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Research Article

Two results on double crossed biproducts

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Abstract

Let H be an algebra with a distinguished element $\varepsilon_H \in H^*$ and C, D two coalgebras. Based on the construction of Brzeziński's crossed coproduct, under some suitable conditions, we introduce a coassociative coalgebra $C \times^G_T H^\beta_R \times D$ which is a more general two-sided coproduct structure including two-sided smash coproduct. Necessary and sufficient conditions for $C \times^G_T H^\beta_R \times D$ equipped with two-sided tensor product algebra $C \otimes H \otimes D$ to be a bialgebra (Hopf algebra) are provided. On the other hand, we obtain an improved version of the double crossed biproduct $C \star^\alpha H^\beta \star D$ in [An extended form of Majid's double biproduct, J. Algebra Appl. **16** (4), 1760061, 2017] which induces a description of $C \star^\alpha H^\beta \star D$ similar to Majid double biproduct $C \star H \star D$ and also present some related structures.

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1. Introduction and preliminary

Let H be a commutative Hopf algebra, C a cocommutative left H-comodule coalgebra and $\alpha : C \longrightarrow H \otimes H$ a normalized 2-cocycle. Then a coassicitative coalgebra $C \times^\alpha H$ named crossed coproduct was introduced by Lin [4] to develop the cohomology theory. Here we mainly consider Lin's crossed coproduct coalgebra $C \times^\alpha H$ for the preconditions that H is a Hopf algebra with a coaction on the coalgebra C through a linear map α (see Example 1.3). More generally, if H is a vector space with a distinguished element $\varepsilon_H \in H^*$ and C is a coalgebra, then Brzeziński provided a coassociative coalgebra $C \times^G_T H$ (here we call it left Brzeziński crossed coproduct) under some conditions [1]. For the research of crossed coproduct, see [2, 3, 5, 8, 11, 14].

Radford biproduct [12] is one of the important objects in the theory of Hopf algebras, which is also related to Rota-Baxter bialgebras introduced by Ma and Liu in [6]. Majid realized a categorical interpretation of Radford's biproduct [10]: $C \star H$ is a (left) Radford biproduct if and only if C is a bialgebra in the Yetter-Drinfel'd category ${}^H_H\mathbb{YD}$. In [14], Wang, Wang and Yao found the necessary and sufficient conditions for crossed coproduct coalgebra $C \times^\alpha H$ and smash product algebra $C\#H$ to become a bialgebra generalizing $C \star H$ for the case that α is trivial.

Let $C \star H$ be a left Radford biproduct and $H \star D$ a right Radford biproduct. Then the sufficient condition Eq.(3.28) for a two-sided smash product algebra $C\#H\#D$ and a

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two-sided smash coproduct coalgebra $C \times H \times D$ to form a bialgebra, named the Majid double biproduct and denoted by $C \star H \star D$, was given by Majid [10]. Double-bosonization (more general Majid double biproduct) construction provides a canonical way of building up quantum groups from smaller ones by repeatedly extending their positive and negative root spaces by linear braided groups [10].

This paper explores the structure related to Majid double biproduct from the following two aspects:

(1) If we substitute $C \times H$ and at the same time $H \times D$ in $C \times H \times D$ for left Brzeźnski crossed coproduct $C \times_T^G H$ and $H_R^\beta \times D$ (a special case of right Brzeźnski crossed coproduct, see below), respectively, then when the new two-sided coproduct structure is a coassociative coalgebra? When this coalgebra equipped two-sided tensor product algebra $C \otimes H \otimes D$ turns to be a bialgebra (Hopf algebra)?

(2) When substituting left smash coproduct $C \times H$ and right smash coproduct $H \times D$ in $C \star H \star D$ for left crossed coproduct $C \times^\alpha H$ and right crossed coproduct $H^\beta \times D$, respectively, the necessary and sufficient conditions for two-sided smash product algebra $C\#H\#D$ and two-sided crossed coproduct coalgebra $C \times {}^\alpha H^\beta \times D$ to be a bialgebra were given in [8] which extends Majid double biproduct $C \star H \star D$ when α and β are trivial. But unfortunately, one of the conditions, i.e., Eq.(3.13), is very complicated. Is there a good substitute for Eq.(3.13)? And whether the beautiful symmetric condition Eq.(3.28) in Majid double biproduct should be included?

In this paper, we discuss the above questions. The outline of the paper is as follows.

