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The Evolution of Fractional Calculus

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ABSTRACT Fractional Calculus started in 1695 with Leibniz discussing the meaning of $D^n y$ for $n = 1/2$. Many mathematicians developed the theoretical concepts, but the area remained somewhat unknown from applied sciences. During the eighties FC emerged associated with phenomena such as fractal and chaos and, consequently, in nonlinear dynamical. In the last years, Fractional Calculus became a popular tool for the modeling of complex dynamical systems with nonlocality and long memory effects.

KEYWORDS
Fractional calculus
Non-locality
Long range memory

INTRODUCTION

The generalization of the concept of derivative $D^{\alpha} f(x)$ to non-integer values of α goes back to the beginning of the theory of differential calculus in the follow-up of the brilliant ideas of Gottfried Leibniz (Machado and Kiryakova 2019). The development of this area of knowledge is due to the contributions of important scientists such as Euler, Liouville and Riemann (Machado *et al.* 2010; Valério *et al.* 2014) as represented in Fig. 1. In the fields of physics and engineering, Fractional Calculus (FC) is presently associated with the modelling of complex phenomena with nonlocality and long memory effects (Tarasov 2019a,b; Băleanu and Lopes 2019a,b). This paper introduces the fundamentals of this tool, its application in the control of dynamical systems, and present day state of development.

MATHEMATICAL FUNDAMENTALS OF FRACTIONAL CALCULUS

The most used definitions of a fractional derivative of order α are the Riemann-Liouville (RL, $t > a$, $Re(\alpha) \in]n-1, n[$), Grünwald-Letnikov (GL, $t > a$, $\alpha > 0$) and Caputo (C, $t > a$, $n-1 < \alpha < n$) formulations (Kochubei and Luchko 2019a,b; Karniadakis 2019):

$$ {}_{a}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1a} $$

$$ {}_{a}^{GL}D_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(t-kh), \tag{1b} $$

$$ {}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1c} $$

where $\Gamma(\cdot)$ is Euler's gamma function, $[x]$ means the integer part of x, and h is the step time increment.

These operators capture the history of all past events, in opposition to integer derivatives that are 'local' operators. This means that fractional order systems have a memory of the dynamical evolution. This behaviour has been recognized in several natural and man made phenomena and their modelling becomes much simpler using the tools of FC, while the counterpart of building integer order models leads often to complicated expressions Machado and Lopes (2020b,a). The geometrical interpretation of fractional derivatives has been the subject of debate and several perspectives have been proposed (Machado 2003, 2021).

