Avrupa Bilim ve Teknoloji Dergisi
Özel Sayı 28, S. 241-245, Kasım 2021

Araştırma Makalesi

www.ejosat.com ISSN:2148-2683

European Journal of Science and Technology
Special Issue 28, pp. 241-245, November 2021

Research Article

Computing Gauss-Bonnet Theorem of σ-Bezier Curves

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(1st International Conference on Applied Engineering and Natural Sciences ICAENS 2021, November 1-3, 2021)

(**DOI:** 10.31590/ejosat.996339)

ATIF/REFERENCE: Ertem Kaya, F. (2021). Computing Gauss-Bonnet Theorem of σ-Bezier Curves. *European Journal of Science and Technology*, (28), 241-245.

Abstract

The energy is one of the most important source for people and the other living things. We know that the energy of sun is the most energy source in universe. The sun is the source of the life. When the science and technology began to progressed, people found a lot of types of energy, but most of them are from the nature. For example energy of sun is naturel but energy of electric is produced by sun and using a lot of other nature sources (wind, water,…etc). The amount and capasity of the energy of the electric, water, wind,…etc. are calculated. For calculating energy we have to use mathematics and physics. Geometricians calculated the functions and energies of the curves by using Willmore and Horn' method lately. In this paper the Wilmore function of the σ-Bezier curves in H^3 and Gauss-Bonnet theorem of the σ-Bezier curves are calculated by using Willmore method.

Keywords: Bezier curves, curvature, torsion, Gauss curvature, Mean Curvature, Willmore energy.

σ-Bezier Eğrilerinin Gauss-Bonnet Teoreminin Hesaplanması

Öz

Enerji, insanlar ve diğer canlı nesneler için en önemli kaynaklardan biridir. Güneş enerjisinin kainattaki en önemli enerji kaynağı olduğunu biliyoruz. Güneş yaşamın kaynağıdır. Fen ve teknoloji gelişmeye başladığında, insanlar pek çok çeşit enerji buldular, ancak bunların pek çoğu doğadan olanlardı. Örneğin Güneş enerjisi doğaldır fakat elektrik enerjisi güneşten veya başka doğal kaynaklardan (rüzgar, su,…vb.) üretilmektedir. Elektrik enerjisi, su, rüzgar,…vb. kapasite miktarları hesaplanabilmektedir. Enerjiyi hesaplamak için matematik ve fizik kullanmak gereklidir. Son dönemlerde geometriciler eğrilerin Willmore ve Horn metoduyla fonksiyon ve enerjilerini hesaplamışlardır. Bu makalede σ-Bezier eğrilerinin H^3' te Willmore fonksiyonu ve Willmore metodu kullanılarak Gauss-Bonnet teoremi hesaplanmıştır.

Anahtar Kelimeler: Bezier eğrileri, eğrilik, torsiyon, Willmore enerji

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1. Introduction

In Differential Geometry Horn found the curve which passes through two specified points with specified orientation while minimizing

$$\varepsilon = \int \kappa^2 ds$$

where κ is the curvature and s the arc distance, has a number of interesting applications in 1983. He applied and introduced the energy with Differential Geometry at first. [Willmore,1965;Yan, 2011].

What makes the Bezier curves so popular and why we use bezier curves?

The mathematical Bezier curves as known Bernstein Polynommial has studied since 1912 but discovered in 1960 by French engineer Pierre Bezier, especially use in automobile design.

The aim of this study is to observe the energy of the cubic σ-Bezier curves by Willmore method.

Now we repeat preliminaries on geometry of cubic σ-Bezier curves in E^3.

2. Preliminaries

Bezier curve is known as a curve $Q(t)$ that called Berstein polynomials is given by:

$$Q(t) = \sum_{i=0}^{l} P_i^l(t) Q_i$$

where $P_i^l(t)$ is a basis function for Bezier curve Q_i refers to the control points of the curve and they constitute B-spline curve. The function of the $P_i^l(t)$ can be defined as the following:

$$P_i^l(t) = \frac{l!}{(l-1)!i!}(1-t)^{l-i} t^i, \quad i = 0,1,2,...,n$$

The curve can be expressed as any degree l with $l+1$ control points [Erkan and Yüce,2018-…- Hertrich-Jeromin, 2003 (all references)].

