 = -2j$ we can factorize two kinds as

$$P(\mathbf{z}) = (\mathbf{z}-2)(\mathbf{z}+2) = (\mathbf{z}-2j)(\mathbf{z}+2j).$$

Also, the characteristic polynomial of a 2×2 split-complex matrix has 4 roots and can be factored into 2 different forms. Thus, a 2×2 matrix has 2 sets of eigenvalues and eigenvectors. Detailed information on this subject can be found in Poodiack's and LeClair's excellent article [19]. For example, characteristic polynomial of

$$\mathbf{A} = \begin{bmatrix} 3-j & 1+2j \\ 2-2j & j \end{bmatrix}$$

is $P(\mathbf{z}) = \mathbf{z}^2 - 3\mathbf{z} + j + 1$, and it can be written as

$$P(\mathbf{z}) = \left(\mathbf{z} - \frac{5-j}{2}\right)\left(\mathbf{z} - \frac{1+j}{2}\right) = (\mathbf{z}-1-j)(\mathbf{z}-2+j).$$

Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{P})$$

be given. A 2×2 matrix $\mathbf{A}$ has two different eigenvalues set $S_1 = \{\lambda_1, \lambda_2\}$ and $S_2 = \{\mu_1, \mu_2\}$ where

$$\boldsymbol{\lambda}_{1,2} = \frac{\mathrm{tr}\mathbf{A}}{2} \pm \frac{\sqrt{\Delta}}{2} \text{ and } \mu_{1,2} = \frac{\mathrm{tr}\mathbf{A}}{2} \pm \frac{\sqrt{\Delta}}{2} j$$

and $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4 \det \mathbf{A}$. Here, $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are called primary roots of $\mathbf{A}$.

3. Exponential of a Split-Complex Matrix

In this section, we will examine exponential of a 2 by 2 split-complex matrix. First, let's give the exponential of an upper triangular $2{\times}2$ matrix. These formulas are given in Bernstein's article for complex matrices [3]. The following lemmas can be proved similarly. However, as stated in Lemma 1, if the split-complex number $\mathbf{a}_{11}{-}\mathbf{a}_{22}$ is null, the formulas for the complex numbers will not work. For example, exponential of the matrix

$$\mathbf{A} = \begin{bmatrix} 2+3j & 2+j \\ \mathbf{0} & 1+2j \end{bmatrix}$$

cannot be found by this method, since $j+1$ is not an inverse.

Lemma 1. *Let $\mathbf{A} = [\mathbf{a}_{ij}]$ be an upper triangular 2×2 split-complex matrix with $\mathbf{a}_{21} = 0$.*
i. *If $\mathbf{a}_{11} = \mathbf{a}_{22}$, then we have*

$$e^{\mathbf{A}} = e^{\mathbf{a}_{11}} \begin{bmatrix} 1 & \mathbf{a}_{12} \\ \mathbf{0} & 1 \end{bmatrix};$$

ii. *If $\mathbf{a}_{11} \neq \mathbf{a}_{22}$ and $\mathbf{a}_{11} - \mathbf{a}_{22}$ are not null, then we have*

$$e^{\mathbf{A}} = \begin{bmatrix} e^{\mathbf{a}_{11}} & \dfrac{\mathbf{a}_{12}\left(e^{\mathbf{a}_{11}} - e^{\mathbf{a}_{22}}\right)}{\mathbf{a}_{11}-\mathbf{a}_{22}} \\ \mathbf{0} & e^{\mathbf{a}_{22}} \end{bmatrix}.$$

Proof. Both formulas can be proved by induction, similar to complex numbers. □

In the above theorem, a solution is not given when $\mathbf{a}_{11} - \mathbf{a}_{22}$ is null. We will use a different method for such matrices in the next section. In the most general case, we will calculate the exponential of matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{P})$$

with diagonalization, and in which cases this method will fail for split-complex numbers.

We will examine the exponential of a 2×2 split-complex matrix $\mathbf{A}$ in three cases

i. $\Delta = 0$,

ii. $\Delta \neq 0$ and Δ is not null split-complex number,

iii. $\Delta \neq 0$ and Δ is a null split-complex number

where $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4\det\mathbf{A}$.

3.1. **Case 1 : $\Delta = 0$.** In the case $\Delta = 0$ for the matrix $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$, we encounter 4 special cases that will change the result.

i. $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}, \mathbf{a}_{21}$ are not null
ii. $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{21}$ are null, $\mathbf{a}_{12}$ is not null
iii. $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}$ are null, $\mathbf{a}_{21}$ is not null
iv. $\mathbf{a}_{11} - \mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ are null

Lemma 2 and Theorem 1 can be easily proved using elementary linear algebra knowledge for first three cases. We will only give proofs for the fourth case.

