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Finslerian Viewpoint to the Rectifying, Normal and Osculating Curves

Zehra Özdemir*, Fatma Ateş and F. Nejat Ekmekci

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ABSTRACT

The theory of Finsler metric was introduced by Paul Finsler, in 1918. The author defines this metric using the Minkowski norm instead of the inner product. Therefore, this geometry is a more general metric and includes the Riemannian metric. In the present work, using the Finsler metric, we investigate the position vector of the rectifying, normal and osculating curves in Finslerian 3-space $\mathbb{F}^3$. We obtain the general characterizations of these curves in $\mathbb{F}^3$. Furthermore, we show that rectifying curves are extremal curves derived from the Finslerian spherical curve. We also plotted various examples by using the Randers metrics.

Keywords: *Special curves, Finsler space, Randers metric, centrodes, extremal curves.*
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1. Introduction

Finsler geometry has various applications in thermodynamics, optics, ecology, evolution, biology, etc. Also, the Finsler distance on this structure corresponds to the phase of an optical wave-guide. Finsler geometry appears in field theory and particle physics. Moreover, it has an extraordinary research field in geometry, biology, physics, engineering, and computer sciences [1, 2, 7, 3, 4, 18]. Remizov [17] investigated the singularities of geodesics flows in two-dimensional Finsler space. In [19, 20], Yıldırım et al. defined the helices in Finsler space and gave some characterizations for these curves. Ergüt et al. studied $AW(k)$-type curves in three dimensional Finsler manifold [10].

On the other hand, a curve in three-dimensional space is characterized through the Frenet frame $\{T, N, B\}$. The planes, spanned by the vectors $\{T, B\}$, $\{N, B\}$, and $\{T, N\}$, are called as the rectifying plane, normal plane, and osculating plane, respectively. A curve is called as a rectifying (resp. normal, osculating) curve if its position vector always lies in its rectifying (resp. normal, osculating) plane. In Euclidean geometry, the position vector of a planar curve always lies in its osculating plane. The position vector of a spherical curve always lies in its normal plane. The notion of the rectifying curve was introduced by Chen in [7]. Since the position vector of the rectifying curve determines the instantaneous rotation axis at each point of the curve, the author also showed that these curves are necessary for mechanics, kinematics and differential geometry. Then the characterizations of the osculating and normal curve in 3-dimensional and 4-dimensional Euclidean space were presented by İlarslan et al. [12, 13, 14]. Furthermore, many authors have studied the rectifying, normal and osculating curve in three and four-dimensional spaces: The normal, osculating and rectifying curve in Lie group was defined in [5]. The normal and rectifying curves in Galilean space was defined by Öztekin et al. [15, 16]. The quaternionic normal curves in the Euclidean and the semi-Euclidean spaces were introduced in [21]. Then, in [11], the authors generate the ruled surfaces whose base curve is rectifying curve. Deshmukh et al. derived the rectifying curves via the dilation of unit speed curves on the unit sphere $\mathbb{S}^2$ in the Euclidean space $\mathbb{E}^3$. Also, the authors gave some new characterizations related to the rectifying curves in Euclidean $3-$space [9].

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**** Corresponding author***

In this study, we examine the characterizations of the position vector of the rectifying, normal and osculating curves in Finsler space F^3. Then, we find various characterizations for these curves. Furthermore, we visualized some examples using the Randers metric which is a special Finslerian metric.

2. Basic Concepts

Let $\mathbb{M}$ be a 3-dimensional C^∞ manifold and $T_x\mathbb{M}$ be the tangent space of $\mathbb{M}$. Then the tangent bundle $T\mathbb{M}$ is denoted by $T\mathbb{M} =: \cup_{x\in\mathbb{M}} T_x\mathbb{M}$. Here, if $(x, y) \in T\mathbb{M}$ then $x = (x^i)$ is a point in $\mathbb{M}$ and $y \in T_x\mathbb{M}$. Let $0(\mathbb{M})$ be the image of the null section of the tangent bundle, then we denote $T_0\mathbb{M} := T\mathbb{M} \setminus 0(\mathbb{M})$.Then, $\mathbb{F}^{m+1} = (\mathbb{M}, \mathbb{M}', F)$ is a Finsler manifold and $\mathbb{F}^1 = (\xi, \xi', F_1)$ be a 1-dimensional Finsler submanifolds. Here, ξ is a smooth curve in $\mathbb{M}$ and ξ' is the tangent bundle of the curve ξ in $\mathbb{M}'$. Moreover, $\{\frac{\partial}{\partial s}, \frac{\partial}{\partial v}\}$ is a natural field of frames on ξ', where $\frac{\partial}{\partial v}$ is a unit Finsler vector field, (see, [3]).

