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Research Article

Subdirectly irreducible semilattices with endomorphism

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Abstract

In this paper we initiate an investigation into the class of meet semilattices endowed with an endomorphism. A consideration of the subdirectly irreducible algebras leads to a description of a subclass of those algebras $(S;\wedge,k)$ in which $(S;\wedge)$ is a meet semilattice and k is an endomorphism on S characterised by the property $k \geqslant id_S$. We particularly show that such an algebra is subdirectly irreducible if and only if it is a chain with one of the following forms:

(1) $\cdots < a_j < a_{j-1} < \cdots < a_0$;
(2) $0 < \cdots < a_j < a_{j-1} < \cdots < a_0$

in which $k(a_j) = a_{j-1}$ for $j \geqslant 1$, $k(0) = 0$ and $k(a_0) = a_0$.

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1. Introduction

An ordered set $(S;\leqslant)$ is a *meet-semilatice* if for any $x, y \in S$, the greatest lower bound $\inf\{x,y\}$ of x and y exists, denoted by $x \wedge y$. A mapping $k\colon S \to S$ is said to be *endomorphism* if $k(x \wedge y) = k(x) \wedge k(y)$ for every $x, y \in S$. For any additional background see, for example, either of the texts Blyth [1] or Grätzer [5].

Throughout what follows, we shall use the terminology $(S;\wedge)$ to denote a meet- semilattice.

In 1991, Ježek [8] initiated a study of the class of semilattices with an automorphism by characterising its subdirectly irreducible members. In 2004, Jackson [7] introduced a class of closure semilattices. In particular, he gave a representation of semilattices by means of topological Boolean algebras. Furthermore, the other related topics can be found in [2,6].

Here our objective is to initiate an investigation into the class of semilattices $(S;\wedge)$ endowed with an endomorphism k. We shall say that such an algebra $(S;\wedge,k)$ is an SLE-*algebra*; namely, an SLE-algebra is an algebra $(S;\wedge,k)$ of type $\langle 2,1\rangle$ where $(S;\wedge)$ is a meet-semilattice, and k is an endomorphism on S.

In what follows we shall denote by **SLE** the class of SLE-algebras.

Example 1.1. Every meet-semilattice gives to an SLE-algebra. For example, if $(S;\wedge)$ is a semilattice, then $(S;\wedge,\mathrm{id}_S) \in \mathbf{SLE}$.

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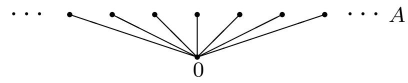
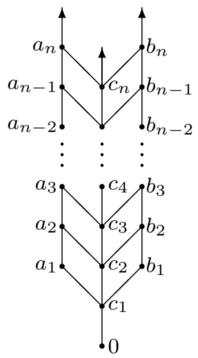
Example 1.2. Let $(S;\wedge)$ be a meet-semilattice and $a \in S$. If define $k\colon S \to S$ by $k(x) = a \wedge x$, then clearly $(S;\wedge,k) \in \mathbf{SLE}$.

Example 1.3. Consider a semilattice $(S;\wedge)$ described as the following Hasse diagram:



in which $S = A \cup \{0\}$ where $A = \{a_i \mid i \in \mathbb{N}\}$ is an anti-chain, and $0 < a_i$ for all $i \in \mathbb{N}$. Define $k\colon S \to S$ given by $k(0) = 0$ and $k(a_i) = a_{i+1}$ for each $i \in \mathbb{N}$. Then $(S;\wedge,k) \in \mathbf{SLE}$.

Example 1.4. Consider an infinite SLE-algebra $(S;\wedge,k)$ depicted as follows:



in which $c_i = a_i \wedge b_i$ $(i = 1, 2, \cdots)$, the endomorphism $k\colon S \to S$ is defined by $k(0) = 0$ and $k(x_i) = x_{i+1}$ where $x_i \in \{a_i, b_i, c_i\}$.

2. Congruences

By a *semilattice congruence* we shall mean an equivalence relation φ on a semilattice $(S;\wedge)$ satisfying the following condition:

$$(\forall x, y \in S)\,(\forall z \in S) \quad (x, y) \in \varphi \Longrightarrow (x \wedge z, y \wedge z) \in \varphi.$$

As usual we shall denote by ω and ι the equality relation and the universal relation, respectively. For $a, b \in S$ with $a \leqslant b$, we denote by $\theta_S(a,b)$ the principal semilattice congruence that collapses a and b; i.e., the smallest semilattice congruence on S generated by a and b.