In Section 2, we firstly give the right version of Brzeziński crossed coproduct $H_R^F \times D$ (Proposition 2.1). Let H be an algebra with a distinguished elment $\varepsilon_H \in H^*$ such that $\varepsilon_H(xy) = \varepsilon_H(x)\varepsilon_H(y)$ and C, D two coalgerbras. Assume that $G : C \otimes H \longrightarrow H \otimes H$, $T : C \otimes H \longrightarrow H \otimes C$, $R : H \otimes D \longrightarrow D \otimes H$, $\beta : D \longrightarrow H \otimes H$ are linear maps satisfying some additional conditions. Then we get the necessary and sufficient conditions for $C \times_T^G H_R^\beta \times D$ to be a coassociative coalgebra with counit $\varepsilon_C \otimes \varepsilon_H \otimes \varepsilon_D$ and comultiplication given by Eq.(2.10) (see Theorem 2.4). This coalgebra contains the two-sided crossed coproduct $C \times {}^\alpha H^\beta \times D$ ([8, Lemma 2.1]) and the two-sided twisted tensor coproduct $C \times_T H_R \times D$ (of course two-sided smash coproduct $C \times H \times D$ [10]) as special cases. Also we obtain the necessary and sufficient conditions for $C \times_T^G H_R^\beta \times D$ equipped with the two-sided tensor product algebra $C \otimes H \otimes D$ to become a bialgebra (see Theorem 2.9), we call this bialgebra Brzeziński's two-sided crossed coproduct bialgebra and denote it by $C \diamond_T^G H_R^\beta \diamond D$. We also present the antipode for $C \diamond_T^G H_R^\beta \diamond D$ (see Proposition 2.12).

In Section 3, similar to Majid double biproduct, we have: Under the assumption of Lemma 3.1. Suppose that $C \star^\alpha H$ is a left crossed biprodcut and $H^\beta \star D$ is a right crossed biproduct. Then the two-sided crossed coproduct $C \times^\alpha H^\beta \times D$ equipped with the two-sided smash product $C\#H\#D$ becomes a bialgebra (denoted by $C \star {}^\alpha H^\beta \star D$) if and only if Eqs.(3.26)-(3.29) hold (see Proposition 3.6). We note here that based on the proof of Theorem 3.2 Eqs.(3.26)-(3.29) are the simplified decomposition of Eq.(3.13) in [8] and Eq.(3.28) in [10] is one of the necessary conditions. And when α and β are trivial, $C \star {}^\alpha H^\beta \star D$ is exactly Majid double biproduct $C \star H \star D$. At last, we list some special cases.

Throughout this paper, we follow the definitions and terminologies in [12], with all algebraic systems supposed to be over the field K. Now, let C be a coalgebra. We use Sweedler's notation for the comultiplication: $\Delta(c) = c_1 \otimes c_2$ for any $c \in C$. Denote the category of left H-comodules by ${}^H\mathrm{Mod}$. For $(M, \varphi) \in {}^H\mathrm{Mod}$, write: $\varphi(x) = x_{-1} \otimes x_0 \in H \otimes M$, for all $x \in M$. Denote the category of right H-comodules by Mod^H. For $(M, \psi) \in \mathrm{Mod}^H$, write: $\psi(x) = x_{(0)} \otimes x_{(1)} \in M \otimes H$, for all $x \in M$. We denote the left-left Yetter-Drinfel'd category by ${}_H^H\mathbb{YD}$, and the right-right Yetter-Drinfel'd category by $\mathbb{YD}_H^H$. Given a K-space M, we write id_M for the identity map on M.

Let us recall from [1] the left Brzeziński's crossed coproduct as follows. Here we slightly modify the description of this general crossed coproduct.

Proposition 1.1. (Left Brzeziński's crossed coproduct) *Let C be a coalgebra, H a vector space and a distinguished element $\varepsilon_H \in H^*$. Let $G : C \otimes H \longrightarrow H \otimes H$ (write $G(c\otimes x) = c_G \otimes x^G$) and $T : C \otimes H \longrightarrow H \otimes C$ (write $T(c \otimes x) = x_T \otimes c_T$) be linear maps such that*

$$\varepsilon_H(x_T)c_T = \varepsilon_H(x)c, \; x_T\varepsilon(c_T) = x\varepsilon(c). \tag{1.1}$$

Then $C \times^G_T H$ ($= C \otimes H$ as vector space) is a coalgebra with counit $\varepsilon_C \otimes \varepsilon_H$ and a coproduct

$$\Delta(c \otimes x) = c_1 \otimes c_{3GT} \otimes c_{2T} \otimes x^G$$

if and only if the following conditions hold:

$$x_{tT} \otimes c_{1T} \otimes c_{2t} = x_T \otimes c_{T1} \otimes c_{T2}; \tag{1.2}$$

$$c_G\varepsilon_H(x^G) = x^G\varepsilon(c^G) = x\varepsilon(c); \tag{1.3}$$

$$c_{1G} \otimes {c_{2g}}^G \otimes x^g = c_{2gT} \otimes c_{1TG} \otimes x^{gG}; \tag{1.4}$$

$$c_{2GT} \otimes {x^G}_T \otimes c_{1Tt} = c_{1G} \otimes {x_T}^G \otimes c_{2T}. \tag{1.5}$$

for all $c \in C, x \in H$ and $g = G, t = T$.

Remark 1.2. When Eq.(1.1) and Eq.(1.3) are interchanged, Proposition 1.1 still holds.