Definition 2.1. A metric on $T_0\mathbb{M}$ is called a Finslerian metric, if there exists a function $F : T\mathbb{M} \to \mathbb{R}_+$, smooth on $T_0\mathbb{M}$ and continuous on $0(\mathbb{M})$. The function F is positively homogeneous of degree 1 according to y, that is,

$$F(x, \lambda y) = \lambda F(x, y),\ \lambda > 0,\ y \neq 0,$$

where the function F is called as fundamental Finsler function and satisfies

$$g_{ij}(x, y) = \frac{\partial^2 F^2}{2\partial y^i \partial y^j}.$$

The three dimensional Finsler space is denoted by $\mathbb{F}^3 = (\mathbb{M}, \mathbb{M}', F)$.

Then we have

$$G^i(x, y) = \frac{1}{4} g^{ih}(x, y)(\frac{\partial^2 F^x}{\partial y^h \partial x^j} y^j - \frac{\partial F^x}{\partial x^h})(x, y),$$

where

$$\frac{|\partial F^*|}{\partial x^i} = 2\frac{\partial F}{\partial x^i} \ and\ \frac{|\partial F^*|}{\partial y^i} = 2\frac{\partial F}{\partial y^i}.$$

Let $\mathbb{F}^3 = (\mathbb{M}, \mathbb{M}', F)$ be three dimensional Finsler space and ξ be a smooth curve in $\mathbb{M}$. Then the locally parametric equations of ξ are given as

$$\xi^i = \xi^i(s);\ (\xi'^1(s), \xi'^2(s), \xi'^3(s)) \neq (0, 0, 0),$$

where s is the arclength parameter on ξ. Then the moving Frenet frame along the curve ξ is denoted by $\{e_1, e_2, e_3\}$. The vector fields e_1, e_2 and e_3 are Finslerian tangent, principal normal and binormal vector fields of ξ, respectively, defined as follows;

$$\begin{aligned} e_1 &:= \frac{\partial}{\partial v} = \frac{d\xi^i}{\partial s}\frac{\partial}{\partial y^i}, \\ e_2 &= (\frac{1}{\kappa(s)}\xi''^i(s) + 2G^i(s))\frac{\partial}{\partial y^i} = n^i \frac{\partial}{\partial y^i}, \\ e_3 &= \frac{1}{\sqrt{c}} c^i \frac{\partial}{\partial y^i} = b^i \frac{\partial}{\partial y^i}, \end{aligned}$$

where

$$\xi'_i = g_{ij}\xi'^j, \quad n_i = g_{ij} n^j, \quad c = g_{ij} c^i c^j, \quad c^1 = \begin{vmatrix} \xi'_2 & \xi'_3 \\ n_2 & n_3 \end{vmatrix}, c^2 = \begin{vmatrix} \xi'_3 & \xi'_1 \\ n_3 & n_1 \end{vmatrix}, c^3 = \begin{vmatrix} \xi'_1 & \xi'_2 \\ n_1 & n_2 \end{vmatrix}.$$

The Cartan connection for studying the geometry of the curves in a Finsler manifold satisfy

$$\nabla^*_{\frac{\partial}{\partial s}} X = (\frac{dX^i}{ds} + X^j(s) S^i_j(s))\frac{\partial}{\partial y^i},$$

Where

$$S^i_j(s) = G^i_j(s) + (x''^k + 2G^k(s))g^i_{jk}(s).$$

The Frenet formulas of the curve ξ are given by

$$\begin{aligned} \nabla^*_{\frac{\partial}{\partial s}} e_1 &= \kappa(s)e_2, \\ \nabla^*_{\frac{\partial}{\partial s}} e_2 &= -\kappa(s)e_1 + \tau(s)e_3, \\ \nabla^*_{\frac{\partial}{\partial s}} e_3 &= -\tau(s)e_2, \end{aligned} \tag{2.1}$$

The Finslerian curvature and torsion of the curve ξ are defined by

$$\kappa(s) = \left\{g_{ij}(s)(\xi''^i(s) + 2G^i(s))(\xi''^j(s) + 2G^j(s))\right\}^{\frac{1}{2}},$$

$$\tau(s) = -g_{ij}(s)b^i(s)\left\{\frac{\partial n^j}{\partial s} + n^k(s)S^j_k(s)\right\},$$

[3].

