



Tweaking Ramanujan's Approximation of $n!$

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Abstract

About 1730 James Stirling, building on the work of Abraham de Moivre, published what is known as Stirling's approximation of $n!$. He gave a good formula which is asymptotic to $n!$. Since then hundreds of papers have given alternative proofs of his result and improved upon it, including notably by Burside, Gosper, and Mortici. However, Srinivasa Ramanujan gave a remarkably better asymptotic formula. Hirschhorn and Villarino gave nice proof of Ramanujan's result and an error estimate for the approximation. In recent years there have been several improvements of Stirling's formula including by Nemes, Windschitl, and Chen. Here it is shown (i) how all these asymptotic results can be easily verified; (ii) how Hirschhorn and Villarino's argument allows tweaking of Ramanujan's result to give a better approximation; and (iii) that new asymptotic formulae can be obtained by further tweaking of Ramanujan's result. Tables are calculated displaying how good each of these approximations is for n up to one million.

1. Introduction

About 1730 the Scottish mathematician James Stirling (1692–1770), building on the work of the French mathematician Abraham de Moivre (1667–1754), published what is known as Stirling's approximation of $n!$. In fact, Stirling [1] proved that $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$; that is, $n!$ is asymptotic to $\sqrt{2\pi n} \left(\frac{n}{e}\right)^n$. De Moivre had been considering a gambling problem and needed to approximate $\binom{2n}{n}$ for large n . The Stirling approximation gave a very satisfactory solution to this problem.

The problem of extending the factorial from the positive integers to a wider class of numbers was first investigated by the German mathematicians Daniell Bernoulli (1700–1782) and Christian Goldbach (1690–1764) in the 1720s. In 1729 the Swiss polymath Leonhard Euler (1707–1783) succeeded and in 1730 he proved that for z any complex number with positive real part, $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$, where $\Gamma(n) = (n-1)!$, for any positive integer n . The name gamma function is due to the French mathematician Adrien-Marie Legendre (1752–1833).

In 1774 the French polymath Pierre-Simon Laplace (1749–1827) noticed that Stirling's formula for $n!$ has a generalization to the gamma function, namely that for x a positive real number, $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$. One of the most elementary proofs of Stirling's formula for the gamma function was published in 2008 by Reinhard Michel [2].

Most of the proofs in the literature of Stirling's formula and its extensions prove that they are asymptotic by establishing an error estimate such as

$$\Gamma(x+1) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x (1 + O(x^{-1})).$$

In fact most of the effort goes into proving such error estimates.

In this paper our main result says that once one knows that Stirling's formula is asymptotic to $\Gamma(x+1)$, all of the other known asymptotic formulae can be verified trivially without the need to establish any error estimates.



In 1917 William Burnside [3] published a modest improvement on Stirling’s formula, namely $\Gamma(x+1) \sim \sqrt{2\pi} \left(\frac{x+1/2}{e}\right)^{x+1/2}$. How modest an improvement it is can be ascertained from our Table 1. In 1978 Ralph William (Bill) Gosper Jr, [4], published a significant improvement on Stirling and Burnside’s formulae. It was that $\Gamma(x+1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \sqrt{2x + \frac{1}{3}}$. In a web post in 2002, Robert H. Windschitl, [5], gave an elegant and good asymptotic approximation of $n!$, namely that $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh\left(\frac{1}{x}\right)\right)^{\frac{x}{2}}$. In 2010 Gergő Nemes gave an asymptotic approximation which is almost as good as Windschitl’s but better than all the others at that time. It was that $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x$. An asymptotic formula of a different style, which is much better than Gosper’s, was published in 2011 by Cristinel Mortici [6]. It was $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e} + \frac{1}{12ex}\right)^x$.

