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Research Article

Assessment of dependent risk using extreme value theory in a time-varying framework

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Abstract

Several extreme events in history have shown that the low probability and high impact extreme values may result in catastrophic losses. In this paper, we propose the use of extreme value theory with a time-varying framework to model the bivariate dependent insurance occurrences and provide more reliable risk measures, such as value at risk and expected shortfall. In this paper three models are considered; time series for the underlying volatility of the data, extreme value theory for the tail estimation, and copula to model the dependence structure are combined. The performance of the proposed generalized Pareto-GARCH-Copula model is tested using the violation numbers and backtesting methods. We then aim to assess the combined model in terms of its effectiveness in reducing the ruin probability. Results show that, compared to well-known traditional methods, which may underestimate the extreme risks, the dynamic generalized Pareto-GARCH-Copula model captures better the real-life data's behavior and results in lower ruin probabilities for heavy-tailed and non-conventional dependent insurance data.

Mathematics Subject Classification (2020). 91G70, 60G70, 62P05

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1. Introduction

The increasing frequency and the severity of catastrophic events worldwide have shown that extreme events are non-negligible. Even though these events are probabilistically infrequent, the consequences are severe if caught blindsided [20]. Extreme value theory (EVT) focuses explicitly on the tail of the distribution over a threshold and offers an insight into the severity of the future potential extreme events that can be more extreme than any previous historical event.

EVT is applied to many scientific disciplines, including flood frequency analysis, fire insurance claims, and financial time series. It became favorable in actuarial science in the 1990s and since then, used for the modeling of unusually high and unexpected non-life insurance losses [4], and mortality [5]. Implementation of EVT is also present in

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the pricing of the Swiss Re mortality index bond [2] and reinsurance layers [22]. New approaches develop based on the EVT to find the capital requirements set by the Basel III framework [28].

Even though past studies in EVT used static parameters, in some cases, the shape of the tail can change over time due to exogenous factors. Based on this, an approach is suggested where the threshold itself is included in the model as an unknown [3]. Later on, a time-dependent parametric form is proposed to capture the data volatility [30], at which a generalized autoregressive conditional heteroscedasticity (GARCH) model is used to forecast the time-varying parameters of the tail distribution. The addition of time series models to the EVT allows some eventual asymmetry and historical data changes to reflect on the model. In the presence of heavier volatility than that of in insurance data, for example, financial time series, a Bayesian non-parametric form of EVT is proposed [6].

Considering that insurance companies usually have business in more than one branch (life insurance, motor insurance, etc.), creating correlated loss sources, the interrelations and diversification effects in assessing the risk should be examined. Generally, the normal distribution method is not suitable for measuring the tail dependence in extreme value data sets [31]. Therefore, from the perspective of risk modeling, extreme value copula models are frequently used as it allows discovering the unique dependence structure of multivariate extreme distributions [12].

The main aim of this study is to quantify the multidimensional risk covered by insurance contracts and provide robust risk measure estimations by integrating copula and time series to the time-varying EVT. The sliding windows approach is used in parameter estimation to incorporate the time-varying changes in data and to consider the threshold as a model parameter. The proposed generalized Pareto distribution (GPD)-GARCH-Copula model considers volatility, time-varying changes in data and the potential dependence on the extreme values on the right tail. We utilize value at risk (VaR) and expected shortfall (ES) risk measures to quantify the risk for the selected confidence level, which are also used in practice to determine the capital requirements. We test the model in univariate and bivariate cases to analyze the effect of multidimensionality and compare it with other traditional estimation methods; historical simulation (HS), normal approximation (NA), and exponentially weighted moving average (EWMA). We implement backtesting methods to compare the accuracy of violation numbers generated from the risk measure estimation methods.

The secondary target of the paper is to investigate the benefits of the GPD-GARCH-Copula model on the ruin theory for the heavy-tailed data. We analyze whether the proposed model will make a change in the probability of ruin by including extreme values in the modeling. By implementing the risk measure estimations as initial surplus, we study the effects of each competing method on the heavy-tailed and time-varying asymptotic ruin probability.

The rest of the paper is organized as follows. Section 2 introduces the methodology for EVT, risk measure estimation methods, ARMA-GARCH, and copula model. The time-varying structure is briefly explained in Section 3 which is followed by Section 4 describing the use of the backtesting methods to compare the validity of the risk measure results. Section 5 reviews the ruin probability under the heavy-tailed data assumption. Section 6 presents the implementation of the method to real-life data and its numerical findings. Section 7 concludes the study.

2. Risk measures and dependence in extreme value theory

From the insurance point of view, risk management is principally interested in high losses in the right tail. EVT uses statistical methods to extract information from the extreme values in a given data. The methodology is divided mainly into two principal approaches: *(i)* block maxima method (BMM), and *(ii)* peaks over threshold (POT). Both have the purpose of modeling the series of maxima or minima, depending on the interest of the study.

BMM divides the stream of data into several intervals that have the same fixed length. The highest value in each interval is considered the extreme value. Depending on the length of the interval, the number of extreme values can be very few. The uncertainty of the interval length is the main shortcoming of BMM. Considering that we are interested in extreme events that do not occur frequently, data loss becomes an issue. On the other hand, POT overcomes this drawback by fitting a GPD to the values exceeding the determined threshold. It is considered to be the more useful method for real-world applications. However, the POT has its drawback in determining a sufficiently high threshold to start modeling.

POT models the exceed values over a predefined threshold value. Let X_i, $i = 1, \dots, n$, be independent random variables with a common distribution function F. The maximum of these n random variables are defined as $M_n = \max(X_1, \dots, X_n)$. If there exists the constants $a_n > 0$, $b_n \in \mathbb{R}$ and non-degenerate distribution function H, such that

$$\lim_{x\to\infty} \left(\frac{M_n - b_n}{a_n} < x \right) = H(x), \tag{2.1}$$

then the distribution function F belongs to the maximum domain of attraction of the extreme value distributions H ($F \in \mathrm{MDA}(H)$).

Given that X having a cumulative distribution function (cdf) F, where $x_F = \inf\{x : F(x) = 1\}$, the conditional distribution of exceedances over a sufficiently high threshold, $F_u(y)$ $u < x_F$, is

$$\begin{aligned} F_u(y) &= P(X - u \le y \mid X > u) \\ &= \frac{F(u+y) - F(u)}{1 - F(u)}, \end{aligned} \tag{2.2}$$

where, $0 \le y \le x_F - u$, and $x_F \le \infty$.

Theorem 2.1 (Pickands, Balkema De Haan (PBH)). *For a large family of distributions, as the sufficiently large threshold value u is progressively raised, F_u can be approximated by a two parameter GPD with parameters ξ for shape and σ for scale, such that*

$$F \in MDA(H) \leftrightarrow \lim_{u\to x_F} \sup_{0\le y\le x_F - u} |F_u(y) - G_{\xi,\sigma}(y)| = 0. \tag{2.3}$$

and here, the cdf of the GPD is defined as

$$G_{\xi,\sigma}(y) := \begin{cases} 1 - \left(1 + \xi \frac{y}{\sigma}\right)^{-1/\xi} & \text{for } \xi \neq 0 \\ 1 - \exp\left(-\frac{y}{\sigma}\right) & \text{for } \xi = 0 \end{cases} \tag{2.4}$$

where, $0 \le y \le x_F - u$ if $\xi \ge 0$, and $0 \le y \le -\sigma/\xi$ if $\xi < 0$. The proof of PBH theorem can be found in [1, 23].

Based on the real-life data, we can employ the PBH theorem to determine the distributional behavior of the tail, which is expected to cause the insurance company to pay extreme losses. Having this as an important setup, we need to make a plausible selection on the threshold that is decisive in the form of tail distribution through parameter estimation. By setting a low threshold (u), one can expect to have a larger data set which becomes advantageous for better parameter estimation. However, the POT method is based on the convergence and assumption of the selected threshold approaching the x_F. It is known that the higher the threshold is, the better the GPD fits [13]. Therefore, choosing an appropriate threshold refers to the assurance of a balance between bias and variance. However, there is no exact norm for selecting the right threshold [15]. For this reason, we use the conventional quantile-quantile (QQ), mean excess (ME), and Hill plots to examine and determine the optimal threshold level.

A static model, which is generally suitable for independent and identically distributed (iid) data, applies to the EVT unconditionally to the observations. However, the iid condition does not hold for most real-world cases, especially, under high volatility and dependent data structure. In recent years, the idea that has become widespread in risk measure calculations is the necessity to consider the time-varying changes in the observations.

To capture the possible changes in data; including volatility, autocorrelation, and drift, we propose a comprehensive ARMA(p, q)-GARCH(m, s) model which is given as

$$\begin{aligned} X_t =& \sum_{i=1}^{p} \varphi_i X_{t-i} + \sum_{j=1}^{q} \varepsilon_{t-j} + \varepsilon_t, \\ \varepsilon_t =& \sigma_t \nu_t, \\ \sigma_t^2 =& \alpha_0 + \sum_{i=1}^{m} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2, \end{aligned} \tag{2.5}$$

where $\nu_t > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, σ_t^2 is the variance process, and ε_t is residual at time t.

The claims over the time frame can be captured well by the time series model whose variance is known to be varying over time. The ARMA-GARCH model explained above, builds up the main foundation of the loss models whose residuals are used to develop a sophisticated model for the tail behavior. In the bivariate case, where both of the variables following ARMA-GARCH, yield residuals, whose tails also imitate the similar extreme loss behavior as the original data. The well-known risk measures VaR and ES can be taken as leading indicators to determine the boundary for the threshold value. Given the loss random variable X and its distribution function F, VaR_q and ES_q are defined as

$$\mathrm{VaR}_q := \min\{x : F(x) \geq q\}, \tag{2.6}$$

$$\mathrm{ES}_q := E\left[X | X > \mathrm{VaR}_q\right], \tag{2.7}$$

respectively. The number of exceedances above a pre-specified threshold u, N_u over n observations, is $N_u = \#\{1 \leq i \leq n : x_i > u\}$. Therefore, the empirical distribution function at u, $F(u)$, becomes

$$\mathrm{F}(u) = \frac{n - \mathrm{N}_u}{n}. \tag{2.8}$$

When $\mathrm{F}_u(y)$ is replaced by $\mathrm{G}_{\xi,\sigma}(y)$, the distribution function is obtained as

$$\mathrm{F}(x) = 1 - \frac{\mathrm{N}_u}{n}\left(1 + \xi \frac{x - u}{\sigma}\right)^{-1/\xi}. \tag{2.9}$$

Given the quantile q, the risk measures can be derived such that

$$\mathrm{VaR}_q = u + \frac{\sigma}{\xi}\left[\left(\frac{n}{\mathrm{N}_u}(1 - q)\right)^{-\xi} - 1\right], \tag{2.10}$$

$$\mathrm{ES}_q = \frac{\mathrm{VaR}_q}{1 - \xi} + \frac{\sigma - \xi u}{1 - \xi}. \tag{2.11}$$

The credibility and reliability of these risk measures should be evaluated due to extreme value behaviors. Therefore, we propose a comparative analysis using the examination of the performance of risk measures obtained by GPD-GARCH-Copula with traditional methods. The reason behind this is that traditional methods have their own drawbacks. For example, the HS method is based on the convergence of the empirical distribution to the actual loss distribution, which relies on the stability of the historical data set. The NA method provides a straightforward methodology and application, but it overlooks the heavy tail and skewed distributional properties. The EWMA method can explain the time-varying changes in the volatility to an extent by assigning exponentially decreasing weights to each past data. Although these models are naive, they are still the most commonly used ones in practice. We expect the proposed approach overcomes the shortcomings of these well-known methods.

When we consider the existence of an association between two loss variables, the tail portions of those are expected to keep the same dependence. Along with conventional correlation measures, the theoretical flexibility of copula in capturing the dependence should be taken as an advantage to tackle the dependence among risks. If the underlying distribution is other than an elliptic distribution, the linear correlation is insufficient to measure the dependence. Without requiring any assumptions about the underlying distribution, the copula allows the risk manager to separate an n-dimensional joint distribution function into its n marginal distributions and a copula function.

A d-dimensional copula is a multivariate distribution function denoted by $C(u)$, $u = \{u_1, \dots, u_d\}$, that is defined on $[0,1]^d$. The behavior of d random variables $X_1, \dots, X_d$ with marginal distribution functions $F_1(x_1) = u_1, \dots, F_d(x_d) = u_d$ is described by their joint distribution function F, and there exists a copula C, such that [26]

$$F(x_1, \dots, x_d) = C(u_1, \dots, u_d). \tag{2.12}$$

Based on the dependence structure, there are two families of copulas, elliptical and Archimedean. Along with these two families, we consider the extreme value copula family, which can capture the various dependence structures occurring at the most tail portions. We study the selection of optimal copula model based on Akaike and Bayesian information criteria among Gaussian, Student's-t, Clayton, Frank, Gumbel, Joe-Clayton, Placket copula, and their variations.

3. Time-varying model parameters

The threshold in the traditional approach is generally chosen as a high enough percentile of the data [10], and it is fixed during the analysis period. However, the literature shows that for the POT method the selected threshold value significantly affects the parameter estimates [9].

Typically if we consider a financial or actuarial time series, non-stationarity and changes in tail behavior are observed. For this reason, an approach that accounts for time-varying changes in the parameters needs to be developed. A new model is proposed in which the threshold itself is a parameter [3], and the possible approaches to model the data below the threshold are shown [25]. Nevertheless, all these models consider static and GPD parameters.

Failure to recognize time-dependent changes can cause the premium and risk measures used by the insurance companies to determine the capital amounts in the long term to be inaccurate and nonstable. The dynamic GPD-GARCH-Copula model connects the threshold parameter to other model parameters via a moving windows approach. The window length in which the parameters are re-estimated should be selected as not too broad or narrow to ensure enough extreme values above the threshold for the GPD fit.

4. Backtesting methods

Backtesting the violation numbers are considered in the analysis of the forecasting performance of the risk measure estimation methods. The violation number is defined as the number of times the observed value exceeds the forecasted risk measure in the testing period. The proportion of the violation number to the length of the testing period should be close to the selected confidence level. If this is not the case, the proposed method of risk measure calculation is not valid or reliable. Widely used tests in literature are Kupiec's proportion of failures (POF), Christoffersen's independence and interval forecast (IND), and composite (CC) tests.

Kupiec's POF Test [21] is the most commonly used backtest in literature. Given that the number of violations is v and the sample size is n, it follows a binomial distribution with parameters n and q. The null hypothesis is; the observed violation rate, (v/n), equal to the expected violation rate, $(1-q)$, with the log-likelihood ratio, $\mathrm{LR}_{\mathrm{POF}}$,

$$\mathrm{LR}_{\mathrm{POF}} = 2\left[\ln \frac{\left(\frac{v}{n}\right)^{v}\left(1-\frac{v}{n}\right)^{n-v}}{(1-q)^{v} q^{n-v}}\right] \tag{4.1}$$

which follows the χ_1^2 distribution. Thus the model is rejected if the violation number is too high or low. The drawback of this test is that it may fail to reject a model having a violation clustering.

Christoffersen's independence test depicts if an outcome of an estimate (violation or not) is dependent or independent of the previous estimate. The test first categorizes the outcomes via an indicator, $I_t = \{0,1\}$, where 0 refers to no violation and 1 refers to a violation at time t. For the two following days, similar to a Markov chain, there are four possible categories, $I_{t,t+1} = \{00, 01, 10, 11\}$, whose number of outcomes is represented by n_{ij} as described in Table 1.

Table 1. Christoffersen's Markov chain representation [8].

	$I_{t-1}=0$	$I_{t-1}=1$	Total
$I_t = 0$	n_{00}	n_{10}	$n_{00}+n_{10}$
$I_t = 1$	n_{01}	n_{11}	$n_{01}+n_{11}$
Total	$n_{00}+n_{01}$	$n_{10}+n_{11}$	N

The violation probabilities π_i for $i = 0, 1$, are expressed as

$$\begin{aligned} \pi_0 &= \frac{n_{01}}{n_{00}+n_{01}}, \quad \pi_1 = \frac{n_{11}}{n_{10}+n_{11}}, \\ \pi &= \frac{n_{11}+n_{01}}{n_{00}+n_{01}+n_{10}+n_{11}}. \end{aligned} \tag{4.2}$$

Hence, the resulting log-likelihood ratio becomes

$$\mathrm{LR}_{\mathrm{IND}} = -2\left[\ln \frac{(1-\pi)^{(n_{00}+n_{10})}\,\pi^{(n_{01}+n_{11})}}{(1-\pi_0)^{n_{00}}\,\pi_0^{n_{01}}\,(1-\pi_1)^{n_{10}}\,\pi_1^{n_{11}}}\right]. \tag{4.3}$$

Similar to the POF test, the $\mathrm{LR}_{\mathrm{IND}}$ function follows χ_1^2 distribution. The disadvantage of this test is that it only tests the independence of two consecutive days. Therefore, it ignores the violations above or below two days.

To address the problem of the independence test, a composite independence test is proposed [7]. The simplified format for the composite test statistic is defined as

$$\mathrm{LR}_{\mathrm{CC}} = \mathrm{LR}_{\mathrm{POF}} + \mathrm{LR}_{\mathrm{IND}}. \tag{4.4}$$

The combined $\mathrm{LR}_{\mathrm{CC}}$ function follows χ_2^2 distribution. The composite test can detect a violation from POF and IND tests combined. It comes with a decreased ability to detect a violation of only one of the two tests.

5. Ruin probability for heavy-tailed distributions

Most of the studies focus on the EVT model and compare its result with more traditional method; hardly any of them investigates its effects on the ruin probabilities. In insurance, we often deal with heavy-tailed distributions. Therefore in EVT, its asymptotic behavior is used to represent the actual probability [17, 19]. Similarly, the ruin probability can be approached asymptotically [14].

We now aim to study the effects of EVT on the asymptotic ruin probability under some heavy-tailed distributional assumptions. The classical CramérLundberg model is built upon the relationship between the initial surplus, premiums collected, and the loss paid. The number of losses, $N(t)$, follows a homogeneous Poisson(λ_t) counting process, and it is assumed to be independent from the loss $X_t \geq 0$, which is iid with common distribution function, F, $\mathrm{E}(X) = \mu_X < \infty$ and $\mathrm{Var}(X) = \sigma_X^2 < \infty$. The cumulative loss process, $S(t)$, is defined as

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{5.1}$$

and it forms an independent compound Poisson process with the expected value of $\mathrm{E}[S(t)] = \mu_X \lambda_t$, under independence assumption. Here, EVT is emphasized since $S(t)$ is mostly affected by the few extremely large claims in the tail portion.

Assuming that $\sigma_X^2 < \infty$ in the Cramér-Lundberg model, by CLT $\{S(t) - \mathrm{E}[S(t)]\}$ fluctuates in the order of $\sqrt{t}$. By definition, the extreme values in the loss process only occur if $\sigma_X^2 = \infty$, which creates a contrast to the classical theory designed for the framework of the small claims. On the other hand, the $\sigma_X^2 = \infty$ condition is not a rarely seen circumstance for insurance data. The extreme portion of the distributions, which by nature does not have a finite moment generating function (mgf) and tails follow a power law and can be modeled by EVT.

Consider a large initial capital denoted with $u_0 \geq 0$ and a premium rate denoted with $c(t) > \lambda_t \mu_X$ earned continuously with time. The surplus process following the Cramér-Lundberg model is defined as

$$U(t) = u_0 + c(t) - S(t), \quad t \geq 0, \tag{5.2}$$

where, the premium is considered under the expected value principle, such that $c(t) = (1+\rho)\lambda_t \mu_X$, and $\rho > 0$ denotes the premium safety loading factor. By taking $c(t) > \mathrm{E}[S(t)]$, the ruin with probability one is avoided, and $U(t)$ is ensured to have a positive drift. The infinite time horizon ruin probability, $\psi(u_0)$, is the probability that the first time the surplus falls below 0, such that

$$\begin{aligned} \psi(u_0) &= P\left(\inf_{t\geq 0} (U(t) < 0)\right) \\ &= P\left(\inf_{t\geq 0}\left((1+\rho)\mu_X\lambda_t - \sum_{i=1}^{N(t)} X_i < -u_0\right)\right). \end{aligned} \tag{5.3}$$

Pollaczek-Khintchine formula [18, 24] defines the ladder heights as

$$L_t = \sup_{t\geq 1}(u_0 - U(t)) = \sup_{t\geq 1}(S(t) - c(t)), \tag{5.4}$$

where the process L_t has stationary and independent increments, and the number of the ladder heights have Geometric distribution with parameter $\theta = \rho/(1+\rho)$ and $t \geq 1$. Under the net profit condition of the CramérLundberg approximation, where $c > \mu_X \lambda_t$ for 'small claims' X_t has a finite moment generating function. Then, there exists a Lundberg constant (adjustment coefficient) $R \in (0, \infty)$ that is derived from the Esscher transform of F, which leads to

$$\int_0^\infty e^{-Rx}\overline{F}(x)dx = \frac{c}{\lambda}, \tag{5.5}$$

where $\overline{F} = 1 - F$. Therefore, the ruin is very unlikely and ruin probability has an upper bound such that [16],

$$\psi(u_0) \leq e^{-Ru_0}, \quad u_0 \geq 0, \tag{5.6}$$

which leads to an asymptotic limit, a constant value $C \in (0, 1)$, such that

$$\lim_{u_0 \to \infty} \psi(u) e^{R u_0} = C. \tag{5.7}$$

The smaller claims assumption, providing the existence of unique R, can be shown in terms of $\mathrm{E}(e^{R x_t})$ and this property holds for any distribution function with exponentially bounded tail functions. For heavy-tailed risk losses (with power tail behavior), the smaller claims condition in Equations 5.6 and 5.7 are typically not satisfied. Therefore, for the heavy-tailed distributions ruin probability should be reconsidered. Let $f_I(y)$ be the integrated density function, and F_I be the integrated tail distribution function of F,

$$\frac{d}{dy} F_I(y) = f_I(y) = \frac{1 - F(y)}{\mu_X}, \tag{5.8}$$

$$1 - F_I(u_0) = \frac{1}{\mu_X} \int_{u_0}^{\infty} \overline{F}(y) dy, \quad x \geq 0, \tag{5.9}$$

respectively. Under these conditions by using compound Geometric distribution, the probability of ruin can be obtained as

$$\psi(u_0) = \frac{\rho}{1+\rho} \sum_{n=0}^{\infty} \frac{1}{(1+\rho)^n} \overline{F_I^{n*}}(u_0), \quad u_0 \geq 0. \tag{5.10}$$

To express the n-fold convolution of integrated F, $F_I^{n*}(u_0)$, we let X has a Pareto like distribution tail such that, $\overline{F}(x) \sim c x^{-\alpha}$ for $x \to \infty$, where c and α are positive constants. Then, the n-fold tail convolution of F becomes

$$\begin{aligned} \overline{F^{n*}}(x) &= P(X_1 + \cdots + X_n > x) \\ &\sim P(\max_{1 \leq i \leq n} X_i > x) \\ &\sim n \overline{F}(x), \quad x \to \infty. \end{aligned} \tag{5.11}$$

Equation 5.11 leads to the defining property of a sub-class of heavy-tailed distributions called subexponential distributions, and is denoted by $\mathcal{S}$, $\forall n \geq 2$. For all $F \in \mathcal{S}$

$$\lim_{x \to \infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n, \tag{5.12}$$

where $x > 0$, all widely used heavy-tailed distributions fall under the class of $\mathcal{S}$ and subexponential distributions contain the distributions with regularly varying tails. Dividing both sides of Equation 5.10 to $\overline{F_I}(u_0)$ and using subexponential distribution, one can write

$$\lim_{u_0 \to \infty} \frac{\psi(u_0)}{\overline{F_I}(u_0)} = \frac{\rho}{1+\rho} \sum_{n=0}^{\infty} \frac{1}{(1+\rho)^n} n = \frac{1}{\rho}, \tag{5.13}$$

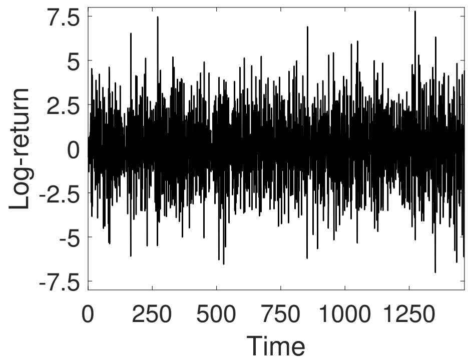
by interchanging $\lim_{u_0 \to \infty}$ with $\sum_{n=0}^{\infty}$. The tail distribution mostly determines the ruin probability and the solvency of the system [14], such that

$$F_I \in S \Leftrightarrow \psi(u_0) \sim \rho^{-1} \overline{F_I}(u_0), u_0 \to \infty \Leftrightarrow (1 - \psi(u_0)) \in \mathcal{S}. \tag{5.14}$$

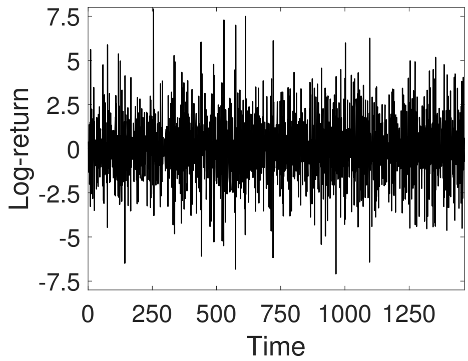
Hence, in heavy-tailed cases, ruin can occur even in the case of a sufficiently large threshold and it highly depends on the integrated tail of the loss distribution [11].

6. Implementation

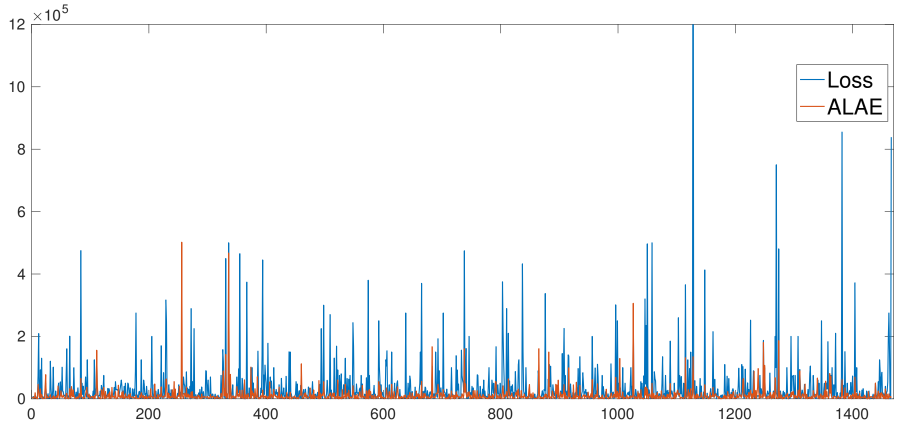
The real-life insurance data from the Copula package of the R programming language, containing 1500 accident indemnity payments (Loss) and corresponding allocated loss adjustment expenses (ALAE) is utilized to illustrate the proposed combined model. ALAE data set covers mainly the company expenses attributable to the settlement of losses, including fees paid to outside attorneys, medical consultancy, insurance experts, legal fees, etc. These general expenses are mostly covered by the insurer's expense reserves which are dependent on claims payment process. After extracting 34 claims, which are left truncated and censored due to policy limits and deductibles, we plot the uncensored 1466 claims in original and log-return scales shown in Figure 1. We find the Loss and ALAE return series are mean reverting and show increased volatility periods.



(a) Loss log-return.



(b) ALAE log-return.



(c) Loss and ALAE original data.

Figure 1. Plots of data in log-return and original scale.

Insurance data generally violates the normal distribution assumptions, and the extreme values at the tail portions require more attention. The descriptive statistics in Table 2 depict that Loss and ALAE yield wide ranges. They both do not have a symmetrical distribution and show heavier tails than a standard normal distribution. Jarque-Bera test validates that both data sets are not normally distributed, and the Augmented Dickey-Fuller test rejects the existence of unit root. To facilitate the implementation of the models, log-return series are used in the application.

The stepwise implementation of the proposed approach is presented in Table 3 in detail. Based on the algorithm, we first assess if the data is heavy-tailed to apply EVT by using QQ plots (Figure 2). The Loss and ALAE data set asymmetrically disperse from the normal distribution in the left and right tails and show convexity, indicating heavier tails and right skewness for the distribution. While the difference is slight for the left tail, we can see that the right tail deviates considerably from the normal distribution, suggesting a heavy right tail.

Next, we employ ME plots (Figure 3) to aid in the selection of threshold values by investigating where the function shows linearity and determining the GPD model's adequacy

Table 2. Descriptive statistics for Loss and ALAE.

	Loss	ALAE
Minimum	10	15
Maximum	2,173,595	501,863
Mean	37,110	12,017
Skewness	10.9656	10.0787
Kurtosis	3.3655	4.0912
Standard Deviation	92,513	26,712
Jarque-Bera Test	$2.6444e^6$	$1.3918e^6$
ADF Unit Root Test	-32.2487	-31.76284
Observations	1466	1466

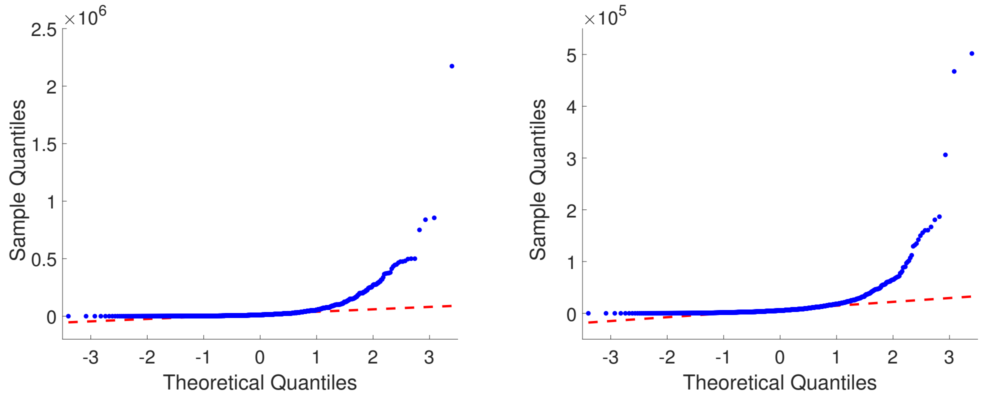
Table 3. Algorithm of the bivariate dynamic EVT-GARCH-Copula model.

```
Compute the portfolio return of the log-return data set.
for j ={Loss, ALAE}
    for p, q, m, s = 0, 1, 2
        Fit ARMA(p, q)-GARCH(m, s)
        Compute AIC, BIC.
    end
end
Optimal models are determined by min{AIC, BIC}.
for j ={Loss, ALAE}
    for k = 1 : (n − 1) − (moving window length).
        for i = k : k + (moving window length)
            ARMA(p, q)-GARCH(m, s) parameters are estimated.
            Standardized residuals (std. res.) are obtained.
            Std. res. are converted to uniform std. res.
            Fit copula model to uniform std. res. and simulate one day ahead estimates.
            Back transform uniform std. res. to std. res. using the inverse cdf.
            EVT applied to simulated std. res.
            Select threshold value, and estimate GPD parameters.
        end
        Use GPD parameters, estimate residual VaR_j and ES_j.
        Use ARMA-GARCH coefficients, estimate VaR_j and ES_j.
    end
    Compute the portfolio VaR and ES.
    Determine if there is a violation.
    Compute the heavy tailed ruin probability, ψ(u_0).
end
```

in practice. Visually, plots show linearity for the threshold value between 3 and 5 for Loss and 2.5 and 4.5 for ALAE.

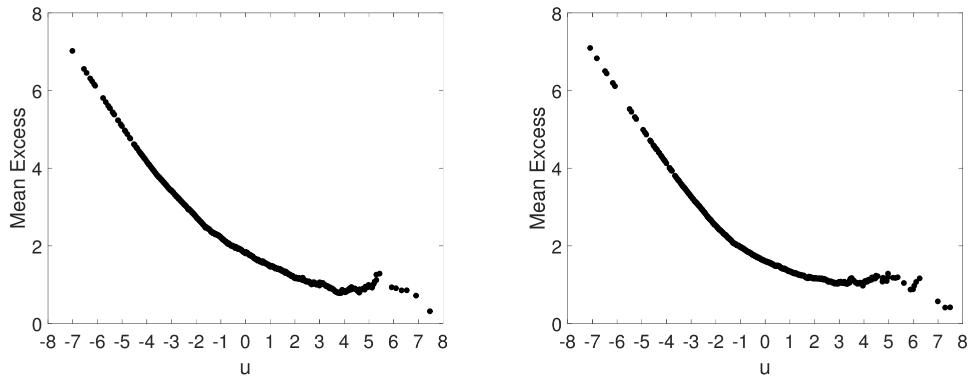
Figure 4 shows that the Hill estimators of the Loss and ALAE expose non-linearity when the ordered statistics for the threshold value are lower than they should be. When the threshold value is increased the Hill estimator becomes linear around the optimal threshold, i.e. the 1400th observation, corresponding to 95.5%. Using the information of QQ, ME, and Hill plots, 95% confidence level for the risk measures for Loss and ALAE are 3.5296 and 3.1219, respectively.

Contrary to the standard implementation of EVT in univariate cases, we consider joint tail behavior of Loss and ALAE under copula dependence structure. The joint behavior



(a) Loss. **(b)** ALAE.

Figure 2. QQ plots.



(a) Loss. **(b)** ALAE.

Figure 3. ME plots.

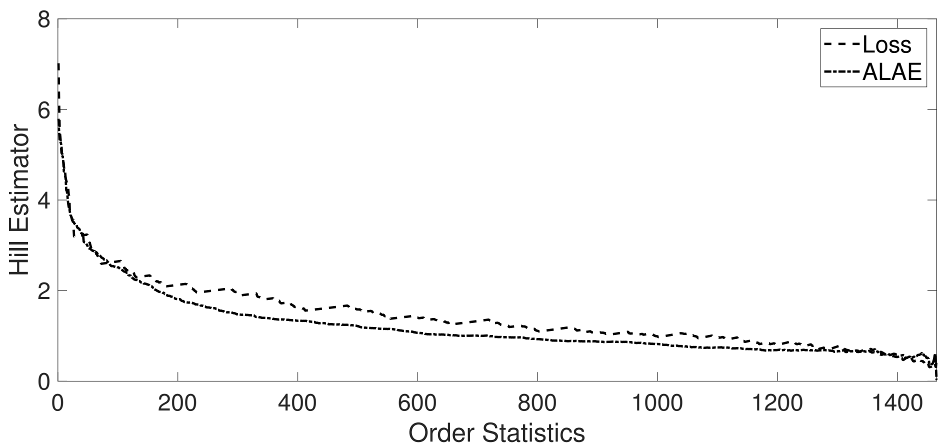


Figure 4. Hill plot for Loss and ALAE.

of Loss and ALAE is examined through copula functions with equally likely weights in a portfolio. For the bivariate case, the dependence measures given in Table 4 range between 0.3087 and 0.4437, suggesting a positive dependence between Loss and ALAE. The scatter plot on a log-scale (Figure 5) shows definite also positive dependence. We can see that in the upper tail of the figure, large values of both variables are highly correlated with each other. In contrast, the lower-left corner displays a more diffused dependence structure.

Table 4. Empirical dependence measures for Loss and ALAE.

Kendall	Pearson	Spearman	Right Tail
0.3087	0.4313	0.4437	0.4313

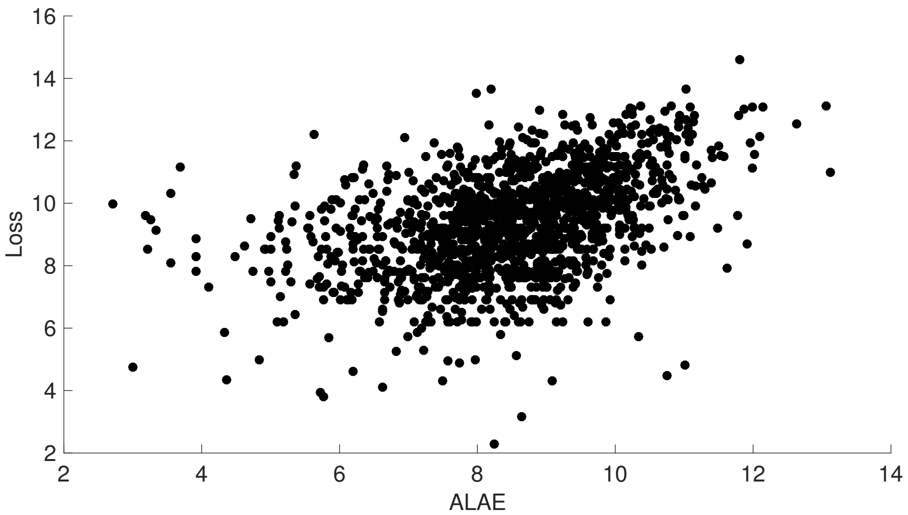


Figure 5. Scatter plot of Loss vs. ALAE observations.

For the dynamic cases, it should be pointed out that daily occurrences do not pose a strong base for implementing the proposed model. However, similar to the literature on the modeling of daily Loss and ALAE, we can state a general framework to apply our approach to larger time units, i.e., monthly, or quarterly, if the data for more extended periods are available.

We consider nine copula models to cover a variety of dependence structures. These are Gaussian, Student's-t, Clayton, rotated Clayton, symmetrized Joe-Clayton (SJC), Frank, Gumbel, rotated Gumbel, and Plackett. As we follow a dynamic approach, a copula fitting procedure is performed for the tails in every sliding window of length 250. The copula model, which most frequently fits the data within $t \in (251, 1465)$ based on log-likelihood, is found to be the SJC followed by the rotated Clayton and the Student's-t copula (Figure 6). The Gaussian copula shows the worst performance of being the optimal copula among others.

As next, we run the GPD-GARCH model for univariate and bivariate cases to utilize their results to set a base for comparison after the incorporation of the copula model. ARMA-GARCH models are fitted for each lag $\{p, d, m, s \leq 2\}$, based on AIC and BIC for the univariate static case. The best-fitting model is found to be AR(1)-GARCH(1,1) for both Loss and ALAE, and the standardized residual distribution is selected as Student's-t distribution to reflect on the heavy tail [29]. The GPD is fitted to the exceedances for both series, and parameter estimates are obtained by maximum likelihood estimation (MLE).

Finally, the univariate static $\mathrm{VaR}_{0.95}$ and $\mathrm{ES}_{0.95}$ values given in Table 5, yield the highest risk measures to be observed from GPD and EWMA methods in which the order of rank changes for VaR and ES. The results of the GPD-GARCH model for Loss can be interpreted as one day ahead claim amount will exceed 4.4209 with a probability of 0.05,

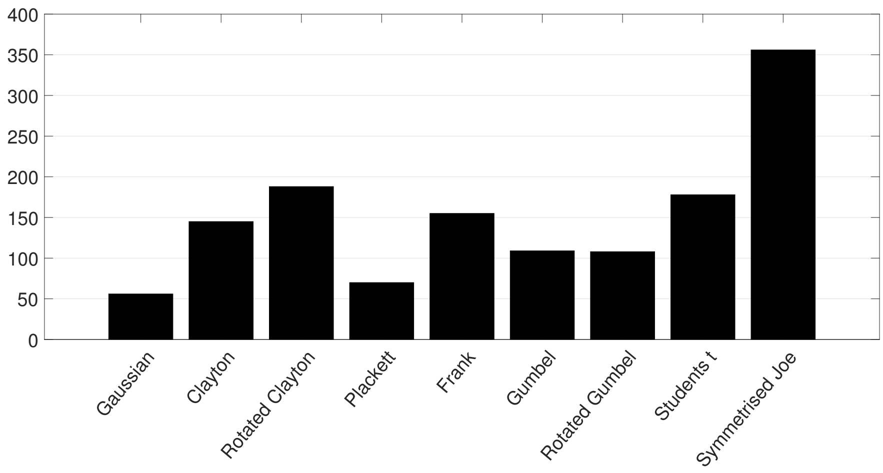


Figure 6. Optimality in hitting numbers of dynamic copula models.

Table 5. Risk measure point estimates for the univariate static case.

	Univariate			
	Loss		ALAE	
Method	VaR	ES	VaR	ES
Historical	3.5296	4.4130	3.1219	4.1712
Normal	3.4809	4.3732	3.1173	4.1594
EWMA	4.2138	5.0801	3.5494	4.6742
GPD-GARCH	4.4209	4.9041	3.3002	4.7215

i.e., given that this level is exceeded, the expected claim amount will be 4.9041. A similar interpretation holds for the ALAE and also for other methods tested.

After the static approach explained above, we test the univariate dynamic case. For the sliding window framework, 250 days of training data are used to re-estimate the parameters, and we simulate 1000 scenarios to calculate one-day ahead risk measure forecasts. Therefore, the estimated values cover 251^{th} to 1465^{th} data resulting in 1215 estimated points. In the estimation period, re-examining the ME plot and determining the threshold value in each moving window is not feasible. Therefore, we assume that the number of exceedances is the top 5% of the investigated data set in each moving window. Optimally, for each moving window, the lags should be re-estimated, which creates a computational challenge in programming. Therefore, regarding the literature [27], we fit an AR(1)-GARCH(1,1) in each moving window.

The violation numbers for the univariate and bivariate dynamic cases are calculated. The expected number of violations for $\text{VaR}_{0.95}$ is $1215 \times 0.05 \approx 61$, hence the GPD-GARCH method provides closer estimates of the actual VaR level with 68 and 67 violations as shown in Table 6. The method yielding the highest violation numbers is EWMA, even though in the static case in Table 5, EWMA has the closest risk measure estimations to the best performing GPD-GARCH method. This proves the need for the dynamic method over the static one and the importance of the moving windows approach for the risk measure estimation.

In the bivariate dynamic case, referred to as Portfolio (Table 6), the decrement in the violation rates from the univariate dynamic (68 and 67 violations) to the bivariate dynamic (65 violations) case means that gathering several insurance branches under one roof can reduce the risk by itself. In the bivariate dynamic copula case for the portfolio, the SJC

Table 6. Violation numbers for the univariate and bivariate dynamic cases.

	Univariate				Bivariate	
	Loss		ALAE		Portfolio	
Method	VaR	ES	VaR	ES	VaR	ES
Historical	80	28	84	33	72	30
Normal	77	29	82	33	72	29
EWMA	80	33	84	35	79	32
GPD-GARCH	68	27	67	29	65	27
GPD-GARCH-Copula	-	-	-	-	62	23

copula is added to the model. The GPD-GARCH-Copula model is found to reduce the number of VaR violations from 65 to 62. However, the most significant change is for the ES measure, which is reduced from 27 to 23 violations by including the copula model.

Forecasted univariate and bivariate dynamic risk measures are given in Figure 7. For space limitations, we only include VaR estimates for the univariate case and VaR and ES estimates for the bivariate portfolio case. It shows that, while HS and NA provide closer results to each other, EWMA responds with a jump after a high loss and decreases towards the average when the subsequent loss is lower. Due to EVT incorporating the extreme values in modeling, the estimated risk measures are overall higher than the other methods.

It is observed that the EVT method can capture extremes more efficiently for the ALAE data since the Loss data contains fewer extremes than ALAE. The bivariate dynamic case shows that the GPD-GARCH-Copula method is more sensitive to high returns and responds immediately with higher VaR and ES values than any other method. As expected, dependence modeling for the most extreme data in the tail contributes to this outcome.

6.1. Backtesting

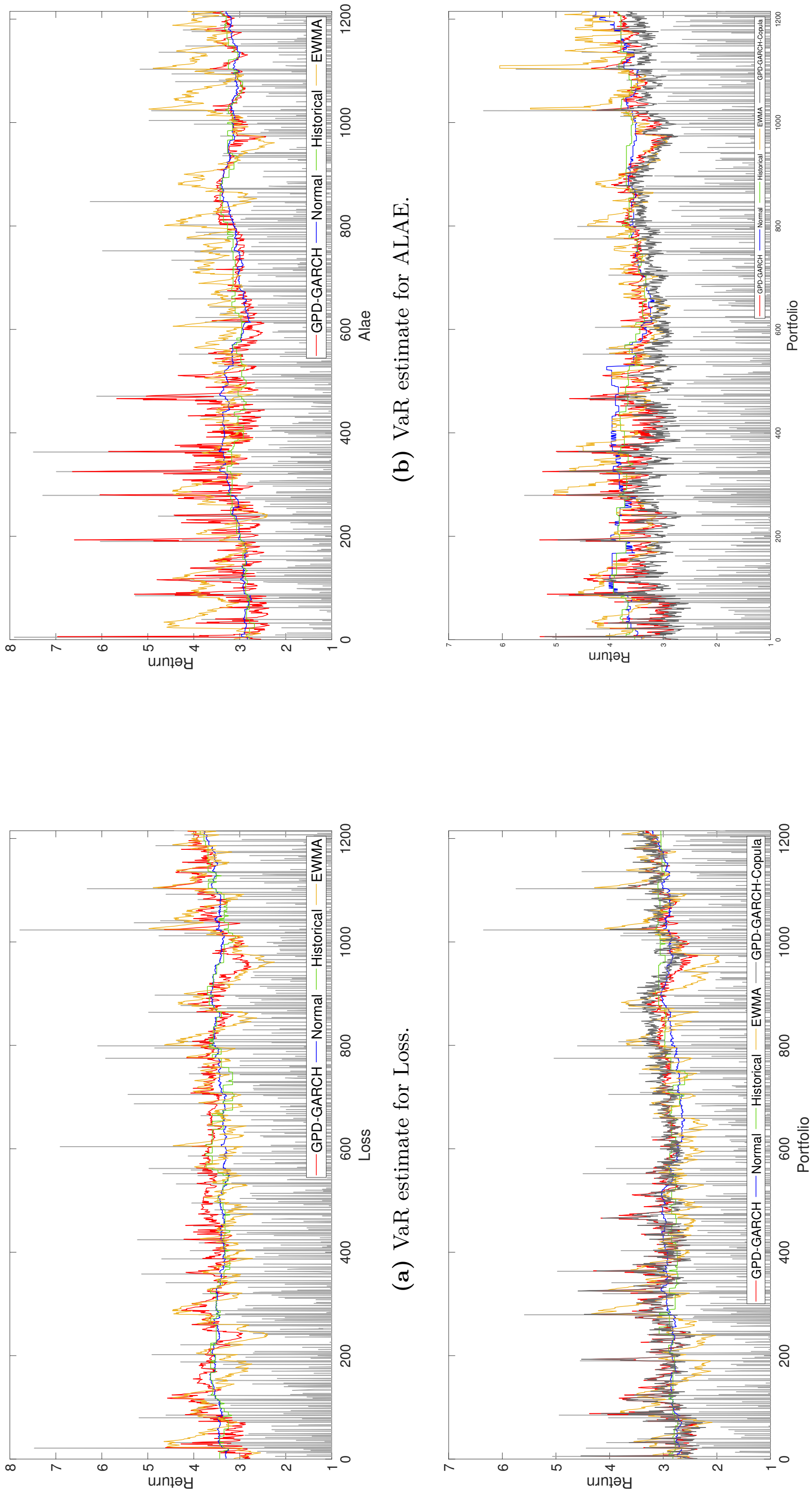
The VaR violation numbers for each method under univariate dynamic and bivariate dynamic cases are tested. POF test results are given in Table 7. In the univariate case, all methods except the GPD-GARCH fails the test (p-value< 0.05) with $\mathrm{LR}_{\mathrm{POF}} > 3.841$. However, in the bivariate case, all methods except the EWMA passes the test (p-value$>$ 0.05).

Table 7. Proportion of failures test results.

	Univariate				Bivariate	
	Loss		ALAE		Portfolio	
Method	$\mathrm{LR}_{\mathrm{POF}}$	p-value	$\mathrm{LR}_{\mathrm{POF}}$	p-value	$\mathrm{LR}_{\mathrm{POF}}$	p-value
Historical	5.8644	0.0155*	8.4119	0.0037*	2.0755	0.1497
Normal	4.2338	0.0396*	7.0858	0.0078*	2.0755	0.1497
EWMA	5.8644	0.0155*	8.4119	0.0037*	5.2936	0.0214*
GPD-GARCH	0.8784	0.3486	0.6559	0.4180	0.3063	0.5800
GPD-GARCH-Copula	-	-	-	-	0.0269	0.8697

*Significant at 5% level.

IND test results are given in Table 8. As expected, State 1 (I_{00}) is the group with the highest number of observations, as none of the models had too many violations, to begin with. In addition, State 4 (I_{11}) is encountered only one time in every method. All methods accept GPD-GARCH, and GPD-GARCH-Copula methods fail the test with $\mathrm{LR}_{\mathrm{IND}} > 3.841$, in both univariate and bivariate cases.



(a) VaR estimate for Loss.

(b) VaR estimate for ALAE.

(c) VaR estimate for Portfolio.

(d) ES estimate for Portfolio.

Figure 7. Univariate and bivariate dynamic risk measure estimates.

Table 8. Christoffersen's independence and interval forecast test results.

Univariate						
	Loss					
Method	State 1	State 2	State 3	State 4	LR_{IND}	p-value
Historical	1054	79	80	1	5.8350	0.0157*
Normal	1060	76	77	1	5.1335	0.0235*
EWMA	1054	79	80	1	5.8350	0.0157*
GPD-GARCH	1079	67	68	1	3.2839	0.0700
	ALAE					
Method	State 1	State 2	State 3	State 4	LR_{IND}	p-value
Historical	1047	83	83	1	6.7071	0.0096*
Normal	1051	81	81	1	6.2019	0.0128*
EWMA	1047	83	83	1	6.7071	0.0096*
GPD-GARCH	1081	66	66	1	3.0134	0.0826

Bivariate						
	Portfolio					
Method	State 1	State 2	State 3	State 4	LR_{IND}	p-value
Historical	1071	71	71	1	3.9578	0.0467*
Normal	1071	71	71	1	3.9578	0.0467*
EWMA	1057	78	78	1	5.4793	0.0192*
GPD-GARCH	1085	64	64	1	2.6693	0.1023
GDP-GARCH-Copula	1091	61	61	1	2.1895	0.1390

*Significant at 5% level.

CC test results are given in Table 9. All methods, accept the GPD-GARCH and GPD-GARCH-Copula methods, fail the test with $LR_{CC} > 5.99$ at significance level 5%.

Table 9. Composite test results.

	Univariate				Bivariate	
	Loss		ALAE		Portfolio	
Method	LR_{CC}	p-value	LR_{CC}	p-value	LR_{CC}	p-value
Historical	11.6994	0.0029*	15.1189	0.0005*	6.0333	0.0490*
Normal	9.3673	0.0092*	13.2877	0.0013*	6.0333	0.0490*
EWMA	11.6994	0.0029*	15.1189	0.0005*	10.7730	0.0046*
GPD-GARCH	4.1622	0.1248	3.6693	0.1597	2.9756	0.2259
GPD-GARCH-Copula	-	-	-	-	2.2164	0.3302

*Significant at 5% level.

6.2. Asymptotic ruin probabilities

In computing $\psi(u_0)$ in all models, the initial surplus u_0 is designated as the estimated VaR and ES risk measures to satisfy the asymptotic convergence and compare the estimation methods. The loading factor, ρ, is chosen as 0.01, which affects the ruin probability linearly as in Equation 5.14. Therefore, choosing a different ρ will not affect the ranking of the models as they are only affected by the distributional assumptions of each model.

Estimated ruin probabilities for the univariate static case are given in Table 10. For Loss data, the lowest ruin probability is obtained in the GPD-GARCH method followed by the HS. For the ALAE data, the lowest ruin probability with VaR initial surplus is obtained in the GPD-GARCH method, whereas with ES, it is obtained in the EWMA method. Overall, average ruin probabilities from lower to higher belong to; GPD-GARCH, EWMA, HS, and NA.

Table 10. Ruin probability estimates for the univariate static case.

	Univariate			
	Loss		ALAE	
Method	VaR	ES	VaR	ES
Historical	0.0656	0.0255	0.0037	0.0014
Normal	0.0914	0.0386	0.0034	0.0011
EWMA	0.0772	0.0310	0.0030	0.0008
GPD-GARCH	0.0234	0.0141	0.0014	0.0010

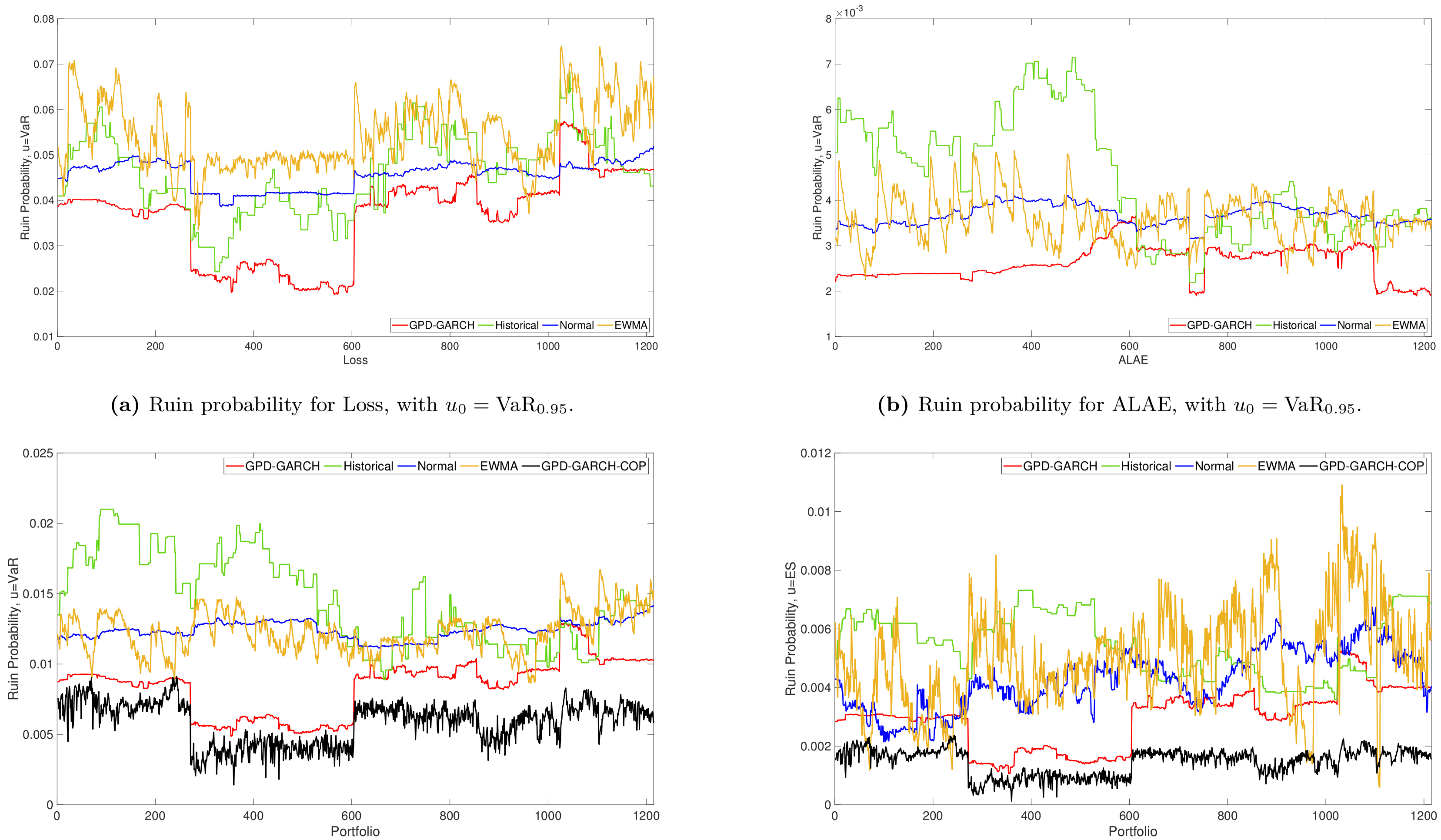
In the univariate dynamic case, 250 data are used in a sliding window framework. By doing that, time variation in data is incorporated into the model and ruin probability as well. Figure 8 presents the ruin probabilities for the Loss and ALAE data sets based on four estimation methods with initial surplus $u_0 = \mathrm{VaR}_{0.95}$. The overall lowest ruin probability is obtained with the GPD-GARCH model for both Loss and ALAE. The highest ruin probability presented is the EWMA method for Loss and the HS for ALAE. As the ruin probability depends on the underlying distribution, time-varying model parameters play an important role along with estimated risk measures used as initial surplus.

Similar results hold for the ruin probabilities with $u_0 = \mathrm{ES}_{0.95}$. The difference with $\mathrm{ES}_{0.95}$ compared to $\mathrm{VaR}_{0.95}$ is the lowered ruin probabilities, as the initial surplus is higher with the ES risk measures. For the bivariate dynamic case, the overall lowest ruin probability is obtained with the GPD-GARCH-Copula model for the portfolio. The highest ruin probability presented is HS for VaR and EWMA for ES. Again, ruin probabilities with initial wealth $\mathrm{ES}_{0.95}$ are lower than $\mathrm{VaR}_{0.95}$ as expected.

To compare the dynamic ruin probabilities over the period $t \in [251, 1465]$, average values of the estimated univariate dynamic and bivariate dynamic ruin probabilities are computed and presented in Table 11. Even though the GPD-GARCH-Copula model estimates higher risk measures, it comes with a reward of lowered ruin probabilities which cannot be said for other models tested. Dynamically analyzing the bivariate actuarial data, instead of univariate case, significantly reduces the risk and ruin probability by considering the time-varying dependence between risks.

Table 11. Average ruin probability estimates for the univariate and bivariate dynamic cases.

	Univariate				Bivariate	
	Loss		ALAE		Portfolio	
Method	VaR	ES	VaR	ES	VaR	ES
Historical	0.0461	0.0159	0.0044	0.0027	0.0146	0.0055
Normal	0.0456	0.0149	0.0037	0.0016	0.0124	0.0043
EWMA	0.0543	0.0174	0.0036	0.0023	0.0123	0.0050
GPD-GARCH	0.0367	0.0101	0.0026	0.0011	0.0085	0.0030
GPD-GARCH-Copula	-	-	-	-	0.0059	0.0015



(a) Ruin probability for Loss, with $u_0 = \mathrm{VaR}_{0.95}$.

(b) Ruin probability for ALAE, with $u_0 = \mathrm{VaR}_{0.95}$.

(c) Ruin probability for Portfolio, with $u_0 = \mathrm{VaR}_{0.95}$.

(d) Ruin probability for Portfolio, with $u_0 = \mathrm{ES}_{0.95}$.

Figure 8. Univariate and bivariate dynamic estimate of ruin probabilities.

7. Concluding comments

This paper investigates the effects of time-varying extreme dependence in the right tail by using the EVT-GARCH-Copula model. The primary motivation is to explicitly determine risk measures under different modeling assumptions for the risk management of actuarial data in case of extremes. VaR and ES risk measures are estimated by the dynamic GPD-GARCH-Copula model and compared with three other commonly used methods; historical simulation, normal approximation, and exponentially weighted moving average. Results are validated by backtesting the violation rates. This study contributes to the literature by expanding the combination of three models to show the effects of EVT on the heavy-tailed asymptotic ruin probabilities by using the estimated risk measures as initial surplus.

Based on the findings, we conclude that the proposed GPD-GARCH-Copula outperforms other models in providing more reliable risk measures. Additionally, we depict that the valuation of multivariate risk sources in a dependent framework provides more robust risk measures and lowered ruin probabilities.

It should be noted that the constraints such as deductibles, limits, and stop-loss contracts which can originate from the reinsurance contracts are taken out of the scope of this paper. These can be added to the risk measure estimations and ruin probabilities by a few adjustments in future studies.

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