Definition 2.2. A *rectifying curve* defined as a curve whose position vector always lies in its rectifying plane. Namely, the position vector of a rectifying curve is characterized by

$$c(s) = \nu(s)e_1(s) + \mu(s)e_3(s) \tag{2.2}$$

where $\nu(s)$ and $\mu(s)$ are differentiable functions, [7, 12].

Definition 2.3. A *normal curve* defined as a curve whose position vector always lies in its normal plane. Namely, the position vector of a normal curve is characterized by

$$c(s) = \nu(s)e_2(s) + \mu(s)e_3(s) \tag{2.3}$$

where $\nu(s)$ and $\mu(s)$ are differentiable functions, [14].

Definition 2.4. An *osculating curve* defined as a curve whose position vector always lies in its osculating plane. Namely, the position vector of a osculating curve is characterized by

$$c(s) = \nu(s)e_1(s) + \mu(s)e_2(s) \tag{2.4}$$

where $\nu(s)$ and $\mu(s)$ are differentiable functions, [13].

3. Finslerian characterization of the rectifying, normal and osculating curves

In this section, the characterizations of the rectifying, normal and osculating curves are investigated in $\mathbb{F}^3$. The similar characterizations of those curves were found in Euclidean and Lorentzian spaces. However, since the Finsler metric is a general metric, the results are quite completely different in apply. To demonstrate these differences some examples are given with the help of the Randers metric. Moreover, the rectifying curves with the help of the Finslerian spherical curve are obtained.

3.1. Rectifying curves in Finsler space

Theorem 3.1. *Let ξ be a smooth curve in the $3-$dimensional Finsler space $\mathbb{F}^3$ and s be an arc length parameter of the curve ξ. Then ξ is a rectifying curve if and only if the curvature and torsion of the curve satisfy the equations*

$$(s + k_1)\kappa(s) - k_2\tau(s) = 0 \tag{3.1}$$

where k_1, k_2 are constants and $\kappa(s) = \left\{g_{ij}(s)(\xi''^i(s) + 2G^i(s))(\xi''^j(s) + 2G^j(s))\right\}^{\frac{1}{2}}$ *and* $\tau(s) = -g_{ij}(s)b^i(s)\left\{\dfrac{\partial n^j}{\partial s} + n^k(s)S^j_k(s)\right\}$.

Proof. Let ξ be an arc length parameterized rectifying curve in $3-$dimensional Finsler space $\mathbb{F}^3$ such that locally parameterized as

$$\xi^i = \xi^i(s);\ (\xi'^1(s), \xi'^2(s), \xi'^3(s)) \neq (0,0,0).$$

Then from eq.(2.2), we have

$$\xi^i \frac{\partial}{\partial y^i} = \nu(s)\frac{d\xi^i}{\partial s}\frac{\partial}{\partial y^i} + \mu(s)b^i\frac{\partial}{\partial y^i},\ i = \{1,2,3\}, \tag{3.2}$$

for some functions $\nu(s)$ and $\mu(s)$. Differentiating eq.(3.2) with relevance s and using the eqs.(2.1), we obtain

$$\begin{aligned}\frac{\partial}{\partial v} = \frac{\partial \xi^i}{\partial s}\frac{\partial}{\partial y^i} &= \left(\nabla^*_{\frac{\partial}{\partial s}}\nu(s)\right)e_1(s) + \nu(s)\nabla^*_{\frac{\partial}{\partial s}}e_1 + \left(\nabla^*_{\frac{\partial}{\partial s}}\mu(s)\right)e_3 + \mu(s)\nabla^*_{\frac{\partial}{\partial s}}e_3 \\ &= \left(\nabla^*_{\frac{\partial}{\partial s}}\nu(s)\right)e_1 + \nu(s)\kappa(s)e_2 - \mu(s)\tau(s)e_2 + \left(\nabla^*_{\frac{\partial}{\partial s}}\mu(s)\right)e_3.\end{aligned}$$