2.1. Serret-Frenet Elemets of Bezier Curves in E^3

Serret-Frenet elements of Bezier curves $\{T, N, B, \kappa, \tau\}$ are found firstly by Samanci [Samanci, 2015], after Erkan and Yüce [Erkan and Yüce, 2018] for Bezier curves as follows:

$$T(t) = \frac{m-2}{m} \frac{\sum_{i=0}^{m-1} B_{i,m-1}(t)\Delta b_i}{(\sum_{i,j=0}^{m-1} B_{i,m-1}(t) B_{j,m-1}(t) \langle \Delta b_i, \Delta b_j \rangle)^{\frac{1}{2}}} \sin\alpha$$

$$N(t) = \frac{\sum_{i=0}^{m-1} B_{i,m-1}(t) J\Delta b_i}{(\sum_{i,j=0}^{m-1} B_{i,m-1}(t) B_{j,m-1}(t) \langle \Delta b_i, \Delta b_j \rangle)^{\frac{1}{2}}}$$

$$\kappa(t) = \frac{m-1}{m} \frac{\sum_{i=0}^{m-2}\sum_{j=0}^{m-1} B_{i,m-2}(t) B_{j,m-1}(t) \langle \Delta^2 b_i, J\Delta b_j \rangle}{(\sum_{i,j=0}^{m-1} B_{i,m-1}(t) B_{j,m-1}(t) \langle \Delta b_i, \Delta b_j \rangle)^{\frac{3}{2}}}$$

2.2. The Definition Cubic of σ-Bezier Curves in E^3

Definition 2.2.1. Let $\sigma \in [-1,1]$: for $s \in [0,1]$, the following polynomial functions [Hu, Ji, Qui and Zhang, 2015],

$$p_{0,2}(s;\sigma) = (1 - 2\sigma s - \sigma s^2)(1 - s^2)$$
$$p_{1,2}(s;\sigma) = 2s(1-s)(1 + \sigma - \sigma s + \sigma s^2)$$
$$p_{2,2}(s;\sigma) = (1 - \sigma s - \sigma s^2)s^2$$

are called the extensions Berstein basis functions of degree 2 associated with the shape parameter σ [Hu, Ji, Qui and Zhang, 2015]; for any integer $m(m \geq 3)$, the functions $p_{i,m}(s;\sigma) = (i = 0,1,2,...,m)$ defined as

$$p_{i,m}(s;\sigma) = (1-s) p_{i,m-1},(s,\sigma) + s p_{i-1,m-1}(s,\sigma)$$
$$s \in [0,1]$$

These are extension Berstein basis functions of degree n [Hu, Ji, Qui and Zhang, 2015].

2.3. Construction of σ-Bezier Curves

Given control points $P_i (i = 0,1,...,r; r \geq 2)$ in R^2 or R^3 then,

$$p(s;\sigma) = \sum_{i=0}^{m} P_i b_{i,m}(s;m), \; s \in [0,1], \; \sigma \in [-1,1]$$

is called σ-Bezier curve of degree n with shape parameter, where basis functions $b_{i,m}(s;\sigma) = (i = 0,1,2,...,m; m \geq 2)$ in [Hu, Ji, Qui and Zhang, 2015] and [Yan and Liang, 2011].

In [Hu, Ji, Qui and Zhang, 2015] contained σ-Bezier curve of degree n with control points $P_0, P_1,..., P_m (m \geq 2)$ is

$$p_1(s;\sigma_1) = \sum_{i=0}^{m} P_i b_{i,m}(s;\sigma_1), \; s \in [0,1]$$

$$p_2(s;\sigma_1) = \sum_{i=0}^{i-1} P_i b_{i,m}(s;\sigma_1) + \sum_{i=1}^{k} (P_i + \delta_i) b_{i,m}(s;\sigma_1) + \sum_{i=k+1}^{m} P_i b_{i,m}(s;\sigma_1)$$

$$p_3(s;\sigma_2)=\sum_{i=0}^{i-1}P_i b_{i,r}(s;\sigma_2)+\sum_{i=1}^{k}(P_i+\delta_i)b_{i,r}(s;\sigma_2)$$
$$+\sum_{i=k+1}^{m}P_i b_{i,m}(s;\sigma_2)$$

and here $\delta_i=(\delta_i^x\delta_i^y\delta_i^z)^T$ $i=(l,l+1,...,m)$ are pesturbations of control points $P_i(i=l,l+1,...,m)$ [Hu, Ji, Qui and Zhang, 2015].

2.4. Curvature κ and torsion τ of the σ-Bezier Curves

Let κ and τ denote principal curvatures of σ-Bezier curve are

$$\kappa(s,\sigma)=\frac{m-1}{m}\frac{\sum_{i=0}^{m-2}\sum_{j=0}^{m-1}B_{i,m-2}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle\Delta^2 b_i(s,\sigma),J\Delta b_j(s,\sigma)\right\rangle}{(\sum_{i,j=0}^{m-1}B_{i,m-1}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle\Delta b_i(s,\sigma),\Delta b_j(s,\sigma)\right\rangle)^{\frac{3}{2}}}$$

and

$$\tau(s,\sigma)=\frac{m-2}{m}\frac{\sum_{i=0}^{m-1}\sum_{j=0}^{m-2}\sum_{k=0}^{m-3}B_{i,m-1}(s,\sigma)B_{j,m-2}(s,\sigma)B_{k,m-3}(s,\sigma)\left\langle\Delta b_i(s,\sigma)\times\Delta^2 b_j(s,\sigma),\Delta^3 b_k(s,\sigma)\right\rangle}{\left\|\sum_{i=0}^{m-1}\sum_{j=0}^{m-2}B_{i,m-1}(s,\sigma)B_{j,m-2}(s,\sigma)\left\langle\Delta b_i(s,\sigma),\Delta^2 b_j(s,\sigma)\right\rangle\right\|^2}.$$

3. Computing the Energy of σ-Bezier Curves by Using Horn's method

Let calculate the energy of σ-Bezier curve by using Horn's Method [Horn, 2013]. So we take,

$$\varepsilon=\int\kappa^2 ds$$

$$=\int\left(\frac{m-1}{m}\frac{\sum_{i=0}^{m-2}\sum_{j=0}^{m-1}B_{i,m-2}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle\Delta^2 b_i(s,\sigma),J\Delta b_j(s,\sigma)\right\rangle}{(\sum_{i,j=0}^{m-1}B_{i,m-1}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle\Delta b_i(s,\sigma),\Delta b_j(s,\sigma)\right\rangle)^{\frac{3}{2}}}\right)^2 ds$$

$$=\frac{(m-1)^2}{m}\int\frac{\left(\sum_{i=0}^{m-2}\sum_{j=0}^{m-1}B_{i,m-2}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle\Delta^2 b_i(s,\sigma),J\Delta b_j(s,\sigma)\right\rangle\right)^2}{(\sum_{i,j=0}^{m-1}B_{i,m-1}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle\Delta b_i(s,\sigma),\Delta b_j(s,\sigma)\right\rangle)^3}ds$$

4. Mean and Gauss Curvatures of the Energy of σ-Bezier Curves

Let $H=\frac{\kappa+\tau}{2}$ and $K=\kappa*\tau$ denote the Mean and Gauss curvatures of σ-Bezier curves;

$$H=\frac{\kappa+\tau}{2}$$

$$=\frac{1}{2}\frac{m-1}{m}\frac{\sum_{i=0}^{m-2}\sum_{j=0}^{m-1}B_{i,m-2}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle\Delta^2 b_i(s,\sigma),J\Delta b_j(s,\sigma)\right\rangle}{(\sum_{i,j=0}^{m-1}B_{i,m-1}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle\Delta b_i(s,\sigma),\Delta b_j(s,\sigma)\right\rangle)^{\frac{3}{2}}}$$

$$+\frac{1}{2}\frac{m-2}{m}\frac{\sum_{i=0}^{m-1}\sum_{j=0}^{m-2}\sum_{k=0}^{m-3}B_{i,m-1}(s,\sigma)B_{j,m-2}(s,\sigma)B_{k,m-3}(s,\sigma)\left\langle\Delta b_i(s,\sigma)\times\Delta^2 b_j(s,\sigma),\Delta^3 b_k(s,\sigma)\right\rangle}{\left\|\sum_{i=0}^{m-1}\sum_{j=0}^{m-2}B_{i,m-1}(s,\sigma)B_{j,m-2}(s,\sigma)\left\langle\Delta b_i(s,\sigma),\Delta^2 b_j(s,\sigma)\right\rangle\right\|^2}$$

and Gauss curvature is;

$$
\mathrm{K} = \kappa * \tau
$$

$$
= \frac{(m-1)(m-2)}{m^2} \frac{\sum_{i=0}^{m-2}\sum_{j=0}^{m-1} B_{i,m-2}(s,\sigma) B_{j,m-1}(s,\sigma) \left\langle \Delta^2 b_i(s,\sigma), J\Delta b_j(s,\sigma) \right\rangle}{\left(\sum_{i,j=0}^{m-1} B_{i,m-1}(s,\sigma) B_{j,m-1}(s,\sigma) \left\langle \Delta b_i(s,\sigma), \Delta b_j(s,\sigma) \right\rangle\right)^{\frac{3}{2}}}
$$

$$
\sum_{i=0}^{m-1}\sum_{j=0}^{m-2}\sum_{k=0}^{m-3} B_{i,m-1}(s,\sigma) B_{j,m-2}(s,\sigma) B_{k,m-3}(s,\sigma) * \frac{\left\langle \Delta b_i(s,\sigma) \times \Delta^2 b_j(s,\sigma), \Delta^3 b_k(s,\sigma) \right\rangle}{\left\| \sum_{i=0}^{m-1}\sum_{j=0}^{m-2} B_{i,m-1}(s,\sigma) B_{j,m-2}(s,\sigma) \left\langle \Delta b_i(s,\sigma), \Delta^2 b_j(s,\sigma) \right\rangle \right\|^2}
$$

In above the operator $*$ is multiplication operator.

5. Willmore Function on Curvatures of σ-Bezier Curves

Let $h : N^2 \to R^3$ be a compact surface immersed in R^3 [Wilmore, 1959, Ertem Kaya, in press, Hertrich-Jeromin, 2003]. Let κ and τ denote principal curvatures of h, Mean curvature $\mathrm{H} = \frac{\kappa + \tau}{2}$ and Gauss curvature $\mathrm{K} = \kappa * \tau$ of h [Wilmore, 1959; Hsu, Husne, Sullivan, 1992; Ertem Kaya, in press; Hertrich-Jeromin, 2003].

In 1965 Willmore [Wilmore, 1959, Willmore, 1965, Horn, 2013, Ertem Kaya, in press, Hertrich-Jeromin, 2003] proposed the study of the functional. So it can be written $\tau(h)$ by generated on σ-Bezier curve. We obtain,

$$
\tau(h) = \frac{1}{2}\int_{N^2} \left(\frac{m-1}{m} \frac{\sum_{i=0}^{m-2}\sum_{j=0}^{m-1} B_{i,m-2}(s,\sigma) B_{j,m-1}(s,\sigma) \left\langle \Delta^2 b_i(s,\sigma), J\Delta b_j(s,\sigma) \right\rangle}{\left(\sum_{i,j=0}^{m-1} B_{i,m-1}(s,\sigma) B_{j,m-1}(s,\sigma) \left\langle \Delta b_i(s,\sigma), \Delta b_j(s,\sigma) \right\rangle\right)^{\frac{3}{2}}} \sum_{i=0}^{m-1}\sum_{j=0}^{m-2}\sum_{k=0}^{m-3} B_{i,m-1}(s,\sigma) B_{j,m-2}(s,\sigma) B_{k,m-3}(s,\sigma) + \frac{m-2}{m} \frac{\left\langle \Delta b_i(s,\sigma) \times \Delta^2 b_j(s,\sigma), \Delta^3 b_k(s,\sigma) \right\rangle}{\left\| \sum_{i=0}^{m-1}\sum_{j=0}^{m-2} B_{i,m-1}(s,\sigma) B_{j,m-2}(s,\sigma) \left\langle \Delta b_i(s,\sigma), \Delta^2 b_j(s,\sigma) \right\rangle \right\|^2} \right)^2 dA
$$

where dA is the area form on $\tau(h)$ induced by the immersion h [Willmore, 1959; Willmore 1965;Horn, 2013]. It is so-called Willmore functional of the σ-Bezier curves as follows;