The following description of the principal semilattice congruences is due to Dean and Oehmke (see [3] or [4]).

Lemma 2.1. [4, Lemma 2.1] *If $(S;\wedge)$ is a semilattice and $a \leqslant b$ in S. Then*

$$(x, y) \in \theta_S(a,b) \iff x = y \text{ or } x \wedge a = y \wedge a \text{ with } x, y \leqslant b.$$

Furthermore, for arbitrary a and b in S, $\theta_S(a,b) = \theta_S(a \wedge b, a) \vee \theta_S(a \wedge b, b)$.

As a consequence of the above, the following lemma is immediate:

Lemma 2.2. *If $(S;\wedge)$ is a semilattice then the following statements hold:*

(1) *If $a, b, c \in S$ with $a \leqslant b$ and $a \leqslant c$ then*

$$\theta_S(a,b) \wedge \theta_S(a,c) = \theta_S(a, b \wedge c).$$

(2) *If $a, b, c, d \in S$ with $a \leqslant b \leqslant c \leqslant d$ then*

$$\theta_S(a,b) \wedge \theta_S(c,d) = \omega.$$

By a *congruence* on an SLE-algebra $(S;k)$ we shall mean a semilattice congruence ϑ that satisfies the condition

$$(x,y)\in\vartheta \Longrightarrow (k(x),k(y))\in\vartheta.$$

In what follows for $a,b\in S$ with $a\leqslant b$, we shall denote by $\theta(a,b)$ the principal congruence on $(S;k)$ that collapses a and b; i.e., the smallest semilattice congruence on S generated by a and b

A description of principal congruences on an SLE-algebra $(S;\wedge,k)$ can be given as follows.

Theorem 2.3. *If $(S;\wedge,k)\in\mathbf{SLE}$ and $a\leqslant b$ in S then we have*

$$\theta(a,b)=\bigvee_{i\geqslant 0}\theta_S(k^i(a),k^i(b)).$$

Proof. Let $\varphi(a,b)$ be the right side of the stated equality. Then clearly $\varphi(a,b)$ is a semilattice congruence that collapses a and b. To see that $\varphi(a,b)$ is a SLE-congruence, it suffices to verify that for every $i\geqslant 0$,

$$(x,y)\in\theta_S(k^i(a),k^i(b))\Longrightarrow (k(x),k(y))\in\theta_S(k^{i+1}(a),k^{i+1}(b)).$$

In fact, if $(x,y)\in\theta_S(k^i(a),k^i(b))$ then we have either $x=y$ or $x\wedge k^i(a)=y\wedge k^i(a)$ with $x,y\leqslant k^i(b)$, so either $k(x)=k(y)$ or $k(x)\wedge k^{i+1}(a)=k(y)\wedge k^{i+1}(a)$ with $k(x),k(y)\leqslant k^{i+1}(b)$. It then follows that $(k(x),k(y))\in\theta_S(k^{i+1}(a),k^{i+1}(b))$. Hence, $\varphi(a,b)$ is a SLE-congruence.

If now ψ is a congruence on $(S;\wedge,k)$ that collapses a and b, then for every $i\geqslant 0$, we have $(k^i(a),k^i(b))\in\psi$, so $\theta_S(k^i(a),k^i(b))\leqslant\psi$, and so

$$\varphi(a,b)=\bigvee_{i\geqslant 0}\theta_S(k^i(a),k^i(b))\leqslant\psi.$$

Consequently, it follows that $\varphi(a,b)=\theta(a,b)$, as our required. □

By Theorem 2.3, the following corollary is immediate.

Corollary 2.4. *Let $(S;\wedge,k)\in\mathbf{SLE}$ and $a\leqslant b$ in S. Then we have the followings*:

(1) *If $k^{m+1}(x)=k^m(x)$ for some $m\geqslant 0$ where $x=a$ or $x=b$, then*

$$\theta(k^m(a),k^m(b))=\theta_S(k^m(a),k^m(b);$$

(2) *If $k^n(a)=k^n(b)$ for some $n\geqslant 1$, then we have*

$$\theta(a,b)=\bigvee_{i=0}^{n-1}\theta_S(k^i(a),k^i(b)).$$

Here we shall be concerned with those SLE-algebras $(S;\wedge,k)$ in which k satisfies the property $k\geqslant \mathrm{id}_S$. Clearly, if $(S;\wedge,k)$ is such an algebra then we have

$$x\leqslant k(x)\leqslant k^2(x)\leqslant\cdots\leqslant k^n(x)\leqslant\cdots$$

for every $x\in S$.

Example 2.5. The constructions in Examples 1.1 and 1.4 give the SLE-algebras $(S;\wedge,k)$ for which $k\geqslant\mathrm{id}_S$.

For the later purpose, we require the following basic properties.

Theorem 2.6. *Let $(S;\wedge,k)\in\mathbf{SLE}$ with $k\geqslant\mathrm{id}_S$. If $a,b\in S$ with $a\leqslant b$ are such that $k^{n+1}(x)=k^n(x)$ for some $n\geqslant 0$ where $x=a$ or $x=b$, then we have the following properties*:

(1) $\theta(a,k^n(a))=\theta_S(a,k^n(a))$;

(2) *if $k(b)=b$ then $\theta(a,b)=\theta_S(a,b)$;*

(3) *if $k(a)=a$ then $\theta(a,b)=\theta_S(a,k^n(b))$.*

Proof. (1) Since $k^i(k^n(a)) = k^n(a)$ for each i, we have by Theorem 2.3 that

$$\theta(a, k^n(a)) = \bigvee_{i\geqslant 0} \theta_S(k^i(a), k^i(k^n(a))) = \bigvee_{i=0}^{n} \theta_S(k^i(a), k^n(a)).$$

Note that $a \leqslant k^i(a)$ for each i, then we have $\theta_S(k^i(a), k^n(a)) \leqslant \theta_S(a, k^n(a))$. Thus it follows that $\theta(a, k^n(a)) = \theta_S(a, k^n(a))$.

(2) If $k(b) = b$ then, since $a \leqslant k^i(a)$, we have $\theta_S(k^i(a), b) \leqslant \theta_S(a, b)$ for each i. It follows by Theorem 2.3 that

$$\theta(a, b) = \bigvee_{i=0}^{n} \theta_S(k^i(a), b) = \theta_S(a, b).$$

(3) If $k(a) = a$ then $\theta(a, b) = \bigvee_{i=0}^{n} \theta_S(a, k^i(b))$. Since $k^i(b) \leqslant k^n(b)$, we have $\theta_S(a, k^i(b)) \leqslant \theta_S(a, k^n(b))$ for each i. Thus the stated equality holds. □

3. Subdirectly irreducible

Our main interest here will be in the subdirectly irreducible algebras. We recall (see [1] or [5]) that an algebra $\mathcal{A}$ is said to be *subdirectly irreducible* if for any family $\{\vartheta_i \mid i \in I\}$ of congruences on $\mathcal{A}$, $\bigwedge_{i\in I} \vartheta_i = \omega$ implies $\vartheta_i = \omega$ for some i; equivalently, there exists a smallest nontrival congruence ϑ on $\mathcal{A}$ such that $\varphi \geqslant \vartheta$ for every congruence $\varphi \neq \omega$ on $\mathcal{A}$.

The following result will play an important rôle.

Theorem 3.1. *Let $(S; \wedge, k) \in$ **SLE** with $k \geqslant \mathrm{id}_S$. If S is subdirectly irreducible algebra then for every $a \in S$ there exists some $n \geqslant 0$ such that $k^{n+1}(a) = k^n(a)$.*

Proof. Let S be subdirectly irreducible, and suppose, by the way of contradiction, that there exists some $a \in S$ such that $k^{n+1}(a) \neq k^n(a)$ for all n. Then we have

$$a < k(a) < k^2(a) < \cdots < k^n(a) < \cdots .$$

Thus for all $i \geqslant 0$, $\theta(k^i(a), k^{i+1}(a)) > \omega$. If

$$(\natural) \qquad (x, y) \in \bigwedge_{i\geqslant 0} \theta(k^i(a), k^{i+1}(a))$$

then for each i, we have

$$(x, y) \in \theta(k^i(a), k^{i+1}(a)) = \bigvee_{j\geqslant i} \theta_S(k^j(a), k^{j+1}(a)).$$