From the last equation, we get the following three equations

$$\begin{cases}\nabla^*_{\frac{\partial}{\partial s}}\nu(s) = 1, \\ \nu(s)\kappa(s) - \mu(s)\tau(s) = 0, \\ \nabla^*_{\frac{\partial}{\partial s}}\mu(s) = 0.\end{cases} \tag{3.3}$$

Then, the functions $\nu(s)$ and $\mu(s)$ are calculated as follows

$$\nu(s) = s + k_1 \text{ and } \mu(s) = k_2 \tag{3.4}$$

where k_1 and k_2 are constants. By combining the eq.(3.3) and eq.(3.4), we obtain

$$-\frac{g_{ij}(s)B^i(s)\left\{\dfrac{\partial N^j}{\partial s} + N^k(s)S^j_k(s)\right\}}{\{g_{ij}(s)(\xi''^i(s) + 2G^i(s))(\xi''^j(s) + 2G^j(s))\}^{\frac{1}{2}}} = \frac{s+k_1}{k_2}$$

for some constants k_1 and k_2.

Conversely, suppose that ξ is a smooth curve in $\mathbb{F}^3$ and the equality $(s+k_1)\kappa(s) - k_2\tau(s) = 0$ is satisfied for some constants k_1 and k_2. Let us consider the vector $X \in M' \subset \mathbb{F}^3$ given as

$$X^i(s) = \xi^i\frac{\partial}{\partial y^i} - (s+k_1)\frac{d\xi^i}{\partial s}\frac{\partial}{\partial y^i} - k_2 b^i\frac{\partial}{\partial y^i},\ i = \{1,2,3\}.$$

the covariant derivative the $X^i(s)$ becomes zero, i.e. $\nabla^*_{\frac{\partial}{\partial s}}X^i(s) = 0$. Thus, the curve ξ is congruent to a rectifying curve. □

Corollary 3.1. *Let ξ be a rectifying curve in the Finslerian $3-$space $\mathbb{F}^3$. Then ξ can not be a general helix.*

Proof. It is obvious from the eq.(3.1). □

3.2. *Rectifying curves with the help of the Finslerian spherical curve*

This subsection plays a basic role during this work. Sice Finsler metric is a more general metric and includes the Riemannian metric, we give a general approach for Chen's work (see [6]) by using the Finsler metric.

Theorem 3.2. *Let γ be a curve in Finslerian $3-$space $\mathbb{F}^3$ with positive curvature such that locally parameterized as*

$$\gamma^i = \gamma^i(s);\ (\gamma'^1(s), \gamma'^2(s), \gamma'^3(s)) \neq (0,0,0).$$

Then the curve γ is a rectifying curve if and only if it has the following parametric representation:

$$\gamma^i\frac{\partial}{\partial y^i} = \sigma(t)\xi^i\frac{\partial}{\partial y^i}$$

where $\xi^i = \xi^i(s)$ is locally parametrization of the unit speed curve ξ on the Finslerian $2-$sphere $\mathbb{FS}^2$, and the differentiable function $\sigma(t)$ satisfy the following differential equation

$$\nabla^*_{\frac{\partial}{\partial t}}\left(\frac{\nabla^*_{\frac{\partial}{\partial t}}\sigma}{F(\frac{\partial\gamma^i}{\partial t}\frac{\partial}{\partial y^i})}\right) - \frac{\sigma}{F(\frac{\partial\gamma^i}{\partial t}\frac{\partial}{\partial y^i})} = 0. \tag{3.5}$$

Proof. Let σ be a positive function and ξ be a unit speed curve on the Finslerian $2-$sphere $\mathbb{FS}^2$ such that locally parameterized as

$$\xi^i = \xi^i(s);\ (\xi'^1(s), \xi'^2(s), \xi'^3(s)) \neq (0,0,0).$$

If we take the derivative of the equation $\gamma^i \frac{\partial}{\partial y^i} = \sigma(t)\xi^i \frac{\partial}{\partial y^i}$, i={1,2,3}, then we obtain

$$\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i} = \xi^i \frac{\partial}{\partial y^i}\nabla^*_{\frac{\partial}{\partial t}}\sigma + \sigma \frac{\partial \xi^i}{\partial t}\frac{\partial}{\partial y^i},\ i = \{1,2,3\}.$$