In what follows, we list two special cases of left Brzeziński's crossed coproduct $C \times^G_T H$, although it has other very broad examples.

Let H be a Hopf algebra and C a coalgebra. By a **left weak coaction of H on C**, we mean a linear map $\varphi : C \to H \otimes C$ such that, for $c \in C$,

$$\begin{aligned} &c_{-1} \otimes c_{01} \otimes c_{02} = c_{1-1}c_{2-1} \otimes c_{10} \otimes c_{20}, \\ &c_{-1}\varepsilon_C(c_0) = \varepsilon_C(c)1_H \;\text{ and }\; \varepsilon_H(c_{-1})c_0 = c. \end{aligned} \tag{1.6}$$

By a **left coaction of H on C** we mean a left weak coaction such that

$$c_{-11} \otimes c_{-12} \otimes c_0 = c_{-1} \otimes c_{0-1} \otimes c_{00}$$

holds for all $c \in C$.

Example 1.3. (Left crossed coproduct)[8] Let H be a Hopf algebra with a left weak coaction φ on the coalgebra C and $\alpha : C \longrightarrow H \otimes H$ a linear map (write $\alpha(c) = \alpha(c)^1 \otimes \alpha(c)^2$). Then $C \times^\alpha H$(=$C \otimes H$ as vector space) is a coassociative coalgebra with counit $\varepsilon_C \otimes \varepsilon_H$ and comultiplication

$$\Delta(c \otimes x) = c_1 \otimes c_{2-1}\alpha(c_3)^1x_1 \otimes c_{20} \otimes \alpha(c_3)^2x_2.$$

for all $c \in C$, $x \in H$ if and only if α satisfies the following conditions ($\forall c \in C$):

$$\varepsilon_H(\alpha(c)^1)\alpha(c)^2 = \varepsilon_C(c)1_H = \alpha(c)^1\varepsilon_H(\alpha(c)^2); \tag{1.7}$$

$$\begin{aligned} &c_{1-1}\alpha(c_2)^1 \otimes \alpha(c_{10})^1{\alpha(c_2)^2}_1 \otimes \alpha(c_{10})^2{\alpha(c_2)^2}_2 \\ &\quad = \alpha(c_1)^1{\alpha(c_2)^1}_1 \otimes \alpha(c_1)^2{\alpha(c_2)^1}_2 \otimes \alpha(c_2)^2; \end{aligned} \tag{1.8}$$

$$\begin{aligned} &c_{1-1}\alpha(c_2)^1 \otimes c_{10-1}\alpha(c_2)^2 \otimes c_{100} \\ &\quad = \alpha(c_1)^1c_{2-11} \otimes \alpha(c_1)^2c_{2-12} \otimes c_{20}. \end{aligned} \tag{1.9}$$

Proof. Let $T(c \otimes x) = c_{-1}x \otimes c_0,\; G(c \otimes x) = \alpha(c)^1x_1 \otimes \alpha(c)^2x_2$ in Proposition 1.1. □

Remark 1.4. (1) If H is commutative and C is cocommutative and φ is a coaction in Example 1.3, then Eq.(1.9) holds automatically.

(2) Eqs.(1.7) and (1.8) imply that α is a normalized 2-cocycle in the sense of Lin's in [4].

(3) By (1) and (2), we know that left crossed coproduct here is just Lin's crossed coproduct [4] when H is commutative and C is cocommutative.

Example 1.5. (Left twisted tensor coproduct)[7] Let H, C be two coalgebras and $T : C\otimes H \longrightarrow H\otimes C$ be a linear map. Then $C\times_T H$ is a coassociative coalgebra with counit $\varepsilon_C\otimes\varepsilon_H$ and comultiplication

$$\Delta(c\otimes x) = c_1\otimes x_{1T}\otimes c_{2T}\otimes x_2$$

for all $c\in C$, $x\in H$ if and only if Eqs.(1.1), (1.2) and the following condition hold:

$$x_{T1}\otimes x_{T2}\otimes c_T = x_{1T}\otimes x_{2t}\otimes c_{Tt}, \tag{1.10}$$

where $c\in C$, $x\in H$ and $T=t$.

Proof. Let G be trivial, i.e., $G(c\otimes x)=\varepsilon(c)x_1\otimes x_2$ in Proposition 1.1. □

We recall, from [9, 10], the construction of the so-called double biproduct.