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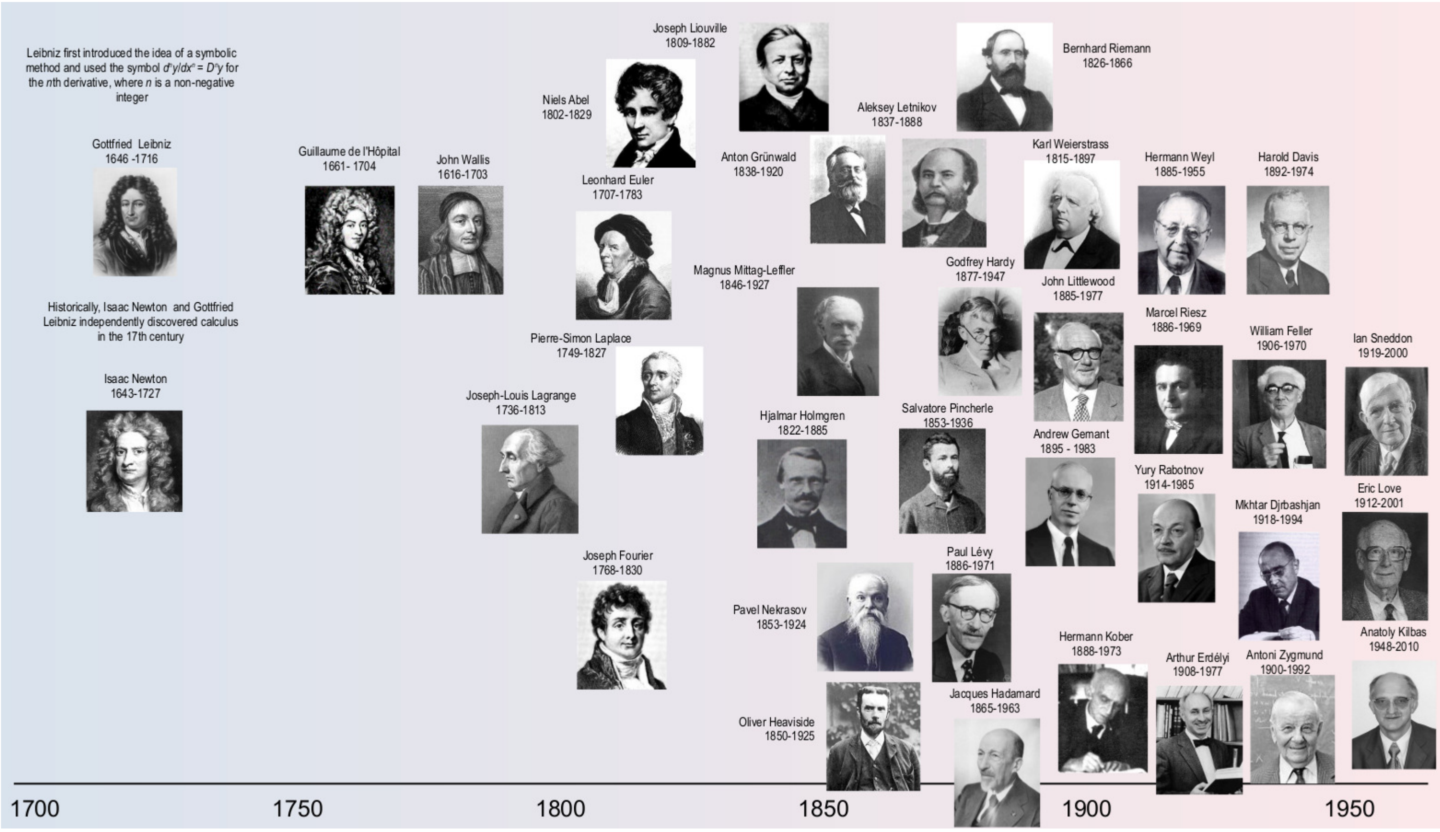

Figure 1 The FC timeline

Using the Laplace transform we have the expressions:

$$\mathcal{L}\left\{{}_0^{RL}D_t^{\alpha} f(t)\right\} = s^{\alpha}\mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^k\, {}_0^{RL}D_t^{\alpha-k-1} f\left(0^+\right), \tag{2a}$$

$$\mathcal{L}\left\{{}_0^{C}D_t^{\alpha} f(t)\right\} = s^{\alpha}\mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, f^{(k)}(0), \tag{2b}$$

where s and $\mathcal{L}$ denote the Laplace variable and operator, respectively.

The Mittag-Leffler function (MLF), $E_{\alpha}(t)$, is defined as:

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)},\ \alpha \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0. \tag{3}$$

The MLF represents a bridge between the exponential and the power law functions. In particular, when $\alpha = 1$ the MLF simplifies and we have $E_1(t) = e^t$, while, for large values of t, the asymptotic behaviour yields $E_{\alpha}(-t) \approx \frac{1}{\Gamma(1-\alpha)}\frac{1}{t}$, $\alpha \neq 1, 0 < \alpha < 2$.

Since the Laplace transform leads to:

$$\mathcal{L}\{E_{\alpha}(\pm a t^{\alpha})\} = \frac{s^{\alpha-1}}{s^{\alpha} \mp a} \tag{4}$$

we observe a generalization of the Laplace transform pairs from the exponential towards the ML, namely from integer up to fractional powers of s. The more general MLF, often called two-parameter MLF, is given by:

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)},\ \alpha, \beta \in \mathbb{C},\ \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0. \tag{5}$$

The function defined by (3) gives a generalization of (5), since $E_{\alpha}(t) = E_{\alpha,1}(t)$.