Then the unit tangent vector of the curve γ is written as

$$e_1 = \frac{\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i}}{F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})} = \frac{\nabla^*_{\frac{\partial}{\partial t}}\sigma}{F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})}\xi^i \frac{\partial}{\partial y^i} + \frac{\sigma}{F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})}\frac{\partial \xi^i}{\partial t}\frac{\partial}{\partial y^i}. \tag{3.6}$$

If we differentiate the eq.(3.6), we calculate

$$\nabla^*_{\frac{\partial}{\partial t}} e_1 = F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})\kappa n \tag{3.7}$$

$$= \nabla^*_{\frac{\partial}{\partial t}}(\frac{\nabla^*_{\frac{\partial}{\partial t}}\sigma}{F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})})\xi^i \frac{\partial}{\partial y^i} + (\frac{\nabla^*_{\frac{\partial}{\partial t}}\sigma}{F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})} + \nabla^*_{\frac{\partial}{\partial t}}\frac{\sigma}{F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})})\frac{\partial \xi^i}{\partial t}\frac{\partial}{\partial y^i}$$

$$+\frac{\sigma}{F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})}\nabla^*_{\frac{\partial}{\partial t}}\frac{\partial}{\partial v}. \tag{3.8}$$

Since the curve ξ is the unit speed curve on the Finslerian $2-$sphere $\mathbb{FS}^2$, we have $g_{ij}(\nabla^*_{\frac{\partial}{\partial t}}\frac{\partial}{\partial v}, \xi^i \frac{\partial}{\partial y^i}) = -1$. Taking the scalar product of eq.(3.6) with $\gamma^i \frac{\partial}{\partial y^i} = \sigma(t)\xi^i \frac{\partial}{\partial y^i}$, we obtain

$$\nabla^*_{\frac{\partial}{\partial t}}(\frac{\nabla^*_{\frac{\partial}{\partial t}}\sigma}{F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})}) - \frac{\sigma}{F(\frac{\partial \gamma^i}{\partial t}\frac{\partial}{\partial y^i})} = 0.$$

□

3.3. *Rectifying curves as centrodes in Finslerian 3 space*

The curve defined by the Darboux vector $d = \tau e_1 + \kappa e_3$ of a regular curve ξ in $\mathbb{E}^3$ with $\kappa \neq 0$ is called the centrode of the curve ξ, [6].

Theorem 3.3. *Let ξ be a unit speed curve in $\mathbb{F}^3$ with non-zero constant curvature κ and non-constant torsion τ. Then, the centrode $d = \tau e_1 + \kappa e_3$ of ξ is a rectifying curve. Conversely, each rectifying curve in $\mathbb{F}^3$ is the centrode of the unit speed curves with non-zero constant curvature and non-constant torsion.*

Proof. Assume that $\xi(s)$ be a unit speed curve with non-zero constant curvature and non-constant torsion. Then, if we differentiate the centrode d of the curve $\xi(s)$ and using the Finslerian Frenet frame equations, we obtain

$$\nabla^*_{\frac{\partial}{\partial s}} d = (\nabla^*_{\frac{\partial}{\partial s}}\tau)e_1.$$

Therefore, we obtain that the tangent vector of the centrode d parallels to the tangent vector of the curve ξ at the corresponding points. From the Frenet frame equations in $\mathbb{F}^3$, we can say that the principal normal vector of d and the principal normal vector of the curve ξ are also parallel at the corresponding points. Thus, the binormal vector field of d and the binormal vector field of ξ are parallel as well. These imply that the position vector of the centrode of γ always lies in its rectifying plane, that is, the centrode d is a rectifying curve.
Conversely, let ξ be a unit speed rectifying curve in $\mathbb{F}^3$. From Theorem 3.1, we know that the ratio $\frac{\tau_\xi}{\kappa_\xi} = c_1 s + c_2$, $c_1 \neq 0, c_2$ are constants. Thus, we may write $\frac{\tau_\xi}{\kappa\xi} = \frac{s}{c}$ here $c \in \mathbb{R}^+$. If we define the functions f and g by $f(s) = \frac{1}{c}\int\limits_{s_0}^{s}\kappa_\xi(u)du$ and $g(s) = f^{-1}(s)$. Then there exists a unit speed curve γ with the curvature $\kappa_\gamma = c$ and $\tau_\gamma = g(s)$. Therefore, the centrode of γ is obtained as

$$d = g\,(s)\,t_\gamma + cb_\gamma.$$

It is easily shown that the curves ξ and γ are congruent. □

In Finslerian $3-$space $\mathbb{F}^3$, we have the following two theorems which have the similar proof.

Theorem 3.4. *Let ξ be a unit speed curve in $\mathbb{F}^3$ with non-constant curvature κ and non-zero constant torsion τ. Then, the centrode $d = \tau e_1 + \kappa e_3$ of ξ is a rectifying curve. Conversely, each rectifying curve in $\mathbb{F}^3$ is the centrode of the unit speed curves with non-zero constant curvature and non-constant torsion.*

Theorem 3.5. *The curves $C_\pm = \xi \pm d$ with non-zero curvature κ are called the co-centrodes of ξ in $\mathbb{F}^3$. Let ξ be a unit speed curve in $\mathbb{F}^3$ with non-zero constant curvature and non-constant torsion. Then, ξ is a rectifying curve if and only iff its co-centrodes is a rectifying curve.*

3.4. Normal curves in Finsler space

Theorem 3.6. *Let ξ be a smooth curve in the $3-$dimensional Finsler space $\mathbb{F}^3$ with an arc length parameter s. Then ξ is a normal curve if and only if the curvature and the torsion of ξ satisfies the following differential equation*

$$-\frac{\tau(s)}{\kappa(s)} + \nabla^*_{\frac{\partial}{\partial s}}\left(\frac{1}{\tau(s)}\left(\nabla^*_{\frac{\partial}{\partial s}}\frac{1}{\kappa(s)}\right)\right) = 0.$$

for all $s \in I$.

Proof. Let ξ be a unit speed normal curve in $\mathbb{F}^3$ such that locally parameterized as

$$\xi^i = \xi^i(s);\ (\xi'^1(s), \xi'^2(s), \xi'^3(s)) \neq (0,0,0).$$

Then from eq.(2.3), we have following equation

$$\xi^i\frac{\partial}{\partial y^i} = \mu(s)n^i\frac{\partial}{\partial y^i} + \zeta(s)b^i\frac{\partial}{\partial y^i},\ i = \{1,2,3\}, \tag{3.9}$$

for some functions $\mu(s)$ and $\zeta(s)$. If we combine the derivative of the eq.(3.9) and eqs.(2.1), then we obtain

$$\begin{aligned} \frac{\partial}{\partial v} &= \left(\nabla^*_{\frac{\partial}{\partial s}}\mu(s)\right)e_2 + \mu(s)\nabla^*_{\frac{\partial}{\partial s}}e_2 \\ &\quad + \left(\nabla^*_{\frac{\partial}{\partial s}}\zeta(s)\right)e_3 + \zeta(s)\nabla^*_{\frac{\partial}{\partial s}}e_3 \\ &= -\kappa(s)\mu(s)e_1 + \left(\nabla^*_{\frac{\partial}{\partial s}}\mu(s) - \zeta(s)\tau(s)\right)e_2 \\ &\quad + \left(\mu(s)\tau(s) + \nabla^*_{\frac{\partial}{\partial s}}\zeta(s)\right)e_3. \end{aligned} \tag{3.10}$$

The eq.(3.10) gives that $\mu(s)$ and $\zeta(s)$ satisfy the following equations

$$\begin{cases} -\kappa(s)\mu(s) = 1, \\ \nabla^*_{\frac{\partial}{\partial s}}\mu(s) - \zeta(s)\tau(s) = 0, \\ \mu(s)\tau(s) + \nabla^*_{\frac{\partial}{\partial s}}\zeta(s) = 0. \end{cases} \tag{3.11}$$

The solutions of the differential equations in eq.(3.11) calculated as follows

$$\mu(s) = -\frac{1}{\kappa(s)} \quad \text{and} \quad \zeta(s) = \frac{1}{\tau(s)}\nabla^*_{\frac{\partial}{\partial s}}\left(\frac{1}{\kappa(s)}\right). \tag{3.12}$$

If we use the third equation in eq.(3.11), we find

$$-\frac{\tau(s)}{\kappa(s)} + \nabla^*_{\frac{\partial}{\partial s}}\left(\frac{1}{\tau(s)}\left(\nabla^*_{\frac{\partial}{\partial s}}\frac{1}{\kappa(s)}\right)\right) = 0.$$