It follows that there exists some $r \geqslant i$ such that

$$(x, y) \in \bigvee_{j=i}^{r-1} \theta_S(k^j(a), k^{j+1}(a)) = \theta_S(k^i(a), k^r(a)).$$

By $(\natural)$, we have also $(x, y) \in \theta(k^r(a), k^{r+1}(a))$, and similarly, $(x, y) \in \theta_S(k^r(a), k^t(a))$ for some $t \geqslant r$. Thus

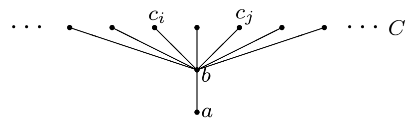
$$(x, y) \in \theta_S(k^i(a), k^r(a)) \wedge \theta_S(k^r(a), k^t(a)) = \omega$$

whence we obtain $x = y$. It therefore follows that

$$\bigwedge_{i\geqslant 0} \theta(k^i(a), k^{i+1}(a)) = \omega.$$

This contradicts the assumption of the subdirectly irreducibility. Hence, there is some $n \geqslant 0$ such that $k^{n+1}(a) = k^n(a)$. □

Example 3.2. Consider an infinite SLE-algebra $(S;\wedge,k)$ depicted as follows:



in which $S=\{a,b\}\cup C$ where $C=\{c_i\mid i\in\mathbb{N}\}$ is an anti-chain, and $a<b<c_i$, $c_i\wedge c_j=b$ for every $i,j\in\mathbb{N}$ $(i\neq j)$, and the endomorphism $k\colon S\to S$ is given by $k(a)=k(b)=b$ and $k(c_i)=c_i$ for each $i\in\mathbb{N}$. Then $k\geqslant \mathrm{id}_S$ and clearly $x\leqslant k(x)=k^2(x)$ for every $x\in S$. Observe that for any i,j with $i\neq j$,

$$\begin{aligned}\theta(b,c_i)\wedge\theta(b,c_j)&=\theta_S(b,c_i)\wedge\theta_S(b,c_j)\\&=\theta_S(b,c_i\wedge c_j)\\&=\theta_S(b,b)\\&=\omega.\end{aligned}$$

Then we see that $(S;\wedge,k)$ is not subdirectly irreducible.

For an SLE-algebra $(S;\wedge,k)$, an element $x\in S$ is said to be *fixed point* if $k(x)=x$. We shall denote by $\operatorname{Fix}S$ the set of fixed points of S.

Theorem 3.3. *Let* $(S;\wedge,k)\in\mathbf{SLE}$ *with* $k\geqslant\mathrm{id}_S$. *If* S *is subdirectly irreducible algebra then* $\operatorname{Fix}S$ *is a singleton or an 2-element chain.*

Proof. By Theorem 3.1, we have clearly that $|\operatorname{Fix}S|\geqslant 1$. Observe now that $\operatorname{Fix}S$ is a chain. In fact, if $x,y\in\operatorname{Fix}S$, then we have by Theorem 2.3 and Lemma 2.2 that

$$\begin{aligned}\theta(x\wedge y,x)\wedge\theta(x\wedge y,y)&=\theta_S(x\wedge y,x)\wedge\theta_S(x\wedge y,y)\\&=\theta_S(x\wedge y,x\wedge y)\\&=\omega.\end{aligned}$$

It follows by the subdirectly irreducibility that $\theta(x\wedge y,x)=\omega$ or $\theta(x\wedge y,y)=\omega$, from which it follows that $x\wedge y=x$ or $x\wedge y=y$, whence $x\leqslant y$ or $y\leqslant x$. Hence, $\operatorname{Fix}S$ is a chain.