FRACTIONAL CONTROL

Let us consider an elemental feedback control system of fractional order α, with unit feedback and transfer function $G(s) = \frac{K}{s^{\alpha}}$, $1 < \alpha < 2$, in the direct loop (Machado 1997, 2001). The open-loop Bode diagrams of amplitude and phase have a slope of -20 dB/dec and a constant phase of $-\alpha\frac{\pi}{2}$ rad, respectively. Therefore, the closed-loop system has a constant phase margin of $\pi\left(1 - \frac{\alpha}{2}\right)$ rad, that is independent of the system gain K.

Assume that $K = 1$, so that $G(s) = \frac{1}{s^{\alpha}}$, and that the closed-loop system is excited by an unit step input $R(s) = \frac{1}{s}$. The output response will be $C(s) = \frac{1}{s(s^{\alpha}+1)}$, or, in the time domain, $c(t) = 1 - E_{\alpha}(-t^{\alpha})$. Figure 2 depicts the responses for $\alpha = \{0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2\}$. We observe that the fractional values 'interpolate' the cases of integer orders $\alpha = \{0, 1, 2\}$. We note a fast initial transient followed by a slow convergence for the steady-state value, which is typical of many fractional order systems.

A popular application of FC is in the area of control (Petráš 2019) and corresponds to the generalization of the Proportional, Integral and Derivative (PID) algorithm, namely to the fractional PID. The $PI^{\lambda}D^{\mu}$ control algorithm has a transfer function given by:

$$G_c(s) = K_P + K_I s^{-\lambda} + K_D s^{\mu}, \tag{6}$$

where K_P, K_I and K_D are the proportional, integral and differential gains, and λ and μ are the fractional orders of the integral and derivative actions, respectively. The cases $(\lambda, \mu) = \{(0,0), (1,0), (0,1), (1,1)\}$, correspond to the P, PI, PD and PID, respectively.

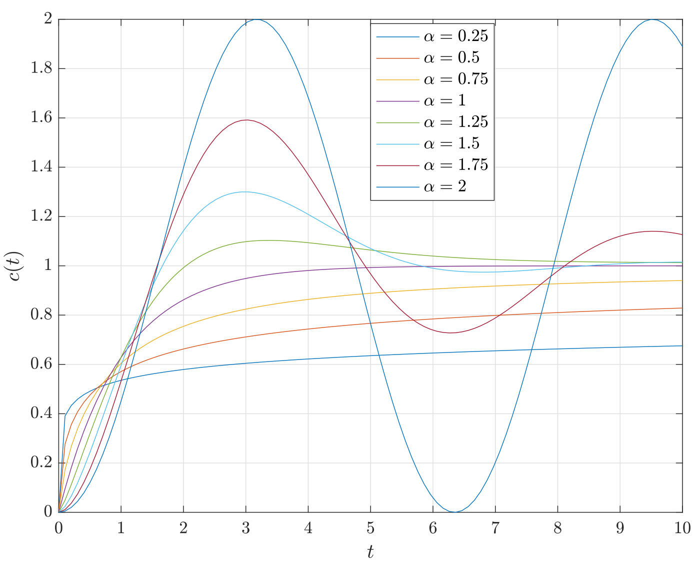

Figure 2 Time response $c(t) = 1 - E_{\alpha}(-t^{\alpha})$ of the fractional closed-loop system for a unit step reference input and $\alpha = \{0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2\}$

PROGRESS OF THE FRACTIONAL CALCULUS

We can estimate of the present day state of FC using publicly available information, just to remind that until 1974 there were only 1 book devoted to FC as a topic, while by 2018 the number of FC books were estimated to be more than 240 Machado and Kiryakova (2017). For that purpose we selected the program VOSviewer van Eck and Waltman (2009, 2017) as the tool for processing bibliographic information.

Let us consider (i) data is available at Scopus database, (ii) papers published during year 2020, and (iii) 8 search keywords, namely {Fractional calculus, Fractional derivative, Fractional integration, Fractional dynamics, Mittag-Leffler, Derivative of non-integer order, Integral of non-integer order, Derivative of complex order, Integral of complex order} that yields 6,589 records. The VOSViewer allows several perspectives of bibliographic analysis, but let us start by considering a network plot for the options 'Co-occurrence', 'All keywords', 'Full counting', 'Minimum number of occurrence of a keyword=4'. This search gives 2,764 keywords, as shown in Fig. 3. On the other hand Fig. 4 depicts the network plot for the options 'Co-authotship', 'Countries', 'Full counting', 'Minimum number of occurrence of a country=4', , 'Minimum number of citations of a country=2' that gives 77 cases. The two network plots show that FC is presently applied in all fields of science, going from the areas of mathematics, physics, engineering and economy, up to medicine, biology and genetics, and the topic is presently very popular in all countries of the globe.

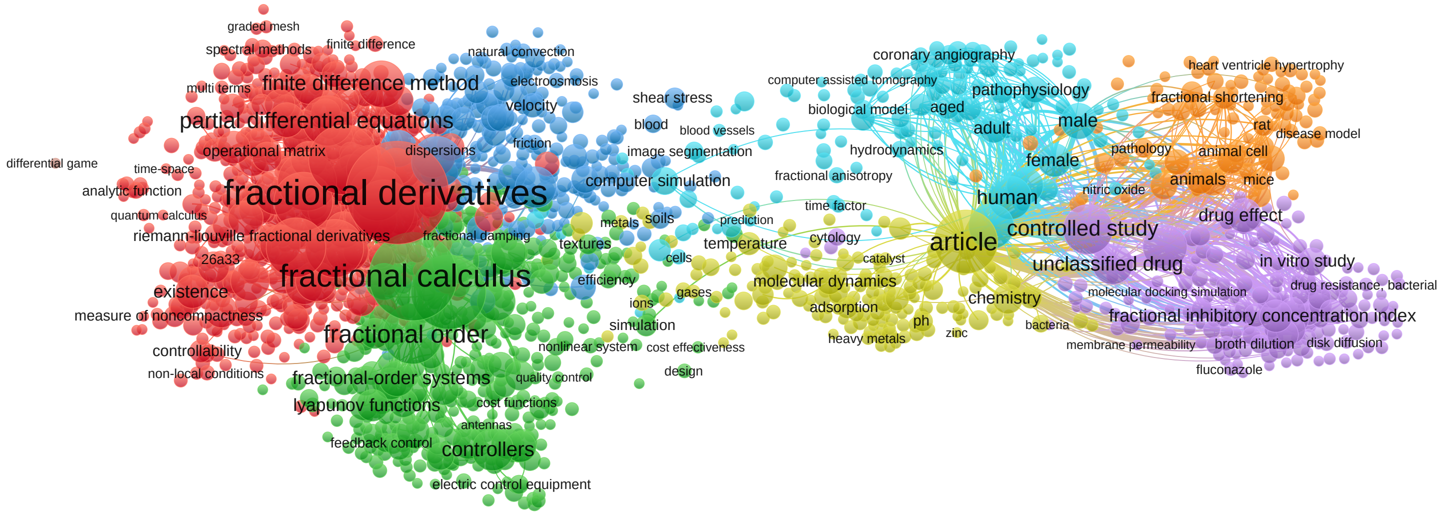

Figure 3 VOSviewer network plot with options ‘Co-occurrence’, ‘All keywords’, ‘Full counting’, ‘Minimum number of occurrence of a keyword=4’