Conversely, suppose that ξ be a smooth curve in $\mathbb{F}^3$ with the curvature and the torsion satisfy

$$-\frac{\tau(s)}{\kappa(s)} + \nabla^*_{\frac{\partial}{\partial s}}\left(\frac{1}{\tau(s)}\left(\nabla^*_{\frac{\partial}{\partial s}}\frac{1}{\kappa(s)}\right)\right) = 0. \tag{3.13}$$

Then, the vector given by

$$X(s) = \xi^i\frac{\partial}{\partial y^i} + \frac{1}{\kappa(s)}e_2(s) - \frac{1}{\tau(s)}\left(\nabla^*_{\frac{\partial}{\partial s}}\left(\frac{1}{\kappa(s)}\right)\right)e_3(s). \tag{3.14}$$

Thus, we reach that $\nabla^*_{\frac{\partial}{\partial s}}X(s) = 0$. Therefore, ξ is congruent to a normal curve. ☐

Corollary 3.2. *Let ξ be a normal curve in the $3-$dimensional Finsler space $\mathbb{F}^3$. Then ξ is a curve on the Finslerian sphere $\mathbb{FS}^2$.*

3.5. Osculating curves in Finsler space

Theorem 3.7. *Let ξ be an arc length parameterized osculating curve in the Finsler space $\mathbb{F}^3$. Then ξ is an osculating curve if and only if ξ is a geodesic curve or the binormal vector of the curve ξ is a constant vector field.*

Proof. Let ξ be a unit speed osculating curve in $\mathbb{F}^3$ such that locally parameterized as

$$\xi^i = \xi^i(s);\; (\xi'^1(s), \xi'^2(s), \xi'^3(s)) \neq (0,0,0).$$

Then from eq.(2.1), we have

$$\xi^i \frac{\partial}{\partial y^i} = \delta(s)\frac{\partial \xi^i}{\partial s}\frac{\partial}{\partial y^i} + \upsilon(s) n^i \frac{\partial}{\partial y^i},\; i = \{1,2,3\}, \tag{3.15}$$

for some functions $\nu(s)$ and $\mu(s)$. Differentiating eq.(3.15) with arc length parameter s, we get

$$\frac{\partial}{\partial v} = \left(\left(\nabla^*_{\frac{\partial}{\partial s}}\delta(s)\right) - \upsilon(s)\kappa(s)\right) e_1(s) +$$

$$\left(\left(\nabla^*_{\frac{\partial}{\partial s}}\upsilon(s)\right) + \delta(s)\kappa(s)\right) e_2(s) + \upsilon(s)\tau(s) e_3(s).$$

Thus the last equation gives

$$\begin{cases} \left(\nabla^*_{\frac{\partial}{\partial s}}\delta(s)\right) - \upsilon(s)\kappa(s) = 1, \\ \left(\nabla^*_{\frac{\partial}{\partial s}}\upsilon(s)\right) + \delta(s)\kappa(s) = 0, \\ \upsilon(s)\tau(s) = 0. \end{cases} \tag{3.16}$$

From the eqs.(3.16) we have $\upsilon(s) = 0$ or $\tau(s) = 0$. If $\upsilon(s) = 0$ is satisfied then it is found $\kappa(s) = 0$, then from the first two equations in eq.(3.16) and this result shows that the curve ξ is a geodesic. If $\tau(s) = 0$ then ξ lies in a plane of $\mathbb{R}^3$. □

4. Examples

Definition 4.1. Let $\alpha(x,y) = [a_{\alpha\beta}(x)y^\alpha y^\beta]^{1/2}$ be a Riemannian metric and $\beta(x,y)$ one-form $b_\alpha(x)y^\alpha$, where $b_i(x)$ is a covariant vector field. Then the metric $F(x,y) = \alpha(x,y) + \beta(x,y)$ and corresponding Finsler space is called a Randers metric and a Randers space , respectively. The coefficients of the Randers metric are given as follows:

$$g_{ij} = \frac{F}{\alpha}\{a_{ij} - \frac{y^i}{\alpha}\frac{y^j}{\alpha} + \frac{\alpha}{F}(b_i + \frac{y_i}{\alpha})(b_j + \frac{y_j}{\alpha})\}$$

where $y_i := a_{ij}y^j$. Since the bilinear form (g_{ij}) is positive definite, then the length of β is less then 1, i.e., $\|\beta\|_\gamma := \sqrt{a^{ij}b_i b_j} < 1$ where $\left(a^{ij}\right) := (a_{ij})^{-1}$[8].

The curves in the examples throughout the article are the unit speed curves according to the Randers metric.

Example 4.1. Let ξ be a unit speed curve on the Finslerian $2-$sphere $\mathbb{FS}^2$ that has the parametric equations as follows

$$x^1(t) = \frac{\cos t \sin t - b}{1 - b^2}, x^2(t) = \frac{\cos^2 t}{\sqrt{1-b^2}}, x^3(t) = \frac{\sin t}{\sqrt{1-b^2}}.$$

Then equation (3.5) gives the following differential equation

$$\sigma \nabla^{2*}_{\frac{\partial}{\partial t}}\sigma - 2(\nabla^*_{\frac{\partial}{\partial t}}\sigma)^2 - \sigma^2 = 0.$$

The solutions of this differential equation are $\sigma(t) = c\sec(t + t_0)$, here t_0 and $c \neq 0$ are constants. Then the parametric equations of the rectifying curve γ related to the curve ξ obtained as the form

$$\gamma^1(t) = \frac{a\sec(t+t_0)\cos t \sin t - a\sec(t+t_0)b}{1-b^2}$$

$$\gamma^2(t) = \frac{\sec(t+t_0)\cos^2 t}{\sqrt{1-b^2}}$$
$$\gamma^3(t) = \frac{\sec(t+t_0)\sin s}{\sqrt{1-b^2}}.$$

The image of the rectifying curve γ over the curve ξ illustrated in Figure 1 with the value $a = t_0 = 1$.

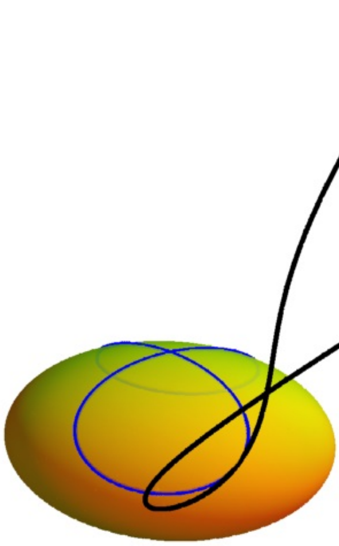

Figure 1. A rectifying curve γ (black) over the curve ξ(blue) on the Finslerian sphere $\mathbb{FS}^2$.

Example 4.2. We consider an arc length parameterized curve ξ in Finsler space $\mathbb{F}^3$. Let we give the locally parametric equations of this curve as follows

$$\xi^1(s) = \frac{R}{2}\frac{\frac{11}{8}\cos\frac{5}{8}s - \frac{5}{8}\cos\frac{11}{8}s - 1.8}{0.19},$$
$$\xi^2(s) = \frac{\frac{11}{8}\sin\frac{5}{8}s - \frac{5}{8}\sin\frac{11}{8}s}{\sqrt{0.19}},$$
$$\xi^3(s) = \frac{R\frac{\sqrt{55}}{8}\cos 0.9s}{\sqrt{0.19}}.$$

Then, from the Corollary 3.2, the curve ξ is a normal curve illustrated in Figure 2.

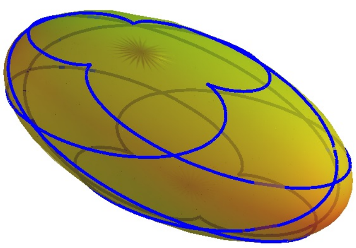

Figure 2. Normal curve ξ on the Finslerian sphere $\mathbb{FS}^2$ for $b = 0.9$.

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Affiliations

ZEHRA ÖZDEMIR
ADDRESS: Amasya University, Dept. of Mathematics, 05189, Amasya-Turkey.
E-MAIL: zehra.ozdemir@amasya.edu.tr
ORCID ID: orcid.org/0000-0001-9750-507X

FATMA ATEŞ
ADDRESS: Necmettin Erbakan University, Dept. of Mathematics-Computer Sciences, 99999, Konya-Turkey.
E-MAIL: second@author.com
ORCID ID: orcid.org/0000-0002-3529-1077

F. NEJAT EKMEKCI
ADDRESS: Ankara University, Dept. of Mathematics, 06100, Ankara-Turkey.
E-MAIL: third@author.com
ORCID ID: orcid.org/0000-0003-1246-2395