Thus the Gauss-Bonnet Theorem states that,

$$
\int_{N^2} \left(\frac{m-1}{m} \frac{\sum_{i=0}^{m-2}\sum_{j=0}^{m-1} B_{i,m-2}(s,\sigma) B_{j,m-1}(s,\sigma) \left\langle \Delta^2 b_i(s,\sigma), J\Delta b_j(s,\sigma) \right\rangle}{\left(\sum_{i,j=0}^{m-1} B_{i,m-1}(s,\sigma) B_{j,m-1}(s,\sigma) \left\langle \Delta b_i(s,\sigma), \Delta b_j(s,\sigma) \right\rangle\right)^{\frac{3}{2}}} * \frac{\sum_{i=0}^{m-1}\sum_{j=0}^{m-2}\sum_{k=0}^{m-3} B_{i,m-1}(s,\sigma) B_{j,m-2}(s,\sigma) B_{k,m-3}(s,\sigma) * \left\langle \Delta b_i(s,\sigma) \times \Delta^2 b_j(s,\sigma), \Delta^3 b_k(s,\sigma) \right\rangle}{\left\| \sum_{i=0}^{m-1}\sum_{j=0}^{m-2} B_{i,m-1}(s,\sigma) B_{j,m-2}(s,\sigma) \left\langle \Delta b_i(s,\sigma), \Delta^2 b_j(s,\sigma) \right\rangle \right\|^2} \right)^2 dA
$$

Note that the operator $*$ multiplication operator.

So this formulae is the Gauss-Bonnet theorem of σ-Bezier curve.

6. Results and Discussion

In this paper we see that computing Gauss-Bonnet theorem for σ-Bezier curves is connected to the curvatures and control points. If the control points' value change, then the value of the Gauss-Bonnet is change. So it is not invariant.

$$W(h)=\frac{1}{2}\int_{N^2}\mathrm{H}^2 dA$$

$$=\frac{1}{2}\int_{N^2}\left(\frac{1}{2}\frac{m-1}{m}\frac{\sum_{i=0}^{m-2}\sum_{j=0}^{m-1}B_{i,m-2}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle \Delta^2 b_i(s,\sigma), J\Delta b_j(s,\sigma)\right\rangle}{\left(\sum_{i,j=0}^{m-1}B_{i,m-1}(s,\sigma)B_{j,m-1}(s,\sigma)\left\langle \Delta b_i(s,\sigma),\Delta b_j(s,\sigma)\right\rangle\right)^{\frac{3}{2}}} + \frac{1}{2}\frac{m-2}{m}\frac{\sum_{i=0}^{m-1}\sum_{j=0}^{m-2}\sum_{k=0}^{m-3}B_{i,m-1}(s,\sigma)B_{j,m-2}(s,\sigma)B_{k,m-3}(s,\sigma)\left\langle \Delta b_i(s,\sigma)\times\Delta^2 b_j(s,\sigma),\Delta^3 b_k(s,\sigma)\right\rangle}{\left\|\sum_{i=0}^{m-1}\sum_{j=0}^{m-2}B_{i,m-1}(s,\sigma)B_{j,m-2}(s,\sigma)\left\langle \Delta b_i(s,\sigma),\Delta^2 b_j(s,\sigma)\right\rangle\right\|^2}\right)^2 dA$$

7. Conclusions and Recommendations

In this work, the Willmore energy and Gauss-Bonnet theorem of the Bezier curves in E^3 or H^3 are computed.

8. Acknowledge

2020 Mathematics Subject Classification is: 53A04, 53A05.

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