If now $|\operatorname{Fix}S|\geqslant 3$, then there exist $x,y,z\in\operatorname{Fix}S$ with $x<y<z$. It then follows the contradiction that

$$\theta(x,y)\wedge\theta(y,z)=\theta_S(x,y)\wedge\theta_S(y,z)=\omega.$$

Therefore, $\operatorname{Fix}S$ must be a singleton or an 2-element chain. □

Corollary 3.4. *A meet semilattice* $(S;\wedge)$ *is subdirectly irreducible if and only if it is a singleton or an 2-element chain.*

Proof. Note that $(S;\wedge)$ can be regarded as $(S;\wedge,\mathrm{id}_S)$, then $\operatorname{Fix}S=S$. Thus by Theorem 3.3, the conclusion is clear. □

In what follows for an ordered set S and $a,b\in S$, we shall write $a\nparallel b$ to denote that a and b are comparable (in the sense that $a\leqslant b$ or $b\leqslant a$), and write $a\parallel b$ to denote that a and b are incomparable (in the sense that $a\nleqslant b$ and $b\nleqslant a$).

Theorem 3.5. *Let* $(S;\wedge,k)\in\mathbf{SLE}$ *with* $k\geqslant\mathrm{id}_S$. *If* S *is subdirectly irreducible then* S *is a chain.*

Proof. Suppose that S is subdirectly irreducible, and let $a,b\in S$. Then by Theorems 3.1 and 3.3, $k^n(a),k^n(b)\in\operatorname{Fix}S$ with $k^n(a)\nparallel k^n(b)$ for some $n\geqslant 0$. We may assume that $k^n(a)\leqslant k^n(b)$. Let $c=a\wedge b$, then $k^n(c)=k^n(a)$. If $k(a)=k(b)$ then $\theta(c,a)\wedge\theta(c,b)=$

$\theta_S(c,a) \wedge \theta_S(c,b) = \omega$, from which it follows by subdirectly irreducibility that $c = a$ or $c = b$, whence $a \leqslant b$ or $b \leqslant a$. Assume now that $k(a) \neq k(b)$. If $k^n(a) \neq k^n(b)$, then since

$$\begin{aligned} &\theta(k^{n-1}(c), k^{n-1}(a)) \wedge \theta(k^n(a), k^n(b)) \\ &= \theta_S(k^{n-1}(c), k^{n-1}(a)) \wedge \theta_S(k^n(a), k^n(b)) \\ &= \omega \end{aligned}$$

it follows by the subdirectly irreducibility that $k^{n-1}(c) = k^{n-1}(a)$. Then $\theta(k^{n-2}(c), k^{n-2}(a)) = \theta_S(k^{n-2}(c), k^{n-2}(a))$. Similar to the above, we can obtain $k^{n-2}(c) = k^{n-2}(a)$. Continuing in this way, we obtain $c = a$, whence $a \leqslant b$. Similarly, if $k^n(b) < k^n(a)$, then we can obtain that $b \leqslant a$.

If, on the other hand, $k^n(a) = k^n(b)$, then since $k(a) \neq k(b)$, we have some m with $1 \leqslant m \leqslant n-1$ such that $k^m(a) \neq k^m(b)$ and $k^{m+1}(a) = k^{m+1}(b)$. Since

$$\begin{aligned} &\theta(k^m(c), k^m(a)) \wedge \theta(k^m(c), k^m(b)) \\ &= \theta_S(k^m(c), k^m(a)) \wedge \theta_S(k^m(c), k^m(b)) \\ &= \theta_S(k^m(c), k^m(a) \wedge k^m(b)) \\ &= \omega \end{aligned}$$

it follows by the subdirectly irreducibility that $k^m(c) = k^m(a)$ or $k^m(c) = k^m(b)$. Thus we have $k^m(a) < k^m(b)$ or $k^m(b) < k^m(a)$. By a similar argument as above we also can show that $a \nparallel b$.

Therefore, we see that S is a chain. $\square$

Example 3.6. Consider a finite chain S described as follows:

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_{n-1} < b_{n-1} < a_n = b_n.$$

If define $k\colon S \to S$ by $k(a_i) = a_{i+1}$, $k(b_i) = b_{i+1}$ for $i = 1, 2, \cdots, n-1$, and $k(a_n) = a_n$, then $(S; \wedge, k) \in \mathbf{SLE}$ with $k \geqslant \mathrm{id}_S$. Since

$$\theta(a_{n-1}, b_{n-1}) \wedge \theta(b_{n-1}, b_n) = \theta_S(a_{n-1}, b_{n-1}) \wedge \theta_S(b_{n-1}, b_n) = \omega,$$

we see that S is clearly not subdirectly irreducible.

In what follows for $a < b$ in S, if the interval $[a, b] = \{x \in S \mid a \leqslant x \leqslant b\}$ is precisely the 2-element set $\{a, b\}$, then we say that b *covers* a or a *is covered by* b, denoted by $a \prec b$. We shall write $a \preceq b$ to denote that $a = b$ or $a \prec b$.

Theorem 3.7. *Let $(S; \wedge, k) \in \mathbf{SLE}$ with $k \geqslant \mathrm{id}_S$. If S is subdirectly irreducible and $a \in S$, then we have $k^{i-1}(a) \preceq k^i(a)$ for every $i \geqslant 1$.*

Proof. Let $a \in S$ and suppose that $b \in S$ is such that $k^{i-1}(a) \leqslant b \leqslant k^i(a)$ for some $i \geqslant 1$. If $k(a) = a$ then there is nothing to do, and if $k(b) = b$, then we have clearly $b = k^i(a)$. We may assume that $k(a) \neq a$ and $k(b) \neq b$. Then by Theorem 3.1, we have $m \geqslant 1$ and $n \geqslant 1$ such that

$$\begin{aligned} &a < k(a) < \cdots < k^{n-1}(a) < k^n(a) = k^{n+1}(a); \\ &b < k(b) < \cdots < k^{m-1}(b) < k^m(b) = k^{m+1}(b). \end{aligned}$$

Let $t = n - i + 1$ then $k^t(b) = k^n(a) \in \operatorname{Fix} S$. Thus we have $t \geqslant m$. Write $c = k^{i-1}(a)$ then $k^t(c) = k^n(a) \in \operatorname{Fix} S$, and since $c \leqslant b \leqslant k(c)$, we have $k^{m+1}(c) = k^m(b) \in \operatorname{Fix} S$, there follows that $m + 1 \geqslant t \geqslant m$. Hence, there are two possibilities to consider:

(1) If $t = m + 1$ then $k^m(c) \notin \operatorname{Fix} S$. Thus we have $k^m(c) \neq k^m(b)$. Since by Corollary 2.4,

$$\begin{aligned} &\theta(k^{m-1}(b), k^m(c)) \wedge \theta(k^m(c), k^m(b)) \\ &= \theta_S(k^{m-1}(b), k^m(c)) \wedge \theta_S(k^m(c), k^m(b)) \\ &= \omega, \end{aligned}$$

it follows by the subdirectly irreducibility that $k^{m-1}(b) = k^m(c)$. Then $\theta(k^{m-2}(b), k^{m-1}(c)) = \theta_S(k^{m-2}(b), k^{m-1}(c))$. Similar to the above, we can obtain $k^{m-2}(b) = k^{m-1}(c)$. Continuing in this way, we have $b = k(c) = k^i(a)$.

(2) If $t = m$ then $k^m(c) = k^m(b)$. Since $k^{m-1}(b) < k^m(b) = k^{m+1}(b)$, we can obtain by a similar argument as in (1) that $b = c = k^{i-1}(a)$.

Therefore, we see that $k^{i-1}(a) \preceq k^i(a)$. □

With combination of Theorems 3.1, 3.3, 3.5 and 3.7, we now can give our main result as follows.

Theorem 3.8. *If* $(S; \wedge, k)$ *with* $k \geqslant \mathrm{id}_S$ *is an* SLE-*algebra then* S *is subdirectly irreducible if and only if it is a chain with one of the following forms*:

(1) $\cdots < a_j < a_{j-1} < \cdots < a_0$;

(2) $0 < \cdots < a_j < a_{j-1} < \cdots < a_0$

in which $k(a_j) = a_{j-1}$ *for* $j \geqslant 1$, $k(0) = 0$ *and* $k(a_0) = a_0$.

Proof. (⇒:) Suppose that S is subdirectly irreducible. Then by Theorem 3.5, S is a chain. If $|S| = 1$; i.e., $S = \{0\}$, then we have clearly $k(0) = 0$. If $|S| = 2$ then S is clearly an 2-element chain as $0 = k(0) < a_0 = k(a_0)$ or $a_1 < a_0$ with $k(a_1) = k(a_0) = a_0$.

We now may assume that $|S| \geqslant 3$. Then since $|\operatorname{Fix} S| \leqslant 2$, there exists $a \in S$ with $k(a) \neq a$, and by Theorem 3.1 we have some $n \geqslant 1$ such that $k^n(a) \in \operatorname{Fix} S$ but $k^{n-1}(a) \notin \operatorname{Fix} S$. It then follows by Theorem 3.7 that

$$(\dagger) \quad a \prec k(a) \prec \cdots \prec k^{n-1}(a) \prec k^n(a).$$

We shall show as follows that if $b \in \operatorname{Fix} S$ with $b \neq k^n(a)$ then $b = 0$, the bottom element of S. Observe first that $b < a$. In fact, if $b \geqslant a$ then $b > k^n(a)$, there follows the contradiction that

$$\begin{aligned} &\theta(k^{n-1}(a), k^n(a)) \wedge \theta(k^n(a), b) \\ &= \theta_S(k^{n-1}(a), k^n(a)) \wedge \theta_S(k^n(a), b) \\ &= \omega. \end{aligned}$$

Thus we have $b < a$. Now for $x \in S$, if $x < b$, then it follows by Theorem 2.6 the contradiction that

$$\theta(x, b) \wedge \theta(a, k^n(a)) = \theta_S(x, b) \wedge \theta_S(a, k^n(a)) = \omega.$$

Hence we have $x \geqslant b$, whence $b = 0$.

To see that S is of one of the stated forms, it suffices to show that for $x \in S$, if $x \notin \operatorname{Fix} S$ and $x \neq k^i(a)$ for each i, then $k^j(x) = a$ for some j. By Theorems 3.1, 3.3 and 3.7, we have some $m \geqslant 1$ such that

$$(\ddagger) \quad x \prec k(x) \prec \cdots \prec k^{m-1}(x) \prec k^m(x) = k^n(a).$$

By (†) we have $x \notin [a, k^n(a)]$. It follows that $x \ngeqslant a$, so $x < a < k^n(a) = k^m(x)$. Thus we obtain by (‡) that $a = k^j(x)$ for some $j \leqslant m$.

Therefore, if $|\operatorname{Fix} S| = 1$, say $\operatorname{Fix} S = \{k^n(a)\}$, then S is of the form (1); and if $|\operatorname{Fix} S| = 2$, say $\operatorname{Fix} S = \{b, k^n(a)\}$, then S is of the form (2).

(⇐:) Suppose that S is one of the stated forms. If $|S| \leqslant 2$ then it is clearly subdirectly irreducible. We may assume that $|S| \geqslant 3$, and let $\varphi \neq \omega$ be a congruence on S. In the form (2), we see $\theta(0, a_i) = \theta(0, a_0) = \iota$ for every i. Thus in the either cases, we have i, j with $j > i > 0$ such that $(a_j, a_i) \in \varphi$, then $(a_{j-i}, a_0) = (k^i(a_j), k^i(a_i)) \in \varphi$, and then, we have

$$(a_1, a_0) \in \theta_S(a_{j-i}, a_0) \leqslant \theta(a_{j-i}, a_0) \leqslant \varphi$$

whence $\theta(a_1, a_0) \leqslant \varphi$. Hence, it follows that $\theta(a_1, a_0)$ is the smallest nontrivial congruence on S, and consequently, S is subdirectly irreducible. □

By Theorem 3.8, the following corollary is clear.

Corollary 3.9. *Let* $(S; \wedge, k) \in \mathbf{SLE}$ *with* $k \geqslant \mathrm{id}_S$. *If* S *is finite, then* S *is subdirectly irreducible if and only if it is a chain with one of the following forms*:

(1) $a_m < a_{m-1} < \cdots < a_0$;

(2) $0 < a_m < a_{m-1} < \cdots < a_0$

in which $k(a_j) = a_{j-1}$ *for* $m \geqslant j \geqslant 1$, $k(0) = 0$, $k(a_0) = a_0$.

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