Pierre-Simon Laplace discovered what is now known as the Stirling series for the gamma function.

$$\Gamma(x+1) \sim e^{-x} x^{x+\frac{1}{2}} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \sum_{n=5}^{\infty} \frac{a_n}{b_n x^n}\right),$$

where the real numbers a_n and b_n are explicitly calculated in [7]. As stated in [8], “the performance deteriorates as the number of terms is increased beyond a certain value.” We show how using up to the term x^{-4} in this divergent series compares with the other approximations.

A major advance in producing an asymptotic formula for $n!$ was made by the extraordinary Indian mathematician Srinivasa Ramanujan (1887–1920) in the last year of his life. Ramanujan’s claim, recorded in [9, p. 339], was that

$$\Gamma(x+1) = \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right)^{\frac{1}{6}},$$

where $\theta_x \rightarrow 1$ as $x \rightarrow \infty$ and $\frac{3}{10} < \theta_x < 1$ and he gave numerical evidence for his claim.

Ramanujan’s approximation is substantially better than all those which were published in the subsequent 80 years. For example, when $n = 1$ million, the percentage error of Ramanujan’s approximation is one million million times better than Gosper’s. In 2013 Michael Hirschhorn and Mark B. Villarino [10] proved the correctness of Ramanujan’s claim above for positive integers. They showed that Ramanujan’s θ_n satisfies for each positive integer n :

$$1 - \frac{11}{8n} + \frac{79}{112n^2} < \theta_n < 1 - \frac{11}{8n} + \frac{79}{112n^2} + \frac{20}{33n^3}.$$

Although they did not explicitly say it, it is clear from their work that $\Gamma(x+1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1 - \frac{11}{8x} + \frac{79}{112x^2}}{30}\right)^{\frac{1}{6}}$,

at least for positive integers. This approximation, as can be seen in Table 3, is better than all that preceded it. Indeed for $n = 1$ million, it has a percentage error at least one million times better than each one.

In 2016 Chao-Ping Chen [11] produced an asymptotic approximation which for $n = 1$ million has a percentage error one million times better than that of Hirschhorn and Villarino. His asymptotic approximation is

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{210}}.$$

A more detailed analysis of Hirschhorn and Villarino’s improvement on that of Ramanujan, suggests a tweaking of their approximation. That tweaking produces an approximation which is stated in Corollary 2.3 and is comparable to Chen’s for $n = 1$ to $n = 10,000$ and much better than Chen’s for $n = 1$ million, as is evidenced in Table 3.

Chen points out in [11] that Burnside’s approximation involves an error of order $O(n^{-1})$, Ramanujan’s approximation involves an error of $O(n^{-4})$, Nemes and Windschitl’s approximations involves an error of $O(n^{-5})$, and his own approximation involves an error of order $O(n^{-7})$. However caution is needed. Consider the following extreme example:

$$\sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{210}} \left(1 + \frac{10^{100}}{n^8}\right) \sim n!$$

and has an error of the order of $O(n^{-7})$ but is an absurdly bad approximation even for $n = 1$ million. The order estimate can be used to compare approximations for “very large” n , but does not tell us how large is “very large”.

2. The approximations of $\Gamma(x+1)$

As suggested in §1, once we know Stirling's asymptotic formula for $\Gamma(x+1)$, all of the others follow trivially. This fact is captured in Theorem 2.1.

Theorem 2.1. *Let f be a function from a subset (a, ∞) to \mathbb{R} , where $a \in \mathbb{R}, a > 0$. If $\lim_{x \rightarrow \infty} f(x) = 1$, then $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \cdot f(x)$.*

Proof. This follows immediately from the Stirling asymptotic approximation, namely that $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$. \square

As an immediate corollary of Theorem 2.1 we obtain that all of the other mentioned approximations are asymptotic to $\Gamma(x+1)$. Some of these were proved by the authors only for x a positive integer.