Definition 1.6. (Majid double biproduct) Let $C\star H$ be a left Radford biproduct and $H\star D$ a right Radford biproduct with the left and right actions, left and right coactions are

$$H\otimes C\longrightarrow C,\quad x\otimes c\mapsto x\triangleright c,\quad C\longrightarrow H\otimes C,\quad c\mapsto c_{-1}\otimes c_0,$$
$$D\otimes H\longrightarrow D,\quad d\otimes x\mapsto d\triangleleft x,\quad D\longrightarrow D\otimes H,\quad d\mapsto d_{(0)}\otimes d_{(1)}$$

for all $x\in H$, $c\in C$, $d\in D$. Then the vector space $C\otimes H\otimes D$ becomes an algebra (called the two-sided smash product, $C\#H\#D$) with unit $1_C\otimes 1_H\otimes 1_D$ and multiplication

$$(c\otimes x\otimes d)(c'\otimes x'\otimes d') = c(x_1\triangleright c')\otimes x_2x'_1\otimes(d\triangleleft x'_2)d' \tag{1.11}$$

and a coalgebra (called the two-sided smash coproduct, $C\times H\times D$) with counit $\varepsilon(c\otimes x\otimes d) = \varepsilon_C(c)\varepsilon_H(x)\varepsilon_D(d)$ and comultiplication

$$\Delta(c\otimes x\otimes d) = c_1\otimes c_{2-1}x_1\otimes d_{1(0)}\otimes c_{20}\otimes x_2d_{1(1)}\otimes d_2. \tag{1.12}$$

Moreover, assume that Eq.(3.28) holds, it follows that $C\otimes H\otimes D$ with two-sided smash product algebra and two-sided smash product coalgebra is a bialgebra, called the **Majid double biproduct**, denoted by $C\star H\star D$.

Remark 1.7. Majid double biproduct here is actually the case of [10, Theorem A.1] with a trivial pairing.

2. Brzeziński's two-sided crossed coproduct

In this section, we will give a class of general two-sided crossed coproduct structure, named "Brzeziński's two-sided crossed coproduct". The necessary and sufficient conditions for two-sided tensor product algebra and Brzeziński's two-sided crossed coproduct coalgebra to become a bialgebra (Hopf algebra) are provided.

Proposition 2.1. (Right Brzeziński's crossed coproduct) *Let H be a vector space and a distinguished element $\varepsilon_H\in H^*$ and D a coalgebra. Let $F: H\otimes D\longrightarrow H\otimes H$ (write $F(x\otimes d) = x_F\otimes d^F$) and $R: H\otimes D\longrightarrow D\otimes H$ (write $R(x\otimes d) = d_R\otimes x_R$) be linear maps such that*

$$\varepsilon_D(d_R)x_R = \varepsilon_D(d)x,\ d_R\varepsilon_H(x_R) = d\varepsilon_H(x) \tag{2.1}$$

holds. Then $H^F_R\times D$ ($= H\otimes D$ as a vector space) is a coalgebra with counit $\varepsilon_H\otimes\varepsilon_D$ and comultiplication

$$\Delta(x\otimes d) = x_F\otimes d_{2R}\otimes {d_1}^F{}_R\otimes d_3$$

for all $d\in D, x\in H$ if and only if the following conditions hold ($r=R, f=F$):

$$d_{R1}\otimes d_{R2}\otimes x_R = d_{1R}\otimes d_{2r}\otimes x_{Rr}; \tag{2.2}$$
$$d^F\varepsilon_H(x^F) = x_F\varepsilon(d^F) = x\varepsilon(d); \tag{2.3}$$
$$x_{fF}\otimes {d_2r}^F\otimes {d_1}^f{}_r = x_f\otimes {d_1}^f{}_F\otimes {d_2}^F; \tag{2.4}$$

$$d_{2Rr}\otimes x_{Fr}\otimes {d_1}^F{}_R=d_{1R}\otimes x_{RF}\otimes {d_2}^F. \tag{2.5}$$

Example 2.2. (Right crossed coproduct) Let H be a Hopf algebra with a right weak coaction ψ on the coalgebra D. Let $\beta: D\longrightarrow H\otimes H$ be a linear map (write $\beta(d)=\beta(d)^1\otimes\beta(d)^2$). Then $H^\beta\times D(=H\otimes D$ as a vector space) is a coassociative coalgebra with counit $\varepsilon_H\otimes\varepsilon_D$ and comultiplication

$$\Delta(x\otimes d)=x_1\beta(d_1)^1\otimes d_{2(0)}\otimes x_2\beta(d_2)^2d_{2(1)}\otimes d_3$$

for all $d\in D$, $x\in H$ if and only if β satisfies the following conditions:

$$\varepsilon_H(\beta(d)^2)\beta(d)^1=\varepsilon_H(\beta(d)^1)\beta(d)^2=\varepsilon_D(d)1_d; \tag{2.6}$$

$$\begin{aligned}&\beta(d_1)^1{}_1\beta(d_{2(0)})^1\otimes\beta(d_1)^1{}_2\beta(d_{2(0)})^2\otimes\beta(d_1)^2d_{2(1)}\\ &\quad=\beta(d_1)^1\otimes\beta(d_1)^2{}_1\beta(d_2)^1\otimes\beta(d_1)^2{}_2\beta(d_2)^2;\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\beta(d_1)^1d_{2(0)(1)}\otimes\beta(d_1)^2d_{2(1)}\otimes d_{2(0)(0)}\\ &\quad=d_{1(1)1}\beta(d_2)^1\otimes d_{1(1)2}\beta(d_2)^2\otimes d_{1(0)}.\end{aligned} \tag{2.8}$$

