Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (4) (2022), 1104 – 1107
DOI : 10.15672/hujms.995747

Research Article

On localization of the eigenvalues of matrices "close" to triangular ones

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Abstract

We suggest a new bound for the eigenvalues of a matrix. For matrices which are "close" to triangular ones that bound is sharper than the well-known results, such as the Ostrowski theorem.

Mathematics Subject Classification (2020). 15A18, 15A42

Keywords. matrices, localization of eigenvalues, Ostrowski theorem

1. Introduction and statement of the main result

Let $A = (a_{jk})_{j,k=1}^n$ be a complex $n \times n$-matrix and

$$U(a;r) = \{z \in \mathbb{C} : |a - z| \le r\} \ \ (a \in \mathbb{C}, r > 0).$$

In the paper [10], Ostrowski has proved that each eigenvalue of A is contained in the set $\cup_{j=1}^n U(a_{jj}; \min\{R_j, Q_j\})$, where

$$R_j = \sum_{i=1, i\ne j}^n |a_{ij}| \text{ and } Q_j = \sum_{i=1, i\ne j}^n |a_{ji}|.$$

That result refines the Gershgorin theorem [3]. In [9] and [1], Ostrowski and Brauer independently have obtained an estimate for the eigenvalues by means of the Cassini ovals. For more details about the Gershgorin, Ostrowski and Brauer theorems see [7, Sections III.2.2, III.2.4 and III.2.5]. These theorems have been refined and extended in many works, cf. [2, 5, 6, 8] and the references given therein.

As is well known, the diagonal entries of a triangular matrix are its eigenvalues. At the same time, in the case of triangular matrices the above pointed results are not attained. Namely, for a triangular matrix A they do not give us the equality $\lambda(A) = a_{kk}$ for each eigenvalue $\lambda(A)$ of A and a positive integer $k \le n$. In this paper we suggest a bound for the eigenvalues which is attained for triangular matrices. To this end introduce the notations.

$$q_{\rm up} := \max_{k=1,\dots,n-1} \Big(\sum_{j=k+1}^n |a_{jk}|^2\Big)^{1/2},$$

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Received: 15.09.2021; Accepted: 26.01.2022

and

$$M_k = 1 + (\sum_{j=1, j\neq k}^{n} |a_{jk}|^2)^{1/2} \ \ (k = 1, ..., n).$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let*

$$q_{\text{up}} < 1. \tag{1.1}$$

Then any eigenvalue of matrix $A = (a_{jk})_{j,k=1}^{n}$ *is located in the set*

$$\cup_{k=1}^{n} U(a_{kk}; \psi_{\text{up}}(k)), \ \ where\ \psi_{\text{up}}(k) := \frac{\sqrt[n]{q_{\text{up}}}\, M_k}{1 - \sqrt[n]{q_{\text{up}}}}.$$

The proof of this theorem is presented in the next section. Theorem 1.1 is sharp: if A is upper triangular, then $\psi_{\text{up}}(k) = 0$ and it implies that any eigenvalue coincides with some diagonal entry.

Combining the Ostrowski theorem and Theorem 1.1, we arrive at

Corollary 1.2. *Let condition (1.1) hold. Then any eigenvalue of* $A = (a_{jk})_{j,k=1}^{n}$ *is located in the set*

$$\cup_{j=1}^{n} U(a_{jj}; \eta_{\text{up}}(j)), \ \ where\ \eta_{\text{up}}(j) := \min\{\psi_{\text{up}}(j), R_j, Q_j\}.$$

Under condition (1.1) Corollary 1.2 refines the Ostrowski theorem in the case

$$\psi_{\text{up}}(j) < \min\{R_j, Q_j\}$$

for at least one $j \leq n$.

Now put

$$q_{\text{low}} := \max_{k=1,\dots,n-1} (\sum_{j=k+1}^{n} |a_{kj}|^2)^{1/2},$$

and

$$L_k = 1 + (\sum_{j=1, j\neq k}^{n} |a_{kj}|^2)^{1/2} \ \ (k = 1, ..., n).$$

Let $\sigma(A)$ denote the spectrum of A and A^* be the matrix adjoint to A. Take into account that for any $\lambda(A) \in \sigma(A)$ we have $\overline{\lambda}(A) \in \sigma(A^*)$ and

$$|\lambda(A^*) - \overline{a}_{kk}| = |\lambda(A) - a_{kk}|.$$

Then, replacing in Theorem 1.1 A by A^*, we get

Corollary 1.3. *Let*

$$q_{\text{low}} < 1. \tag{1.2}$$

Then $\sigma(A)$ *is located in the set*

$$\cup_{k=1}^{n} U(a_{kk}; \psi_{\text{low}}(k)), \ \ where\ \psi_{\text{low}}(k) := \frac{\sqrt[n]{q_{\text{low}}}\, L_k}{1 - \sqrt[n]{q_{\text{low}}}}.$$

Combining Theorem 1.1 and Corollary 1.3, we obtain our next result.

Corollary 1.4. *Let*

$$\max\{q_{\text{low}}, q_{\text{up}}\} < 1. \tag{1.3}$$

Then $\sigma(A)$ *is located in the set*

$$\cup_{k=1}^{n} U(a_{kk}; \psi_0(k)), \ \ where\ \psi_0(k) := \min\{\psi_{\text{low}}(k),\ \psi_{\text{up}}(k)\}.$$

In Corollary 1.2 we can replace $\psi_{\text{up}}(k)$ by $\psi_{\text{low}}(k)$, if instead of (1.1) condition (1.2) holds, and by $\psi_0(k)$, if condition (1.3) holds.

2. Proof of Theorem 1.1

Let A_+ be the upper triangular part of A, i.e. $A_+ = (a_{jk}^+)_{j,k=1}^n$, where $a_{jk}^+ = a_{jk}$ for $j \leq k$ and $a_{jk}^+ = 0$ for $j > k$. Clearly,

$$\det(A_+) = \prod_{j=1}^n a_{jj}.$$

Put

$$t_k^+ := (\sum_{j=1}^n |a_{jk} + a_{jk}^+|^2)^{1/2} \quad (k = 1, ..., n)$$

and

$$t_k^- := (\sum_{j=k+1}^n |a_{jk}|^2)^{1/2} \ (k = 1, ..., n-1), t_n = 0.$$

In this section for the brevity put $q_{\text{up}} = q$. We need the following result proved in [4, Corollary 3.2].

Corollary 2.1. *One has*

$$|\det A - \prod_{j=1}^n a_{jj}| \leq \delta(A),$$

where

$$\delta(A) := q \prod_{k=1}^n \left(1 + \frac{1}{2}(t_k^- + t_k^+)\right).$$

Take into account that

$$(t_k^+)^2 = 2|a_{kk}|^2 + 2\sum_{j=1}^{k-1} |a_{jk}|^2 + (t_k^-)^2.$$

Hence, due to the inequality $(c_1 + c_2)^2 \leq 2(c_1^2 + c_2^2)$ $(c_1, c_2 > 0)$, we get

$$(t_k^+ + t_k^-)^2 \leq 2(2|a_{kk}|^2 + 2\sum_{j=1}^{k-1} |a_{jk}|^2 + (t_k^-)^2) + 2(t_k^-)^2 = 4(|a_{kk}|^2 + \sum_{j=1}^{k-1} |a_{jk}|^2 + (t_k^-)^2)$$

$$= 4(|a_{kk}|^2 + \sum_{j=1, j\neq k}^n |a_{jk}|^2) \leq 4(|a_{kk}| + [\sum_{j=1, j\neq k}^n |a_{jk}|^2]^{1/2})^2 \quad (k = 1, ..., n).$$

Here $\sum_{j=1}^0 = 0$. Now Corollary 2.1 implies the inequality

$$|\det A - \prod_{j=1}^n a_{jj}| \leq q \prod_{k=1}^n (|a_{kk}| + M_k).$$

If

$$\prod_{j=1}^n |a_{jj}| > q \prod_{k=1}^n (|a_{kk}| + M_k), \tag{2.1}$$

then $\det(A) \neq 0$, i.e. A is invertible. Assume that

$$|a_{kk}| > \sqrt[n]{q}(|a_{kk}| + M_k) \tag{2.2}$$

for all $k = 1, ..., n$. Then (2.1) holds and therefore A is invertible.

Let condition (1.1) hold. Then (2.2) is equivalent to the inequality

$$|a_{kk}| > \frac{\sqrt[n]{q_{\text{up}}} M_k}{1 - \sqrt[n]{q_{\text{up}}}} = \psi_{\text{up}}(k). \tag{2.3}$$

Hence we arrive at the following result

Lemma 2.2. *Matrix A is invertible, provided conditions (1.1) and (2.3) hold for all $k = 1, ..., n$.*

Proof of Theorem 1.1: For a $z \in \mathbb{C}$, let $|a_{jj} - z| > \psi_{\text{up}}(k)$ for all $j = 1, 2, ..., n$. Then by Lemma 2.2 $A - zI$ is invertible, where I is the unit matrix. So for any eigenvalue μ of A, there is at least one index $m \leq n$, such that $|a_{mm} - \mu| \leq \psi_{\text{up}}(m)$. This proves the theorem. □

3. Example

Let

$$A = \begin{pmatrix} 2 & 6 & 3 \\ 0 & 5 & 4 \\ 0.008 & 0 & 7 \end{pmatrix}.$$

Then $q_{\text{up}} = 0.008$. So condition (1.1) holds. Besides, $q_{\text{low}} > 1$, $M_1 = 1.008, M_2 = 7, M_3 = 6$. On the other hand $R_1 = 0.008, R_2 = 6, R_3 = 7, Q_1 = 9, Q_2 = 4, Q_3 = 0.008$. Simple calculations show that $\min\{R_1, Q_1\} = 0.008 < \psi_{\text{up}}(1)$ and $\min\{R_3, Q_3\} = 0.008 < \psi_{\text{up}}(3)$, but $\min\{R_2, Q_2\} = 4 > \psi_{\text{up}}(2) = 1.75$. Due to Corollary 1.2, the following the discs contains the eigenvalues: $U(2; 0.008), U(5; 1.75)$ and $U(7; 0.008)$. So in the considered case Corollary 1.2 improves the Ostrowski theorem.

Acknowledgment. I am very grateful to the referee of this paper for his (her) helpful remarks.

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