Corollary 2.2. *For x a positive real number:*

(i) Burnside [3]: $\Gamma(x+1) \sim \sqrt{2\pi} \left(\frac{x+1/2}{e}\right)^{x+1/2}$;

(ii) Gosper [4]: $\Gamma(x+1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \sqrt{2x + \frac{1}{3}}$;

(iii) Mortici [6]: $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e} + \frac{1}{12ex}\right)^x$;

(iv) Ramanujan [9]: $\Gamma(x+1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{\frac{1}{6}}$;

(v) Laplace (n): Fix $n \in \mathbb{N}$. For $a_i, b_i \in \mathbb{N}$,

$$\Gamma(x+1) \sim e^{-x} x^{x+\frac{1}{2}} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \sum_{i=3}^n \frac{a_i}{b_i x^i}\right);$$

(vi) Nemes: $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x$.

(vii) Windschitl [5]: $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh\left(\frac{1}{x}\right)\right)^{\frac{x}{2}}$.

(viii) Hirschhorn & Villarino [10] :

$$\Gamma(x+1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1 - \frac{11}{8x} + \frac{79}{112x^2}}{30}\right)^{\frac{1}{6}}$$

(ix) Chen [11]: $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{210}}$.

Proof. In each case it is sufficient to determine the function f in Theorem 2.1 and observe that $\lim_{x \rightarrow \infty} f(x) = 1$.

(i) Use $f(x) = \left(1 + \frac{1}{2x}\right)^x \left(\frac{1 + \frac{1}{2x}}{e}\right)^{\frac{1}{2}}$.

(ii) Use $f(x) = \sqrt{1 + \frac{1}{6x}}$.

(iii) Use $f(x) = \left(1 + \frac{1}{12x^2}\right)^x$.

(iv) Use $f(x) = \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3}\right)^{\frac{1}{6}}$.

(v) Use $f(x) = \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \sum_{i=3}^n \frac{a_i}{b_i x^i}\right)$.

(vi) Use $f(x) = \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x$.

(vii) Use $f(x) = \left(x \sinh\left(\frac{1}{x}\right)\right)^{\frac{x}{2}}$.

(viii) Use $f(x) = \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1 - \frac{11}{8x} + \frac{79}{112x^2}}{240x^3}\right)^{\frac{1}{6}}$.

(ix) Use $f(x) = \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{210}}$.

□

With this result in hand we see than one can easily tweak any of the known asymptotic approximation to get others and this tweaking can be done to optimize the approximation for any value of n decided in advance. In the spirit of Ramanujan I will not include the details as they are not particularly important.

We can tweak Ramanujan’s approximation again to get an even better approximation for $n = 1$ million which we refer to in Table 2 as the SAM approximation. The proof of the corollary uses an obvious modification of the proof of (ix) above.

Corollary 2.3. For x a positive real number,

$$\Gamma(x + 1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1 - \frac{11}{8x} + \frac{79}{112x^2} + \frac{A}{x^3}}{30}\right)^{\frac{1}{6}},$$

where $A = \frac{380279456577}{722091376690}$.

□

3. Numerical analysis of the approximations

The tables in this section were calculated using the WolframAlpha software package. (See <https://www.wolframalpha.com/>.) They demonstrate the performance of the asymptotic approximations.

Each of the approximations gets further and further from $n!$ as n tends to infinity. So the quality of the approximations is best judged by considering the percentage error, that is $100 \times \frac{\text{approximation} - n!}{n!}$.

In the tables S = Stirling, B = Burnside, G = Gosper, L4 = (Laplace) Stirling series up to x^{-4} , M = Mortici, N=Nemes, W=Windschitl, R = Ramanujan, HV = Hirschhorn and Villarino, C = Chen, and SAM = the author of this paper.

From the tables it is abundantly clear that Gosper’s approximation is a much better approximation than Stirling’s, and Mortici’s elegant approximation is closer in accuracy to Ramanujan’s. Ramanujan’s approximation is amazingly good. The tweaking of Ramanujan’s approximation using the Hirschhorn-Villarino results significantly improves the approximation. Chen’s approximation is better than all that precede it. The SAM approximation obtained by extra tweaking of Ramanujan’s approximation produces an approximation similar to Chen’s up to $n = 10,000$ and much better for $n = 1,000,000$.

Table 1

n	$n!$	S %error	B %error	G %error
2	2	4.0	1.7	1.3×10^{-1}
5	1.2×10^2	1.7	7.6×10^{-1}	2.5×10^{-2}
10	3.6×10^6	8.3×10^{-1}	4.0×10^{-1}	6.6×10^{-3}
20	2.4×10^{18}	4.2×10^{-1}	2.0×10^{-1}	1.7×10^{-3}
50	3.0×10^{64}	1.7×10^{-1}	8.3×10^{-2}	2.7×10^{-4}
100	9.3×10^{157}	8.3×10^{-1}	4.1×10^{-2}	6.9×10^{-5}
10^3	4.0×10^{2567}	8.3×10^{-3}	4.2×10^{-3}	6.9×10^{-7}
10^4	2.8×10^{35659}	8.3×10^{-4}	4.2×10^{-4}	6.9×10^{-9}
10^6	8.3×10^{5565708}	8.3×10^{-6}	4.2×10^{-6}	6.9×10^{-13}

Table 2

n	$n!$	M %error	R % error	L4 %error	N %error
2	2	1.0×10^{-2}	3.3×10^{-3}	1.4×10^{-2}	1.7×10^{-3}
5	1.2×10^2	5.7×10^{-4}	1.2×10^{-4}	3.5×10^{-4}	2.0×10^{-5}
10	3.6×10^6	7.0×10^{-5}	8.6×10^{-6}	7.8×10^{-7}	6.5×10^{-7}
20	2.4×10^{18}	8.7×10^{-6}	5.7×10^{-7}	2.4×10^{-8}	2.0×10^{-8}
50	3.0×10^{64}	5.6×10^{-7}	1.5×10^{-8}	2.5×10^{-10}	2.1×10^{-10}
100	9.3×10^{157}	6.9×10^{-8}	9.5×10^{-10}	7.8×10^{-12}	6.5×10^{12}
10^3	4.0×10^{2567}	6.9×10^{-11}	9.5×10^{-14}	7.8×10^{-17}	6.5×10^{-17}
10^4	2.8×10^{35659}	6.9×10^{-14}	9.5×10^{-18}	7.8×10^{-22}	6.5×10^{-22}
10^6	8.3×10^{5565708}	6.9×10^{-20}	9.5×10^{-26}	7.8×10^{-32}	6.5×10^{-32}

Table 3

n	$n!$	W % error	HV %error	C % error	SAM %error
2	2	1.6×10^{-3}	1.6×10^{-4}	2.2×10^{-4}	2.9×10^{-4}
5	1.2×10^2	1.9×10^{-5}	1.5×10^{-6}	5.0×10^{-7}	6.0×10^{-7}
10	3.6×10^6	6.1×10^{-7}	3.0×10^{-8}	4.1×10^{-9}	4.9×10^{-9}
20	2.4×10^{18}	1.9×10^{-8}	5.2×10^{-10}	3.2×10^{-11}	3.8×10^{-11}
50	3.0×10^{64}	2.1×10^{-10}	2.3×10^{-12}	5.3×10^{-14}	6.3×10^{-14}
100	9.3×10^{157}	6.2×10^{-12}	3.6×10^{-14}	4.2×10^{-16}	4.9×10^{-16}
10^3	4.0×10^{2567}	6.2×10^{-17}	3.7×10^{-20}	4.17×10^{-23}	4.9×10^{-23}
10^4	2.8×10^{35659}	6.2×10^{-22}	3.7×10^{-26}	4.2×10^{-30}	4.9×10^{-30}
10^6	8.3×10^{5565708}	6.2×10^{-32}	3.7×10^{-38}	4.2×10^{-44}	1.3×10^{-50}

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