Proof. Let $R(x\otimes d)=d_{(0)}\otimes xd_{(1)}$, $F(x\otimes d)=x_1\beta(d)^1\otimes x_2\beta(d)^2$ in Proposition 2.1. □

Example 2.3. (Right twisted tensor coproduct) Let H,D be two coalgebras and $R: H\otimes D\longrightarrow D\otimes H$ be a linear map. Then $H_R\times D(=H\otimes D$ as a vector space) with counit $\varepsilon_H\otimes\varepsilon_D$ and comultiplication

$$\Delta(x\otimes d)=x_1\otimes d_{1R}\otimes x_{2R}\otimes d_2$$

for all $d\in D$, $x\in H$, is a coassociative coalgebra if and only if Eqs.(2.1), (2.2) and the following condition hold ($d\in D$, $x\in H$ and $R=r$):

$$d_{Rr}\otimes x_{1R}\otimes x_{2r}=d_R\otimes x_{R1}\otimes x_{R2}. \tag{2.9}$$

Proof. Let F be trivial, i.e., $F(x\otimes d)=x_1\otimes x_2\varepsilon(d)$ in Proposition 2.1. □

Theorem 2.4. *Let H be an algebra and C,D two coalgebras and a distinguished elment $\varepsilon_H\in H^*$ such that $\varepsilon_H(xy)=\varepsilon_H(x)\varepsilon_H(y)$. Assume that $G: C\otimes H\longrightarrow H\otimes H$, $T: C\otimes H\longrightarrow H\otimes C$, $R: H\otimes D\longrightarrow D\otimes H$, $\beta: D\longrightarrow H\otimes H$ are linear maps such that Eqs.(1.1)-(1.3),(2.1),(2.2),(2.6) hold. Then $C\times{}^G_TH^\beta_R\times D$(=$C\otimes H\otimes D$ as vector space) is a coassociative coalgebra with counit $\varepsilon_C\otimes\varepsilon_H\otimes\varepsilon_D$ and comultiplication:*

$$\Delta(c\otimes x\otimes d)=c_1\otimes(c_{3G}\beta(d_1)^1)_T\otimes d_{2R}\otimes c_{2T}\otimes(x^G\beta(d_1)^2)_R\otimes d_3 \tag{2.10}$$

for all $c\in C$,$x\in H$, $d\in D$ if and only if the following conditions hold:

$$\begin{aligned}&c_{1G}\beta(d_{2R})^1\otimes(c_{2g}\beta(d_1)^1)^G\beta(d_{2R})^2\otimes(x^g\beta(d_1)^2)_R\\ &\quad=(c_{2g}\beta(d_1)^1)_T\otimes {c_{1T}}^G\beta(d_2)^1\otimes(x^g\beta(d_1)^2)_G\beta(d_2)^2;\end{aligned} \tag{2.11}$$

$$(c_{2G}\beta(d)^1)_T\otimes(x^G\beta(d)^2)_t\otimes c_{1Tt}=c_{1G}\beta(d)^1\otimes {x_T}^G\beta(d)^2\otimes c_{2T}; \tag{2.12}$$

$$d_{1R}\otimes c_G\beta(d_2)^1\otimes {x_R}^G\beta(d_2)^2=d_{2Rr}\otimes(c_G\beta(d_1)^1)_r\otimes(x_G\beta(d_1)^2)_R; \tag{2.13}$$

$$x_{RT}\otimes c_T\otimes d_R=x_{TR}\otimes c_T\otimes d_R. \tag{2.14}$$

Proof. We only check the coassociativity as follows, and the rest are direct. For all $c\in C,x\in H,d\in D$, we have

$$\begin{aligned}(\Delta\otimes\mathrm{id})\Delta(c\otimes x\otimes d)\ &=\ c_1\otimes(c_{3G}\beta(d_{2r1})^1)_T\otimes d_{2r2R}\otimes c_{2T}\otimes(c_{5g}\beta(d_1)^1{}_t{}^G\beta(d_{2r1})^2)_R\\ &\qquad\otimes d_{2r3}\otimes c_{4t}\otimes(x^g\beta(d_1)^2)_r\otimes d_3\\ &\overset{(2.2)}{=}\ c_1\otimes(c_{3G}\beta(d_{2r})^1)_T\otimes d_{3\bar{r}R}\otimes c_{2T}\otimes(c_{5g}\beta(d_1)^1{}_t{}^G\beta(d_{2r})^2)_R\\ &\qquad\otimes d_{4\overline{R}}\otimes c_{4t}\otimes(x^g\beta(d_1)^2)_{r\overline{R}\bar{r}}\otimes d_5\\ &\overset{(2.12)}{=}\ c_1\otimes(c_{4G}\beta(d_{2r})^1)_{\bar{t}\overline{T}}\otimes d_{3\bar{r}R}\otimes c_{2T}\otimes(c_{5g}\beta(d_1)^{1G}\beta(d_{2r1})^2)_{\overline{T}R}\end{aligned}$$

$$
\begin{aligned}
&\otimes d_{4\overline{R}} \otimes c_{3\bar{t}\overline{T}} \otimes (x^g\beta(d_1)^2)_{r\bar{r}\overline{R}} \otimes d_5 \\
\overset{(2.14)}{=}\; & c_1 \otimes (c_{4G}\beta(d_{2r})^1)_{\bar{t}\overline{T}} \otimes d_{3\bar{r}R} \otimes c_{2T} \otimes (c_{5g}\beta(d_1)^{1^G}\beta(d_{2r1})^2)_{R\overline{T}} \\
&\otimes d_{4\overline{R}} \otimes c_{3\bar{t}\overline{T}} \otimes (x^g\beta(d_1)^2)_{r\bar{r}\overline{R}} \otimes d_5.
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{id}\otimes\Delta)\Delta(c\otimes x\otimes d) \;=\;& c_1 \otimes (c_{3g}\beta(d_1)^1)_t \otimes d_{2r} \otimes c_{2t1} \otimes (c_{2t3G}\beta(d_3)^1)_T \\
&\otimes d_{4R} \otimes c_{2t2T} \otimes ((x^g\beta(d_1)^2)_r{}^G\beta(d_3)^2)_R \otimes d_5 \\
\overset{(1.2)}{=}\;& c_1 \otimes (c_{4g}\beta(d_1)^1)_{t\bar{t}} \otimes d_{2r} \otimes c_{2\bar{t}} \otimes (c_{3t2G}\beta(d_3)^1)_T \\
&\otimes d_{4R} \otimes c_{3t1T} \otimes ((x^g\beta(d_1)^2)_r{}^G\beta(d_3)^2)_R \otimes d_5 \\
\overset{(2.13)}{=}\;& c_1 \otimes (c_{4g}\beta(d_1)^1)_{t\bar{t}} \otimes d_{3r\bar{r}} \otimes c_{2\bar{t}} \otimes (c_{3t2G}\beta(d_2)^1)_{\bar{r}T} \\
&\otimes d_{4R} \otimes c_{3t1T} \otimes ((x^g\beta(d_1)^2)^G\beta(d_{21})^2)_{\bar{r}R} \otimes d_5 \\
\overset{(1.2)}{=}\;& c_1 \otimes (c_{5g}\beta(d_1)^1)_{t\overline{T}\bar{t}} \otimes d_{3r\bar{r}} \otimes c_{2\bar{t}} \otimes (c_{4tG}\beta(d_2)^1)_{\bar{r}T} \\
&\otimes d_{3R} \otimes c_{3\overline{T}T} \otimes ((x^g\beta(d_1)^2)^G\beta(d_2)^2)_{\bar{r}R} \otimes d_5 \\
\overset{(2.11)}{=}\;& c_1 \otimes (c_{4G}\beta(d_{2r})^1)_{\overline{T}\bar{t}} \otimes d_{3r\bar{r}} \otimes c_{2\bar{t}} \otimes (c_{5g}\beta(d_1)^{1^G}\beta(d_{2r})^2)_{\bar{r}T} \\
&\otimes d_{4R} \otimes c_{3\overline{T}T} \otimes (x^g\beta(d_1)^2)_{\overline{R}\bar{r}R} \otimes d_5,
\end{aligned}
$$

finishing the proof. □

Example 2.5. (Two-sided crossed coproduct)([8, Lemma 2.1]) Let H be a bialgebra with a left weak coaction on the coalgebra C and a right weak coaction on the coalgebra D. Assume that $C\times^{\alpha} H$ is a left crossed coproduct and $H\times^{\beta} D$ is a right crossed coproduct. Then $C\times{}^{\alpha}H^{\beta}\times D$ (=$C\otimes H\otimes D$ as a vector space) is a coassociative coalgebra with counit $\varepsilon_C\otimes\varepsilon_H\otimes\varepsilon_D$ and comultiplication

$$\Delta(c\otimes x\otimes d) = c_1\otimes c_{2-1}\alpha(c_3)^1x_1\beta(d_1)^1\otimes d_{2(0)}\otimes c_{20}\otimes\alpha(c_3)^2x_2\beta(d_1)^2d_{2-1}\otimes d_3$$

for all $c\in C$,$x\in H$ and $d\in D$.

Proof. Let $G(c\otimes x)=\alpha(c)^1x_1\otimes\alpha(c)^2x_2, T(c\otimes x)=c_{-1}x\otimes c_0, R(x\otimes d)=d_{(0)}\otimes xd_{(1)}$ in Theorem 2.4. Then by Examples 1.3 and 2.2, we finish the proof. □

Remark 2.6. If α and β are trivial in $C\times{}^{\alpha}H^{\beta}\times D$, then we can get two-sided smash coproduct $C\times H\times D$.

Example 2.7. (Two-sided twisted tensor coproduct) Let $H,\ C,\ D$ be three coalgebras. Let $R: H\otimes D\to D\otimes H, T: C\otimes H\to H\otimes C$ be linear maps and $C\times_T H$ left twisted tensor coproduct, $H_R\times D$ right twisted tensor coproduct. Then $C\times{}_TH_R\times D$ is a coassociative coalgebra with counit $\varepsilon_C\otimes\varepsilon_H\otimes\varepsilon_D$ and comultiplication:

$$\Delta(c\otimes x\otimes d) = c_1\otimes x_{1T}\otimes d_{1R}\otimes c_{2T}\otimes x_{2R}\otimes d_2,$$

for all $c\in C$, $x\in H$, $d\in D$ if and only if Eq.(2.14) holds.

Proof. Let G and β be trivial in Theorem 2.4. Then the proof is completed by Examples 1.5 and 2.3. □

Remark 2.8. Let H be a Hopf algebra with a left coaction on the coalgebra C and with a right coaction on the coalgebra D. Then the two-sided smash coproduct $C\times H\times D$ can be obtained by letting $T(c\otimes x)=c_{-1}x\otimes c_0$ and $R(x\otimes d)=d_{(0)}\otimes xd_{(1)}$ in $C\times{}_TH_R\times D$. In this case Eq.(2.14) holds automatically.

Next we give a class of bialgebra structure on Brzeziński's crossed coproduct.

Theorem 2.9. *Let $H,\ C,\ D$ be bialgebras. Let $G: C\otimes H\to H\otimes H$, $T: C\otimes H\to H\otimes C$, $R: H\otimes D\to D\otimes H$, $\beta: D\to H\otimes H$ be linear maps such that $\beta(1_D)=1_H\otimes 1_H$. Then $C\times {}^G_T H^\beta_R\times D$ equipped with the two-sided tensor product algebra $C\otimes H\otimes D$ becomes a bialgebra if and only if the following conditions hold:*

$$1_G\otimes 1_G=1\otimes 1,\ 1_T\otimes 1_T=1\otimes 1,\ 1_R\otimes 1_R=1\otimes 1; \tag{2.15}$$

$$(xx')_T\otimes (cc')_T=x_Tx'_t\otimes c_Tc'_t; \tag{2.16}$$

$$(dd')_R\otimes (xx')_R=d_Rd'_r\otimes x_Rx'_r; \tag{2.17}$$

$$(cc')_G\beta(dd')^1\otimes (xx')^G\beta(dd')^2=c_g\beta(d)^1c'_G\beta(d')^1\otimes x^g\beta(d)^2x'^G\beta(d')^2, \tag{2.18}$$

where $c,c'\in C,\ x,x'\in H,\ d,d'\in D$ and $g=G,t=T,r=R$. In this case, we call this bialgebra Brzeziński's two-sided crossed coproduct bialgebra and denoted by $C\diamond {}^G_T H^\beta_R\diamond D$.

Proof. We only prove that the comultiplication $\Delta_{C\otimes H\otimes D}$ preserves multiplication and leave the rest to the reader. For all $c,c'\in C,\ x,x'\in H,d,d'\in D$ and $g=G,t=T,r=R$, we have

$$\begin{aligned}
\Delta((c\otimes x\otimes d)(c'\otimes x'\otimes d')) &\overset{(2.18)}{=} c_1c'_1\otimes (c_{3G}\beta(d_1)^1c'_{3g}\beta(d'_1)^1)_t\otimes (d_2d'_2)_r\otimes (c_2c'_2)_t\\
&\quad \otimes(x^G\beta(d_1)^2x'^g\beta(d'_1)^2)_r\otimes d_3d'_3\\
&\overset{(2.16)}{=} c_1c'_1\otimes (c_{3G}\beta(d_1)^1)_t(c'_{3g}\beta(d'_1)^1)_T\otimes (d_2d'_2)_r\otimes c_{2t}c'_{2T}\\
&\quad \otimes(x^G\beta(d_1)^2x'^g\beta(d'_1)^2)_r\otimes d_3d'_3\\
&\overset{(2.17)}{=} c_1c'_1\otimes (c_{3G}\beta(d_1)^1)_t(c'_{3g}\beta(d'_1)^1)_T\otimes d_{2r}d'_{2R}\otimes c_{2t}c'_{2T}\\
&\quad \otimes(x^G\beta(d_1)^2)_r(x'^g\beta(d'_1)^2)_R\otimes d_3d'_3\\
&= \Delta(c\otimes x\otimes d)\Delta(c'\otimes x'\otimes d'),
\end{aligned}$$

as desired. □

In [13], the authors studied the antipode for a class of two-sided crossed product in the setting of Hopf quasigroup. Now we discuss the antipode for Brzeziński's two-sided crossed coproduct bialgebra. Firstly we need the following definition.

Definition 2.10. Let C,D be two coalgebras and H be an algebra with a distinguished element $\varepsilon_H\in H^*$. Let $G: C\otimes H\longrightarrow H\otimes H$, $\beta: D\to H\otimes H$ and $S: H\to H$ be linear maps. Then S is called a (G,β)-antipode of H if for all $x\in H,c\in C,d\in D$, the following conditions hold:

$$S_H(c_G\beta(d)^1)x^G\beta(d)^2=\varepsilon_C(c)\varepsilon_H(x)\varepsilon_D(d)1_H. \tag{2.19}$$

$$c_G\beta(d)^1S_H(x^G\beta(d)^2)=\varepsilon_C(c)\varepsilon_H(x)\varepsilon_D(d)1_H. \tag{2.20}$$

In this case, we call H a (G,β)-**Hopf algebra**.

Remark 2.11. (1) Let H be a Hopf algebra with antipode S and the maps G,β be trivial in Definition 2.10. Then S is a (G,β)-antipode of H. Therefore any Hopf algebras are (G,β)-Hopf algebras.

(2) Let $\alpha: C\to H\otimes H$ be linear map and $G(c\otimes x)=\alpha(c)^1x_1\otimes\alpha(c)^2x_2$ in Definition 2.10, then we call H an (α,β)-**Hopf algebra**.

Proposition 2.12. *Let $C,\ D$ be two Hopf algebras with antipodes $S_C,\ S_D$, H be a bialgebra and $S_H: H\longrightarrow H$ a linear map. Suppose that $C\diamond {}^G_T H^\beta_R\diamond D$ is a Brzeziński's two-sided crossed coproduct bialgebra. Then $C\diamond {}^G_T H^\beta_R\diamond D$ is a Hopf algebra with antipode $\overline{S}$ defined by*

$$\overline{S}(c\otimes x\otimes d)=S_C(c_T)\otimes S_H(x_{TR})\otimes S_D(d_R). \tag{2.21}$$

if and only if H is a (G,β)-Hopf algebra.

Proof. For all $c \in C, x \in H, d \in D$ and $R = r, T = t, G = g$, we have

$$
\begin{aligned}
(\overline{S} * \mathrm{id})(c \otimes x \otimes d) &\overset{(1.2)}{=} S_C(c_{1t1})c_{1t2} \otimes S_H((c_{2G}\beta(d_1)^1)_{tR})(x^G\beta(d_1)^2)_r \otimes S_D(d_{2rR})d_3 \\
&\overset{(1.1)}{=} 1_C \otimes S_H((c_G\beta(d_1)^1)_R)(x^G\beta(d_1)^2)_r \otimes S_D(d_{2rR})d_3 \\
&\overset{(2.13)}{=} 1_C \otimes S_H(c_G\beta(d_2)^1)(x_r^G\beta(d_2)^2) \otimes S_D(d_{1r})d_3 \\
&\overset{(2.19)}{=} 1_C \otimes \varepsilon(c)\varepsilon(x_r)\varepsilon(d_2)1_H \otimes S_D(d_{1r}) \\
&= \varepsilon(c)\varepsilon(x)\varepsilon(d)1_C \otimes 1_H \otimes 1_D
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{id} * \overline{S})(c \otimes x \otimes d) &\overset{(2.14)}{=} c_1S_C(c_{2tT}) \otimes ((c_{3G})\beta(d_1)^1)_tS_H((x^G\beta(d_1)^2)_{rRT}) \\
&\quad \otimes d_{21r}S_D(d_{22R}) \\
&\overset{(2.2)}{=} c_1S_C(c_{2tT}) \otimes ((c_{3G})\beta(d_1)^1)_tS_H((x^G\beta(d_1)^2)_{rT}) \\
&\quad \otimes d_{2r1}S_D(d_{2r2}) \\
&= c_1S_C(c_{2tT}) \otimes ((c_{3G})\beta(d_1)^1)_tS_H((x^G\beta(d_1)^2)_{rT})\varepsilon(d_{2r}) \otimes 1_D \\
&\overset{(2.1)}{=} c_1S_C(c_{2tT}) \otimes ((c_{3G})\beta(d_1)^1)_tS_H((x^G\beta(d_1)^2)_T)\varepsilon(d_2) \otimes 1_D \\
&\overset{(2.12)}{=} c_1S_C(c_{3T}) \otimes (c_{2G})\beta(d_1)^1S_H(x_T{}^G\beta(d_1)^2)\varepsilon(d_2) \otimes 1_D \\
&\overset{(2.20)}{=} c_1S_C(c_{2T})\varepsilon(x_T)\varepsilon(d) \otimes 1_H \otimes 1_D \\
&\overset{(1.1)}{=} c_1S_C(c_2)\varepsilon(x)\varepsilon(d) \otimes 1_H \otimes 1_D \\
&= \varepsilon(c)\varepsilon(x)\varepsilon(d)1_C \otimes 1_H \otimes 1_D.
\end{aligned}
$$

Thus, $\overline{S}$ is the antipode of $C \diamond {}^G_T H^\beta_R \diamond D$. The rest are direct. □

Example 2.13. