Fundamental Journal of Mathematics and Applications, 4 (4) (2021) 258-263
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.992917

On the Framed Normal Curves in Euclidean 4-space

Mahmut Akyiğit[1] and Önder Gökmen Yıldız[2*]

[1] *Department of Mathematics, Faculty of Science and Arts, Sakarya University, Sakarya, Turkey*
[2] *Department of Mathematics, Faculty of Science and Arts, Bilecik Şeyh Edebali University, Bilecik, Turkey*
[*] *Corresponding author*

Article Info

Keywords: *Framed curves, Normal curves, Singular points*
2010 AMS: *53A04, 57R45, 58K05*
Received: *8 September 2021*
Accepted: *8 November 2021*
Available online: *1 December 2021*

Abstract

In this paper, we introduce the adapted frame of framed curves and we give the relations between the adapted frame and Frenet type frame of the framed curve in four-dimensional Euclidean space. Moreover, we define the framed normal curves in four-dimensional Euclidean space. We obtain some characterizations of framed normal curves in terms of their framed curvature functions. Furthermore, we give the necessary and sufficient condition for a framed curve to be a framed normal curve.

1. Introduction

The most basic building blocks of classical differential geometry are curves. There are many studies on Frenet curves and they are useful for investigating the geometric properties of regular curves. Especially, the subject of curves with singular points was discussed in the 20th century. However, recently, the subject of curves with singular points, for which a Frenet frame cannot be formed at a particular point, has been discussed from a different perspective. Honda and Takahashi introduced the concept of framed curves to examine curves with singular points in terms of differential geometry [1]. These curves, called Framed curves expressed by Honda and Takahashi, are a natural generalization of Frenet curves. Moreover, Wang et. al obtained a moving adapted frame to investigate the properties of rectifying curve with singular points in $\mathbb{R}^3$, and this frame was used to analyze some special curves with singular points [2]. For more details on the notion of framed curves, see [3]-[9].

In Euclidean space $\mathbb{R}^3$, curves whose position vector is always in the normal plane are normal curves, and also these curves are spherical curves [10]. Analogously, timelike normal curves in three-dimensional Minkowski space is defined as the curves whose normal planes always contain a fixed point. Therefore, the position vector of such curves with respect to some chosen origin always lies in its normal plane [11]. In particular, timelike normal curves lie in the pseudosphere in $\mathbb{R}^3_1$. In addition, the characterizations of timelike and spacelike normal curves in Minkowski space $\mathbb{R}^3_1$ have been examined recently, [12], [13]. Bahar et. al has been studied framed normal curve in $\mathbb{R}^3$ [4].

In this paper, inspired by [2] and using Euler angles [14], we obtained moving adapted frame for framed curves in $\mathbb{R}^4$. We define generalized Frenet vectors and framed curvatures to investigate the geometric properties of framed curves in $\mathbb{R}^4$. After that, we introduce framed normal curves in $\mathbb{R}^4$. We give some characterizations for framed normal curves. we obtained the necessary and sufficient conditions for such framed curves to be framed normal curves.

Email addresses and ORCID numbers: **makyigit@sakarya.edu.tr, 0000-0002-8398-365X (M. Akyiğit), ogokmen.yildiz@bilecik.edu.tr, 0000-0002-2760-1223 (Ö. G. Yıldız)**

2. Preliminaries

Let $\mathbb{R}^4$ be the 4−dimensional Euclidean space equipped with the inner product

$$< x,y >= \sum_{i=1}^{4} x_i y_i,$$

where $x = (x_1,x_2,x_3,x_4)\,, y = (y_1,y_2,y_3,y_4) \in \mathbb{R}^4$ and norm of $x \in \mathbb{R}^4$ is given by $\|x\| = \sqrt{< x,x >}$. Vector product in $\mathbb{R}^4$ is given by

$$x \times y \times z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

where $x = (x_1,x_2,x_3,x_4)\,, y = (y_1,y_2,y_3,y_4)\,, z = (z_1,z_2,z_3,z_4) \in \mathbb{R}^4$ and e_1,e_2,e_3,e_4 are the canonical basis vectors of $\mathbb{R}^4$. Let Δ_3 be a 6−dimensional smooth manifold as follows

$$\Delta_3 = \left\{ \mu = (\mu_1,\mu_2,\mu_3) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \middle| < \mu_i,\mu_j >= \delta_{ij}, i,j = 1,2,3 \right\}.$$

We can define a unit vector $\nu = \mu_1 \times \mu_2 \times \mu_3$ such that $\det(\nu,\mu_1,\mu_2,\mu_3) = 1$.

Definition 2.1. *$(\gamma,\mu) : I \to \mathbb{R}^4 \times \Delta_3$ is said to be framed curve if $< \gamma'(s)\,, \mu_i(s) >= 0$ for all $s \in I$ and $i = 1,2,3$. $\gamma : I \to \mathbb{R}^4$ is called as a framed base curve if there exists $\mu : I \to \Delta_3$ such that (γ,μ) is a framed curve, [1].*

By following similar way as the curvatures of regular curve, we can define smooth function for a framed curve. Let $\{\nu(s)\,,\mu(s)\}$ be a moving frame along the framed base curve $\gamma(s)$. Then, we have the Frenet-Serret type formula, which is given by

$$\begin{bmatrix} \mu_1'(s) \\ \mu_2'(s) \\ \mu_3'(s) \\ \nu'(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & g(s) & h(s) \\ -f(s) & 0 & j(s) & k(s) \\ -g(s) & -j(s) & 0 & l(s) \\ -h(s) & -k(s) & -l(s) & 0 \end{bmatrix} \begin{bmatrix} \mu_1(s) \\ \mu_2(s) \\ \mu_3(s) \\ \nu(s) \end{bmatrix}$$

where $f(s)\,,g(s)\,,h(s)\,,j(s)\,,k(s)$ and $l(s)$ are smooth curvature functions. Moreover, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that $\gamma'(s) = \alpha(s)\,\nu(s)$. $(f(s)\,,g(s)\,,h(s)\,,j(s)\,,k(s)\,,l(s)\,,\alpha(s))$ are called curvatures of γ at $\gamma(s)$. Clearly, s_0 is singular points of γ iff $\alpha(s_0) = 0$. $(f(s)\,,g(s)\,,h(s)\,,j(s)\,,k(s)\,,l(s)\,,\alpha(s))$ are useful to investigate the framed curve and its singularities.

Theorem 2.2. *Let $(f,g,h,j,k,l,\alpha) : I \to \mathbb{R}^4$ be a smooth mapping. There exists a framed curve $(\gamma,\mu) : I \to \mathbb{R}^4 \times \Delta_3$ whose associated curvature of the framed curve is (f,g,h,j,k,l,α) [1].*

Theorem 2.3. *Let (γ,μ) and $(\widetilde{\gamma},\eta) : I \to \mathbb{R}^4 \times \Delta_3$ be framed curves whose curvatures of the framed curves (f,g,h,j,k,l,α) and $\left(\widetilde{f},\widetilde{g},\widetilde{h},\widetilde{j},\widetilde{k},\widetilde{l},\widetilde{\alpha}\right)$ coincide. Then, (γ,μ) and $(\widetilde{\gamma},\eta)$ are congruent as framed curves [1].*

Let $(\gamma,\mu) : I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with (f,g,h,j,k,l,α). By using Euler angels $\eta = (\eta_1,\eta_2,\eta_3) \in \Delta_3$ is defined by

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & -\cos\varphi\sin\psi + \sin\varphi\cos\psi\sin\theta & \sin\varphi\sin\psi + \cos\varphi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\varphi\cos\psi + \sin\varphi\sin\psi\sin\theta & -\sin\varphi\cos\psi + \cos\varphi\sin\psi\sin\theta \\ -\sin\theta & \sin\varphi\cos\theta & \cos\varphi\cos\theta \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

where θ,φ and ψ are smooth functions. By straightforward calculations,

$$\widetilde{\nu} = \eta_1 \times \eta_2 \times \eta_3 = \mu_1 \times \mu_2 \times \mu_3 = \nu.$$

So, $(\gamma,\eta) : I \to \mathbb{R}^4 \times \Delta_3$ is also a framed curve. Assume that

$$\frac{\tan\psi}{\cos\theta} = l\sin\varphi - k\cos\varphi$$

and

$$h = \cot\theta\,(l\cos\varphi + k\sin\varphi)$$

are satisfied for given smooth function θ, φ and ψ (Euler angle), the adapted frame along $\gamma(s)$ is given by

$$\begin{bmatrix} \nu'(s) \\ \eta_1'(s) \\ \eta_2'(s) \\ \eta_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & p(s) & 0 & 0 \\ -p(s) & 0 & q(s) & 0 \\ 0 & -q(s) & 0 & r(s) \\ 0 & 0 & -r(s) & 0 \end{bmatrix} \begin{bmatrix} \nu(s) \\ \eta_1(s) \\ \eta_2(s) \\ \eta_3(s) \end{bmatrix} \tag{2.1}$$

where $(p(s), q(s), r(s), \alpha(s))$ are framed curvature of $\gamma(s)$ and their expression are

$$\begin{array}{l} p = -h \sec\theta \sec\psi, \\ q = -(j-\varphi')\sin\theta - \psi', \\ r = \frac{\cos\theta}{\cos\psi}(j-\varphi') \end{array}$$

and the following equalities

$$\begin{array}{l} f = -\sin\varphi(\theta' - r\sin\psi), \\ g = -\cos\varphi(\theta' - r\sin\psi), \\ j = r\frac{\cos\psi}{\cos\theta} + \theta' \end{array}$$

hold. We can call the vectors $\nu, \eta_1, \eta_2, \eta_3$ the generalized tangent, the generalized principal normal, the generalized first binormal, and the generalized second binormal vectors of the framed curve, respectively.
In order to give a definition of the framed spherical curve, let us recall that a $4-$dimensional hypersphere S^3 is

$$S^3 = \left\{ x \in \mathbb{R}^4 \middle| < x-m, x-m >= c^2 \right\}$$

where $c \in \mathbb{R}^+$ is the radius and m is the center of hypersphere.

Definition 2.4. *Let $(\gamma, \eta): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve. We call γ a framed spherical curve if the framed base curve γ is a curve on S^3 [2].*

3. Framed normal curves in Euclidean 4-space

In this section, we characterize the framed normal curve with non-zero framed curvatures in $\mathbb{R}^4$.
Let $(\gamma, \eta): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with non-zero framed curvatures. Then its position vector satisfies

$$\gamma(s) = \lambda(s)\eta_1(s) + \mu(s)\eta_2(s) + \rho(s)\eta_3(s). \tag{3.1}$$

By differentiating of (3.1), we obtain

$$\alpha\nu = (-p\lambda)\nu + (\lambda' - q\mu)\eta_1 + (q\lambda + \mu' - r\rho)\eta_2 + (r\mu + \rho')\eta_3$$

and therefore

$$-p\lambda = \alpha, \quad \lambda' - q\mu = 0, \quad q\lambda + \mu' - r\rho = 0, \quad r\mu + \rho' = 0. \tag{3.2}$$

From the first three equations, we obtain

$$\begin{array}{l} \lambda(s) = -\frac{\alpha(s)}{p(s)}, \quad \mu(s) = -\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)', \\ \rho(s) = -\frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)} + \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right). \end{array}$$

By using the above relations, we obtain that

$$\gamma(s) = -\frac{\alpha(s)}{p(s)}\eta_1(s) - \frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\eta_2(s) - \frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)} + \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right)\eta_3(s). \tag{3.3}$$

Then we can give the following theorem.

Theorem 3.1. *Let $(\gamma, \eta): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with non-zero framed curvatures. Then, (γ, η) is congruent to a framed normal curve iff*

$$-\frac{r(s)}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)' = \left(\frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)} + \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right)\right)'. \tag{3.4}$$

Proof. First assume that (γ,η) is congruent to a framed normal curve. Then substituting (3.3) into the fourth equation of (3.2), we obtain imply that (3.4) holds.
Conversely, assume that relation (3.4) holds. Let the vector m be given by

$$m(s)=\gamma(s)+\frac{\alpha(s)}{p(s)}\eta_1(s)+\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\eta_2(s)+\frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)}+\left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right)\eta_3(s). \tag{3.5}$$

By differentiating (3.5) with respect to s and by applying (2.1), we obtain

$$m'(s)=\frac{r(s)}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\eta_3(s)+\left(\frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)}+\left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right)\right)'\eta_3(s).$$

From relation (3.4) it is easily seen that m is a constant vector. So, (γ,η) is congruent to a framed normal curve. □

Theorem 3.2. *Let $(\gamma,\eta): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. If (γ,η) is a framed normal curve, then the following statements hold:*
(i) the generalized principal normal and the generalized first binormal component of the position vector γ are respectively given by

$$\begin{aligned}&<\gamma,\eta_1>=-\tfrac{\alpha(s)}{p(s)},\\ &<\gamma,\eta_2>=-\tfrac{1}{q(s)}\left(\tfrac{\alpha}{p(s)}\right)'\end{aligned} \tag{3.6}$$

(ii) the generalized first binormal and the generalized second binormal component of the position vector γ are respectively given by

$$\begin{aligned}&<\gamma,\eta_2>=-\tfrac{1}{q(s)}\left(\tfrac{\alpha}{p(s)}\right)',\\ &<\gamma,\eta_3>=-\tfrac{1}{r(s)}\left(\tfrac{\alpha(s)q(s)}{p(s)}+\left(\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'\right)'\right).\end{aligned}$$

Conversely, if one of the statements (i) or (ii) holds, then (γ,η) is a framed normal curve.

Proof. Let (γ,η) be a framed normal curve with non-zero framed curvatures. Statements (i) and (ii) are easily obtained from (3.3).
Conversely, if statement (i) holds, differentiating the first equation of (3.6) and by using (2.1), we obtain $<\gamma,\upsilon>=0$ which means that (γ,η) is a framed normal curve. If statement (ii) holds, in a similar way it is seen that (γ,η) is a framed normal curve. □

Theorem 3.3. *Let $(\gamma,\eta): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures and γ has at least one non-singular point. Then (γ,η) is a framed normal curve if and only if γ lies on S^3.*

Proof. Let $(\gamma,\eta): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. By using (3.4), we obtain

$$2\frac{\alpha}{p}\left(\frac{\alpha}{p}\right)'+2\frac{1}{q}\left(\frac{\alpha}{p}\right)'\left(\frac{1}{q}\left(\frac{\alpha}{p}\right)'\right)'+2\frac{1}{\mathrm{r}}\left(\frac{\alpha\mathrm{q}}{\mathrm{p}}+\left(\frac{1}{\mathrm{q}}\left(\frac{\alpha}{\mathrm{p}}\right)'\right)'\right)\left(\frac{1}{\mathrm{r}}\left(\frac{\alpha\mathrm{q}}{\mathrm{p}}+\left(\frac{1}{\mathrm{q}}\left(\frac{\alpha}{\mathrm{p}}\right)'\right)'\right)\right)'=0.$$

The above equation is differential of the equation

$$\left(\frac{\alpha}{p}\right)^2+\left(\frac{1}{q}\left(\frac{\alpha}{p}\right)'\right)^2+\left(\frac{1}{r}\left(\frac{\alpha q}{p}+\left(\frac{1}{q}\left(\frac{\alpha}{p}\right)'\right)'\right)\right)^2=c^2 \quad, c\in\mathbb{R}^+.$$

By using (3.5), it is easily seen that $<\gamma-m,\gamma-m>=c^2$. So, this implies that γ lies on S^3 in $\mathbb{R}^4$.
Conversely, let γ lies on S^3 in $\mathbb{R}^4$, then

$$<\gamma-m,\gamma-m>=c^2 \quad, c\in\mathbb{R}^+, \tag{3.7}$$

where $m\in\mathbb{R}^4$ is the constant vector. By taking the derivative of (3.7) and γ has at least one non-singular point, we get $<\gamma-m,\nu>=0$, which implies that (γ,η) is a framed normal curve. □

Lemma 3.4. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with non-zero framed curvatures. Then, (γ,η) is congruent to a framed normal curve iff there exists a differentiable function $\xi(s)$ such that*

$$\begin{aligned} \xi(s)r(s) &= \tfrac{\alpha(s)q(s)}{p(s)} + \left(\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'\right)', \\ \xi'(s) &= -\tfrac{r(s)}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'. \end{aligned} \tag{3.8}$$

In the following theorem, we obtain the necessary and sufficient conditions for such framed curves to be framed normal curves.

Theorem 3.5. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be framed curve with non-zero framed curvatures. (γ,η) is congruent to framed normal curve iff there exist constants a_0, $b_0 \in \mathbb{R}$ such that*

$$-\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)' = \left(a_0 + \int \frac{\alpha(s)q(s)}{p(s)}\cos\omega(s)ds\right)\cos\omega(s) + \left(b_0 + \int \frac{\alpha(s)q(s)}{p(s)}\sin\omega(s)ds\right)\sin\omega(s), \tag{3.9}$$

where $\omega(s) = \int\limits_0^s r(s)ds$.

Proof. If (γ,η) is congruent to a framed normal curve, according to Lemma 3.4 there exists a differentiable function $\xi(s)$ such that relation (3.8) holds. Let us define differentiable functions $\omega(s)$, $a(s)$ and $b(s)$ by

$$\begin{aligned} &\omega(s) = \int\limits_0^s r(s)ds, \\ &a(s) = -\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'\cos\omega(s) + \xi(s)\sin\omega(s) - \int \tfrac{\alpha(s)q(s)}{p(s)}\cos\omega(s)ds, \\ &b(s) = -\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'\sin\omega(s) - \xi(s)\cos\omega(s) - \int \tfrac{\alpha(s)q(s)}{p(s)}\sin\omega(s)ds. \end{aligned} \tag{3.10}$$

By using (3.8), we easily find $\omega'(s) = r(s)$, $a'(s) = 0$, $b'(s) = 0$ and thus

$$a(s) = a_0\,,\ \ b(s) = b_0\,,\ \ a_0, b_0 \in \mathbb{R}. \tag{3.11}$$

By multiplying the last two equations in (3.10), respectively with $\cos\omega(s)$ and $\sin\omega(s)$, adding the obtained equations and using (3.11), we conclude that relation (3.9) holds.
Conversely, assume that there exist constants a_0, $b_0 \in \mathbb{R}$ such that relation (3.9) holds. By taking the derivative of (3.9), we find

$$-\frac{\alpha(s)q(s)}{p(s)} - \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)' = r(s)\left[\begin{array}{c} -\left(a_0 + \int \frac{\alpha(s)q(s)}{p(s)}\cos\omega(s)ds\right)\sin\omega(s) \\ +\left(b_0 + \int \frac{\alpha(s)q(s)}{p(s)}\sin\omega(s)ds\right)\cos\omega(s) \end{array}\right]. \tag{3.12}$$

Let us define the differentiable function $\xi(s)$ by

$$\xi(s) = \frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)} + \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right). \tag{3.13}$$

Next, relations (3.12) and (3.13) imply

$$\xi(s) = \left(a_0 + \int \frac{\alpha(s)q(s)}{p(s)}\cos\omega(s)ds\right)\sin\omega(s) - \left(b_0 + \int \frac{\alpha(s)q(s)}{p(s)}\sin\omega(s)ds\right)\cos\omega(s).$$

By using the above equation and (3.9), we obtain $\xi'(s) = -\frac{r(s)}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'$. Finally, Lemma 3.4 implies that (γ,η) is congruent to a framed normal curve. □

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] S. Honda, M. Takahashi, *Framed curves in the Euclidean space*, Adv. Geom., **16** (2016), 265-276.

[2] Y. Wang, D. Pei, R. Gao, *Generic properties of framed rectifying curves*, Mathematics, **7**(1) (2019), 37.

[3] A. Kızılay, Ö. G. Yıldız, O. Z. Okuyucu, *Evolution of quaternionic curve in the semi-Euclidean space* E_2^4, Math. Meth. Appl. Sci., **44** (2021), 7577–7587, https://doi.org/10.1002/mma.6374.

[4] B. Doğan Yazıcı, S. Özkaldı Karakuş, M. Tosun, *Framed normal curves in Euclidean space*, Tbilisi Math. J., (2020), 27-37.

[5] B. Doğan Yazıcı, S. Özkaldı Karakuş, M. Tosun, *On the classification of framed rectifying curves in Euclidean space*, Math. Meth. Appl. Sci., (2021), 1-10, https://doi.org/10.1002/mma.7561.

[6] C. Özyurt, *Singular curves and their properties*, Master Thesis, Ankara University, 2021.

[7] S. Honda, T. Masatomo, *Evolutes of framed immersions in the Euclidean space*, Hokkaido Uni. Preprint Ser. Math., **1095** (2016), 1-24.

[8] T. Fukunaga, M. Takahashi, *Existence and uniqueness for Legendre curves*, J. Geom. **104** (2013), 297-307.

[9] K. Eren, Ö. G. Yıldız, M. Akyiğit, *Tubular surfaces associated with framed base curves in Euclidean 3-space*, Math. Meth. Appl. Sci., (2021), 1- 9, https://doi.org/10.1002/mma.7590.

[10] B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?*, Amer. Math. Monthly, **110** (2003), 147-152.

[11] K. Ilarslan, E. Nesovic, *Timelike and null normal curves in Minkowski space* E_1^3, Indian J. Pure Appl. Math., **35**(7) (2004), 881-888.

[12] K. Ilarslan, *Spacelike Normal Curves in Minkowski Space* E_1^3, Turk. J. Math., **29** (2005), 53-63.

[13] K. Ilarslan, E. Nesovic, *Spacelike and timelike normal curves in Minkowski space-time*, Publ. Inst. Math. Belgrade, **85**(99) (2009), 111-118.

[14] F. Gökçelik, Z. Bozkurt, İ. Gök, F. N. Ekmekçi, Y. Yaylı *Parallel transport frame in 4-dimensional Euclidean space*, Caspian J. Math. Sci., **3** (2014), 91- 103.