Lemma 2. *Let $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$ be a split-complex matrix with $\Delta = 0$. Then, the only eigenvalue is $\boldsymbol{\lambda} = \frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}$. So, we have different cases to determine the matrix $\mathbf{P}$, satisfying the equality $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ where,*

$$\mathbf{D} = \begin{bmatrix} \boldsymbol{\lambda} & 1/2 \\ 0 & \boldsymbol{\lambda} \end{bmatrix}$$

i): *If $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}, \mathbf{a}_{21}$ are not null*

$$\mathbf{P} = \begin{bmatrix} \mathbf{a}_{11} - \mathbf{a}_{22} & 1 \\ 2\mathbf{a_{21}} & 0 \end{bmatrix}.$$

ii): *If $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{21}$ are null, $\mathbf{a}_{12}$ is not null*

$$\begin{bmatrix} 2\mathbf{a_{12}} & 0 \\ \mathbf{a}_{22} - \mathbf{a}_{11} & 1 \end{bmatrix}.$$

iii): *If $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}$ are null, $\mathbf{a}_{21}$ is not null*

$$\mathbf{P} = \begin{bmatrix} \mathbf{a}_{11} - \mathbf{a}_{22} & 1 \\ 2\mathbf{a_{21}} & 0 \end{bmatrix}.$$

Theorem 1. *Let $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$ be a split-complex matrix with $\Delta = 0$. Then*

$$e^{\mathbf{A}} = e^{\boldsymbol{\lambda}}\mathbf{P}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}\mathbf{P}^{-1}$$

where $\mathbf{P}$ should be chosen according to the cases given in above Lemma.

Example 1. *Let's calculate exponential of the split-complex matrix*

$$\mathbf{A} = \begin{bmatrix} 3-j & 1+2j \\ 2-2j & 1+j \end{bmatrix}.$$

Since $\Delta = 0$, $\mathbf{a}_{21}$ is null and $\mathbf{a}_{12}$ is not null, we find

$$\mathbf{P} = \begin{bmatrix} 2\mathbf{a}_{12} & 0 \\ \mathbf{a}_{22} - \mathbf{a}_{11} & 1 \end{bmatrix} = \begin{bmatrix} 4j+2 & 0 \\ 2j-\mathbf{2} & 1 \end{bmatrix}.$$

Therefore, using the theorem and $(2j+1)^{-1} = \frac{-1}{3} + \frac{2}{3}j$*, we obtain*

$$e^{\mathbf{A}} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1} = e^2 \begin{bmatrix} 2-j & 2j+1 \\ 2-2j & j \end{bmatrix}.$$

In the case of $\Delta = 0$, if the numbers $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}, \mathbf{a}_{21}$ are null, then there will be a different case other than the three cases given in lemma. Before calculating exponential of a matrix for this case, let's give a lemma.

Lemma 3. *Let* $\mathbf{A} = [\mathbf{a}_{ij}]$ *be a* 2×2 *split-complex matrix with* $\Delta = 0$*. If the numbers* $\mathbf{a}_{11} - \mathbf{a}_{22}$*,* $\mathbf{a}_{12}$*,* $\mathbf{a}_{21}$ *are null, then the matrix* $\mathbf{A}$ *in the form*

$$\begin{bmatrix} x+jy & -uy\,(j+\epsilon) \\ y\dfrac{j+\epsilon}{u} & x-2y\epsilon-jy \end{bmatrix}$$

where $u \neq 0$*,* $x, y \in \mathbb{R}$ *and* $\epsilon = \pm 1$*.*

Proof. If the numbers $\mathbf{a}_{11} - \mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ are null, all of them are same type since $\Delta = 0$. They can be written as $\mathbf{a}_{11} - \mathbf{a}_{22} = \mu_1 (1+\epsilon h)$, $\mathbf{a}_{12} = \mu_2 (1+\epsilon h)$ and $\mathbf{a}_{21} = \mu_3 (1+\epsilon h)$ for $\mu_i \in \mathbb{R}$. So, we have

$$\mathbf{a}_{11} - \mathbf{a}_{22} = \mu_1 (1+\epsilon j) \Leftrightarrow \mu_1^2 (1+\epsilon j)^2 = -4\mu_2\mu_3 (1+\epsilon j)^2 \Leftrightarrow \mu_1^2 = -4\mu_2\mu_3.$$

Let the number be $\mathbf{a}_{11} = x + yj$, then $\mathbf{a}_{22} = x + yj - \mu_1 (1+\epsilon j)$. So, $\mathbf{A}$ will be

$$\mathbf{A} = \begin{bmatrix} x+yj & \mu_2 (1+\epsilon j) \\ \mu_3 (1+\epsilon j) & x+yj-\mu_1 (1+\epsilon j) \end{bmatrix}.$$

Suppose that $\overrightarrow{\mathbf{u}} = (u, 1)$, $0 \neq u \in \mathbb{R}$ is an eigenvector corresponding to the eigenvalue $\lambda = \frac{\mathbf{a}_{11} + \mathbf{a}_{22}}{2}$. So, from the equality $A\overrightarrow{\mathbf{u}} = \boldsymbol{\lambda}\overrightarrow{\mathbf{u}}$ and equality of split-complex numbers, we obtain

$$\mu_2 = -\epsilon uy, \quad \mu_3 = \frac{\mu_1}{u} - \epsilon\frac{y}{u}, \quad x - \epsilon y = \frac{\mathrm{tr}\mathbf{A}}{2}.$$

Moreover, from the equality $\mu_1^2 = -4\mu_2\mu_3$, we find

$$\mu_1^2 = 4\epsilon y\mu_1 - 4y^2 \Rightarrow \mu_1^2 - 4\epsilon y\mu_1 + 4y^2 = 0 \Rightarrow (-\mu_1 + 2y\epsilon)^2 = 0 \Rightarrow \mu_1 = 2y\epsilon.$$

Therefore, we obtain $\mu_1 = 2\epsilon y$, $\mu_2 = -\epsilon uy$, $\mu_3 = \frac{\epsilon y}{u}$ and

$$\mathbf{A} = \begin{bmatrix} x+jy & -uy\,(j+\epsilon) \\ y\dfrac{j+\epsilon}{u} & x-2y\epsilon-jy \end{bmatrix}.$$

Also, the only eigenvalue of this matrix is $x - \epsilon y$. □

Theorem 2. *If the entries* $\mathbf{a}_{11} - \mathbf{a}_{22}$*,* $\mathbf{a}_{12}$*,* $\mathbf{a}_{21}$ *of the matrix* $\mathbf{A}$ *are null, then*

$$e^{\mathbf{A}} = e^{x-\epsilon y} \begin{bmatrix} \epsilon y + jy + 1 & -uy\,(j+\epsilon) \\ \dfrac{1}{u}y\,(j+\epsilon) & 1 - jy - \epsilon y \end{bmatrix}$$

where $\mathbf{a}_{11} = x + yj$ *and* $\overrightarrow{\mathbf{u}} = (u, 1)$ *is the only eigenvector of the matrix* A.

Proof. The matrix $\mathbf{A}$ can be written as $A = \mathbf{PDP}^{-1}$ where

$$\mathbf{P} = \begin{bmatrix} u & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{bmatrix} x - \epsilon y & \frac{y}{u}(j+\epsilon) \\ 0 & x - \epsilon y \end{bmatrix}$$

Therefore, we have

$$e^{\mathbf{D}} = e^{x-\epsilon y} \begin{bmatrix} 1 & \frac{y}{u}(j+\epsilon) \\ \mathbf{0} & 1 \end{bmatrix}$$

and

$$\begin{aligned} e^{\mathbf{A}} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1} &= e^{x-\epsilon y} \begin{bmatrix} u & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{y}{u}(j+\epsilon) \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -u \end{bmatrix} \\ &= e^{x-\epsilon y} \begin{bmatrix} \epsilon y + jy + 1 & -uy(j+\epsilon) \\ \frac{1}{u} y (j+\epsilon) & 1 - jy - \epsilon y \end{bmatrix}. \end{aligned}$$

□

Example 2. *Let's calculate exponential of the split-complex matrix*

$$\mathbf{A} = \begin{bmatrix} 4j + 3 & -8j - 8 \\ 2j + 2 & -4j - 5 \end{bmatrix}.$$

Since $\Delta = 0$ *and* $\mathbf{a}_{11} - \mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ *are null, we use above theorem. From the equalities,* $\mathbf{a}_{11} = x + yj = 3 + 4j$, $\mathbf{a}_{12} = -uy(j+\epsilon) = -8j - 8$, *we get* $x = 3$, $y = 4$, $\epsilon = 1$ *and* $u = 2$, *thus we obtain*

$$e^{\mathbf{A}} = e^{-1} \begin{bmatrix} 5 + 4j & -8(j+1) \\ 2(j+1) & -4j - 3 \end{bmatrix}.$$

Theorem 3. *Let* $\mathbf{A} = [\mathbf{a}_{ij}]$ *be a* 2×2 *split-complex matrix with* $\Delta = 0$. *Then,*

$$e^{\mathbf{A}} = e^{\boldsymbol{\lambda}} \left[(1 - \boldsymbol{\lambda}) I + \mathbf{A} \right]. \tag{2}$$

Proof. In the case $\Delta = 0$, we can write $\mathbf{A} = \mathbf{PDP}^{-1}$ where $\mathbf{P}$ and $\mathbf{D}$ can be chosen as Lemma 2 above. Therefore, according to Lemma 1,

$$\mathbf{A} = \mathbf{P} \begin{bmatrix} \boldsymbol{\lambda} & 1/2 \\ 0 & \boldsymbol{\lambda} \end{bmatrix} \mathbf{P}^{-1} \Rightarrow e^{\mathbf{A}} = e^{\boldsymbol{\lambda}} \mathbf{P} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \mathbf{P}^{-1}$$

is written. Hence, we get

$$\begin{aligned} e^{\mathbf{A}} &= e^{\boldsymbol{\lambda}} \mathbf{P} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \mathbf{P}^{-1} \\ &= e^{\boldsymbol{\lambda}} \mathbf{P} \begin{bmatrix} 1 - \boldsymbol{\lambda} + \boldsymbol{\lambda} & 1/2 \\ 0 & 1 - \boldsymbol{\lambda} + \boldsymbol{\lambda} \end{bmatrix} \mathbf{P}^{-1} \\ &= e^{\boldsymbol{\lambda}} \mathbf{P} (1 - \boldsymbol{\lambda}) I \mathbf{P}^{-1} + e^{\boldsymbol{\lambda}} \mathbf{P} \begin{bmatrix} \boldsymbol{\lambda} & 1/2 \\ 0 & \boldsymbol{\lambda} \end{bmatrix} \mathbf{P}^{-1} \end{aligned}$$

$$= e^{\boldsymbol{\lambda}}\left(\left(1-\boldsymbol{\lambda}\right)I+\mathbf{A}\right).$$

On the other hand, if the entries $\mathbf{a}_{11}-\mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ of the matrix A are null, the only eigenvalue is $\lambda = x - y\epsilon$, and eigenvector is $\overrightarrow{\mathbf{u}} = (u, 1)$. So, we can write as $\mathbf{A} = PDP^{-1}$ where

$$\mathbf{P} = \begin{bmatrix} u & 1 \\ 1 & 0 \end{bmatrix} \qquad ve \qquad \mathbf{D} = \begin{bmatrix} \lambda & \mathbf{k} \\ 0 & \lambda \end{bmatrix}, \quad \mathbf{k} = \frac{y\left(j+\epsilon\right)}{u}.$$

So, we get again

$$e^{\mathbf{A}} = e^{\lambda}\mathbf{P}\begin{bmatrix} 1 & \mathbf{k} \\ 0 & 1 \end{bmatrix}\mathbf{P}^{-1} = e^{\lambda}\left(\mathbf{P}\left(1-\lambda\right)I\mathbf{P}^{-1}+\mathbf{A}\right) = e^{\boldsymbol{\lambda}}\left(\left(1-\boldsymbol{\lambda}\right)I+\mathbf{A}\right).$$

If we write the matrices, we have

$$\begin{aligned} e^{\mathbf{A}} &= e^{\boldsymbol{\lambda}}\left(\left(1-x+y\epsilon\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} x+jy & -uy\left(j+\epsilon\right) \\ y\dfrac{j+\epsilon}{u} & x-2y\epsilon-jy \end{bmatrix}\right) \\ &= e^{x-\epsilon y}\begin{bmatrix} \epsilon y+jy+1 & -uy\left(j+\epsilon\right) \\ \dfrac{1}{u}y\left(j+\epsilon\right) & 1-jy-\epsilon y \end{bmatrix}. \end{aligned}$$

□

Corollary 1. *Let $\mathbf{A} = [\mathbf{a}_{ij}]$ be a 2×2 split-complex matrix with $\Delta = 0$. Then,*

$$e^{\mathbf{A}} = e^{\frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}}\begin{bmatrix} 1+\dfrac{\mathbf{a}_{11}-\mathbf{a}_{22}}{2} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & 1-\dfrac{\mathbf{a}_{11}-\mathbf{a}_{22}}{2} \end{bmatrix}. \tag{3}$$

Proof. According to Theorem 3,

$$\begin{aligned} e^{\mathbf{A}} &= e^{\boldsymbol{\lambda}}\left[\left(1-\boldsymbol{\lambda}\right)\mathbf{I}+\mathbf{A}\right] \\ &= e^{\frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}}\left(\begin{bmatrix} 1-\boldsymbol{\lambda} & 0 \\ 0 & 1-\boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix}\right) \\ &= e^{\frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}}\left(\begin{bmatrix} 1-\dfrac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2} & 0 \\ 0 & 1-\dfrac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2} \end{bmatrix} + \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix}\right) \\ &= e^{\frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}}\begin{bmatrix} 1+\dfrac{\mathbf{a}_{11}-\mathbf{a}_{22}}{2} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & 1-\dfrac{\mathbf{a}_{11}-\mathbf{a}_{22}}{2} \end{bmatrix} \end{aligned}$$

is obtained. □

3.2. **Case 2 : $\Delta \neq 0$ and $\Delta \notin \mathbb{P}_0$.** Now, we will deal with how to find the exponent of a split-complex matrix, when discriminant (Δ) is not a null number. In this case, we have two primary eigenvalues $\boldsymbol{\lambda}_{1,2} = \frac{1}{2}\left(\text{tr}\mathbf{A} \pm \sqrt{\Delta}\right)$, since $\Delta \neq 0$. For the split-complex matrix $\mathbf{A} = [\mathbf{a}_{ij}]_{2\times 2}$, we can write as $\mathbf{A} = \mathbf{PDP}^{-1}$ where

$$\mathbf{P} = \begin{bmatrix} \mathbf{a}_{11} - \mathbf{a}_{22} - \sqrt{\Delta} & 2\mathbf{a}_{21} \\ 2\mathbf{a}_{12} & \mathbf{a}_{22} - \mathbf{a}_{11} + \sqrt{\Delta} \end{bmatrix} \text{ and } \mathbf{D} = \frac{1}{2}\begin{bmatrix} \text{tr}\mathbf{A} - \sqrt{\Delta} & 0 \\ 0 & \text{tr}\mathbf{A} + \sqrt{\Delta} \end{bmatrix}.$$

Notice that $\det \mathbf{P} = 2\sqrt{\Delta}\left(\mathbf{a}_{11} - \mathbf{a}_{22} - \sqrt{\Delta}\right)$, so the matrix $\mathbf{P}$ does not have an inverse, if $\mathbf{a}_{11} - \mathbf{a}_{22} - \sqrt{\Delta} \in \mathbb{P}_0$.

For example, for the split-complex matrix

$$\mathbf{A} = \begin{bmatrix} j+1 & 3+2j \\ 1+2j & j+3 \end{bmatrix},$$

we have

$$\mathbf{P} = \begin{bmatrix} -4j-6 & 4j+2 \\ 4j+6 & 4j+6 \end{bmatrix}$$

and this matrix has no inverse.

Theorem 4. *Let $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ be the eigenvalues of any matrix $\mathbf{A} \in \mathbb{M}_{2\times 2}(\mathbb{P})$. If $\Delta \neq 0$ and $\Delta \notin \mathbb{P}_0$, then, $\boldsymbol{\lambda}_1 \neq \boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1$ is not null. So, exponential of $\mathbf{A}$ is*

$$e^{\mathbf{A}} = \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} I + \frac{e^{\boldsymbol{\lambda}_2} - e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} \mathbf{A}. \tag{4}$$

Proof. Let $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ be the eigenvalues of the matrix $\mathbf{A} \in \mathbb{M}_{2\times 2}(\mathbb{P})$. So we can write

$$\mathbf{A} = \mathbf{P}\begin{bmatrix} \boldsymbol{\lambda}_1 & 0 \\ 0 & \boldsymbol{\lambda}_2 \end{bmatrix}\mathbf{P}^{-1}.$$

Similarly, the matrix $\mathbf{P}$ here is a matrix of eigenvectors whose columns correspond to the eigenvalues $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$. Therefore, we can write

$$e^{\mathbf{A}} = \mathbf{P}\begin{bmatrix} e^{\boldsymbol{\lambda}_1} & 0 \\ 0 & e^{\boldsymbol{\lambda}_2} \end{bmatrix}\mathbf{P}^{-1}.$$

Hence, we get

$$\begin{aligned} e^{\mathbf{A}} &= \mathbf{P}\begin{bmatrix} e^{\boldsymbol{\lambda}_1} & 0 \\ 0 & e^{\boldsymbol{\lambda}_2} \end{bmatrix}\mathbf{P}^{-1} \\ &= \mathbf{P}\begin{bmatrix} \frac{\boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2} + \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2} - \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2} & 0 \\ 0 & \frac{\boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2} - \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} + \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2} \end{bmatrix}\mathbf{P}^{-1} \end{aligned}$$

$$= \mathbf{P}\begin{bmatrix} \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} & 0 \\ 0 & \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} \end{bmatrix}\mathbf{P}^{-1}$$
$$+ \mathbf{P}\begin{bmatrix} \frac{\boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} & 0 \\ 0 & \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_2} - \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} \end{bmatrix}\mathbf{P}^{-1}$$
$$= \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} I + \frac{e^{\boldsymbol{\lambda}_2} - e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1}\mathbf{A}.$$

□

Theorem 5. *Let $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ be the eigenvalues of any matrix $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$. If $\Delta \neq 0$ and $\Delta \notin \mathbb{P}_0$, then, $\boldsymbol{\lambda}_1 \neq \boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1$ is not null. So, exponential of $\mathbf{A}$ is*

$$e^{\mathbf{A}} = \frac{\mathbf{m}}{\Delta}\begin{bmatrix} \sqrt{\Delta}\cosh\sqrt{\Delta} + (\mathbf{a}_{11} - \mathbf{a}_{22})\sinh\sqrt{\Delta} & 2\mathbf{a}_{12}\sinh\sqrt{\Delta} \\ 2\mathbf{a}_{21}\sinh\sqrt{\Delta} & \sqrt{\Delta}\cosh\sqrt{\Delta} - (\mathbf{a}_{11} - \mathbf{a}_{22})\sinh\sqrt{\Delta} \end{bmatrix} \tag{5}$$

where $\mathbf{m} = e^{(\mathrm{tr}\mathbf{A})/2}$ and $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4\det\mathbf{A}$.

Proof. Let $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ be the eigenvalues of the matrix $\mathbf{A} \in \mathbb{M}_{2\times 2}(\mathbb{P})$. We know that

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sin x = \frac{e^x - e^{-x}}{2}.$$

On the other hand, we have $\lambda_1 = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} - \sqrt{\Delta}\right)$, $\lambda_2 = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} + \sqrt{\Delta}\right)$ and $\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1 = \Delta$. Therefore, using the Theorem 4, and the equalities

$$e^{\boldsymbol{\lambda}_2} = e^{\left(\mathrm{tr}\mathbf{A} + \sqrt{\Delta}\right)/2} = e^{(\mathrm{tr}\mathbf{A})/2} e^{\sqrt{\Delta}} = \mathbf{mk},$$
$$e^{\boldsymbol{\lambda}_1} = \mathbf{mk}^{-1},$$

where $\mathbf{k} = e^{\sqrt{\Delta}}$, we find

$$\begin{aligned} e^{\mathbf{A}} &= \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} I + \frac{e^{\boldsymbol{\lambda}_2} - e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1}\mathbf{A} \\ &= \frac{\boldsymbol{\lambda}_2 \mathbf{mk}^{-1} - \boldsymbol{\lambda}_1 \mathbf{mk}}{\Delta} I + \frac{\mathbf{mk} - \mathbf{mk}^{-1}}{\Delta}\mathbf{A} \\ &= \frac{\mathbf{m}}{\Delta}\left(\left(\boldsymbol{\lambda}_2 \mathbf{k}^{-1} - \boldsymbol{\lambda}_1 \mathbf{k}\right) I + \left(\mathbf{k} - \mathbf{k}^{-1}\right)\mathbf{A}\right) \\ &= \frac{\mathbf{m}}{\Delta}\begin{bmatrix} \boldsymbol{\lambda}_2 \mathbf{k}^{-1} - \boldsymbol{\lambda}_1 \mathbf{k} + \mathbf{a}_{11}\left(\mathbf{k} - \mathbf{k}^{-1}\right) & \mathbf{a}_{12}\left(\mathbf{k} - \mathbf{k}^{-1}\right) \\ \mathbf{a}_{21}\left(\mathbf{k} - \mathbf{k}^{-1}\right) & \boldsymbol{\lambda}_2 \mathbf{k}^{-1} - \boldsymbol{\lambda}_1 \mathbf{k} + \mathbf{a}_{22}\left(\mathbf{k} - \mathbf{k}^{-1}\right) \end{bmatrix} \end{aligned}$$

$$= \frac{\mathbf{m}}{\Delta}\begin{bmatrix} \boldsymbol{\lambda}_2\mathbf{k}^{-1}-\boldsymbol{\lambda}_1\mathbf{k}+\mathbf{2a}_{11}\sinh\sqrt{\Delta} & 2\mathbf{a}_{12}\sinh\sqrt{\Delta} \\ 2\mathbf{a}_{21}\sinh\sqrt{\Delta} & \boldsymbol{\lambda}_2\mathbf{k}^{-1}-\boldsymbol{\lambda}_1\mathbf{k}+\mathbf{2a}_{22}\sinh\sqrt{\Delta} \end{bmatrix}.$$

If we write $\lambda_1 = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} - \sqrt{\Delta}\right)$ and $\lambda_2 = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} + \sqrt{\Delta}\right)$, we have

$$\begin{aligned} \boldsymbol{\lambda}_2\mathbf{k}^{-1}-\boldsymbol{\lambda}_1\mathbf{k} &= \frac{1}{2}\left(\mathrm{tr}\mathbf{A}+\sqrt{\Delta}\right)\mathbf{k}^{-1} - \frac{1}{2}\left(\mathrm{tr}\mathbf{A}-\sqrt{\Delta}\right)\mathbf{k} \\ &= (\mathrm{tr}\mathbf{A})\,\frac{\mathbf{k}^{-1}-\mathbf{k}}{2} + \sqrt{\Delta}\frac{\mathbf{k}^{-1}+\mathbf{k}}{2} \\ &= \sqrt{\Delta}\cosh\sqrt{\Delta} - (\mathbf{a}_{11}+\mathbf{a}_{22})\sinh\sqrt{\Delta}. \end{aligned}$$

Thus, we obtain 5. □

Notice that if Δ is a null number, this formula does not work.

3.3. Case 3 : $\Delta \neq 0$ and $\Delta \in \mathbb{P}_0$.

Theorem 6. *Let $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$ be a split-complex matrix with $\Delta \in \mathbb{P}_0$ and $\Delta \neq 0$. If eigenvectors of $\mathbf{A}$ are not a real vector, $\mathbf{A}$ cannot be diagonalized.*

Proof. If $\Delta \in \mathbb{P}_0$, then we can write that $\Delta = x(1+\varepsilon h)$, $x \in \mathbb{R}$. Therefore we have

$$\sqrt{\Delta} = \frac{\sqrt{2x}}{2}(1+\varepsilon j).$$

So, eigenvalues of $\mathbf{A}$ will be

$$\lambda_{1,2} = \frac{2\mathrm{tr}\mathbf{A} \pm \sqrt{2x}(1+\varepsilon j)}{4}.$$

Also, the eigenvector matrix $\mathbf{P}$ can be found as

$$\mathbf{P} = \begin{bmatrix} 2(\mathbf{a}_{11}-\mathbf{a}_{22}) - \sqrt{2x}(1+\varepsilon j) & 2(\mathbf{a}_{11}-\mathbf{a}_{22}) + \sqrt{2x}(1+\varepsilon j) \\ 4\mathbf{a}_{21} & 4\mathbf{a}_{21} \end{bmatrix}.$$

Determinant of this matrix is

$$\det\mathbf{P} = -8\sqrt{2x}a_{21}(1+j\varepsilon) \in \mathbb{P}_0.$$

So, $\mathbf{P}^{-1}$ is not defined. □

Example 3. *For the matrix,*

$$\mathbf{A} = \begin{bmatrix} 1+2j & 1-j \\ 2+2j & j \end{bmatrix}, \tag{6}$$

we have $\Delta = 2j+2 \in \mathbb{P}_0$. Although the primary eigenvalues of this matrix are different from each other, it cannot be diagonalized. We have to use another method to find the exponential of this matrix.

Remark 1. *Using the fact that* $\boldsymbol{\lambda}(1+j)=0 \Leftrightarrow \boldsymbol{\lambda}=0$, *we can write the eigenvectors of some matrices as real vectors. Some of this case,* $\mathbf{A}$ *can be diagonalized. For example, the matrix*

$$\begin{bmatrix} 3 & -j-1 \\ 2j+2 & -3j \end{bmatrix}$$

can be written as,

$$\begin{bmatrix} 3 & -j-1 \\ 2j+2 & -3j \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2-j & 0 \\ 0 & 1-2j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}.$$

Notice that $a_{11}-a_{22}$, a_{11}, a_{21} *and* Δ *are null split-complex numbers in the form* $x(1+j)$, $x \in \mathbb{R}$ *and we do not need to use the identity* $j^2=1$. *Here,* Δ *is also a square of a split-complex number.*

Remark 2. *The exponentials of the* $n \times n$ *split-complex matrix* $\mathbf{A}=X+Yj$ *can be computed by converting them to* $2n \times 2n$ *real matrices with the isomorphism*

$$\mathcal{P}(\mathbf{A}) = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix},$$

and using the help of Jordan form and the useful property $e^{\mathcal{P}(\mathbf{X})} = \mathcal{P}\left(e^{\mathbf{X}}\right)$. *The matrix* $\mathcal{P}(\mathbf{A})$ *is called the real matrix representation of the split-complex matrix* $\mathbf{A}$. *It is known that any* $n \times n$ *complex matrix* A *can be written as sum of a diagonalizable matrix* B *and nilpotent matrix* N_0 *where the matrices* B *and* N *commute. Remember that if* N *is a nilpotent matrix, then* N^k *is zero matrix for* $k \in \mathbb{Z}^+$. *The Jordan matrix decomposition of a square matrix* A *is* $A=PJP^{-1}$ *where* J *is a Jordan matrix [29]. It means that a square complex matrix* A *is similar to a block diagonal matrix* J. *In this case, we can write as*

$$J=D+N$$

where D *is the diagonal and* N *is strictly triangular and thus nilpotent matrix. Then, we have*

$$A=P(D+N)P^{-1}=PDP^{-1}+PNP^{-1}.$$

Therefore, any $n \times n$ *complex matrix* A *can be written as the diagonalizable matrix* $B=PDP^{-1}$ *and nilpotent matrix* $N_0=PNP^{-1}$, *since*

$$(PNP^{-1})^k = PN(P^{-1}P)N(P^{-1}N\cdots NP)NP^{-1} = PN^kP^{-1}=0.$$

Also, the matrices $B=PDP^{-1}$ *and* $N_0=PNP^{-1}$ *commute. This property allows us to simplify the calculation of a matrix exponential.*

$$e^A = e^{PJP^{-1}} = e^{P(D+N)P^{-1}} = Pe^{D+N}P^{-1} = Pe^De^NP^{-1}.$$

Example 4. *Let's find the exponential of the matrix 6*

$$\mathbf{A} = \begin{bmatrix} 1+2j & 1-j \\ 2+2j & j \end{bmatrix}.$$

We know that it is not diagonalized and $\Delta = 2j + 2 \in \mathbb{P}_0$. *Therefore, we convert it to the* 4×4 *real matrix*

$$\mathcal{P}(\mathbf{A}) = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 0 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

This matrix also cannot be diagonalized. But, we can write in Jordan form as

$$\mathcal{P}(\mathbf{A}) = P^{-1} J P = P^{-1}(D + N) P$$

where $N^2 = 0$. *Therefore, we obtain*

$$\begin{aligned} e^{\mathcal{P}(\mathbf{A})} &= P e^N e^D P^{-1} \\ &= \begin{bmatrix} 0 & 1 & 1/2 & -1 \\ 1 & 2 & -1/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 1 & 2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} e^1 & 0 & 0 & 0 \\ 0 & e^3 & 0 & 0 \\ 0 & 0 & e^{-1} & 0 \\ 0 & 0 & e^{-1} & e^{-1} \end{bmatrix} \begin{bmatrix} -1 & 1/2 & -1 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-1} + e^3 & 2e^{-1} & e^3 - e^{-1} & -2e^{-1} \\ 2e^3 - 2e & e^{-1} + e & 2e^3 - 2e & e - e^{-1} \\ e^3 - e^{-1} & -2e^{-1} & e^{-1} + e^3 & 2e^{-1} \\ 2e^3 - 2e & e - e^{-1} & 2e^3 - 2e & e^{-1} + e \end{bmatrix} \end{aligned}$$

since $e^N = I + N$. *As a result, according to equality* $e^{\mathcal{P}(\mathbf{A})} = \mathcal{P}\left(e^{\mathbf{A}}\right)$, *we find*

$$e^{\mathbf{A}} = \frac{1}{2} \begin{bmatrix} e^{-1} + e^3 - j\left(e^{-1} - e^3\right) & 2e^{-1} - 2je^{-1} \\ 2e^3 - 2e - j\left(2e - 2e^3\right) & e^{-1} + e - j\left(e^{-1} - e\right) \end{bmatrix}.$$

Conclusion Let $\mathbf{A} = [\mathbf{a}_{ij}]$ be a 2×2 split-complex matrix, we can compute exponential of $\mathbf{A}$ using the formulas :

- If $\triangle = 0$, then $e^{\mathbf{A}} = e^{\boldsymbol{\lambda}}\left[(1 - \boldsymbol{\lambda}) I + \mathbf{A}\right]$, where $\boldsymbol{\lambda}$ is only eigenvalue of $\mathbf{A}$.
- If $\triangle \neq 0$ and $\triangle$ is not a null split-complex number, then
$$e^{\mathbf{A}} = \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} I + \frac{e^{\boldsymbol{\lambda}_2} - e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} \mathbf{A}.$$
- If $\triangle$ is a null split-complex number, we do not give a direct computation formula without converting it to a real matrix.

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