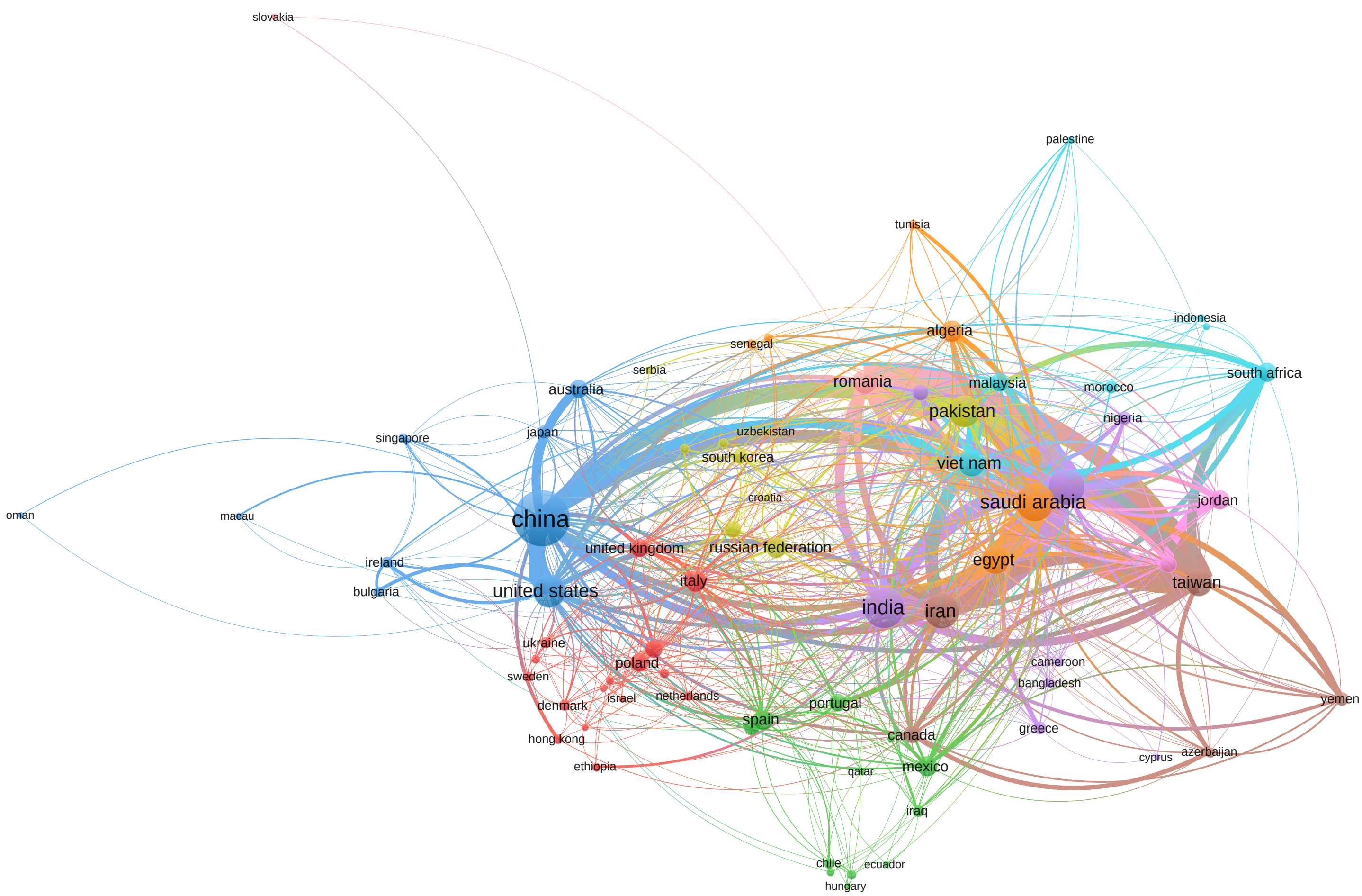

Figure 4 VOSviewer network plot with options ‘Co-occurrence’, ‘All keywords’, ‘Full counting’, ‘Minimum number of occurrence of a country=4’, , ‘Minimum number of citations of a country=2’

CONCLUSIONS

This work introduced and discussed several aspects of the FC. The history, fundamentals and the use of FC in control were described. The present day areas of application of FC and its evolution were also analyzed using a computer package for processing bibliographic information.

Conflicts of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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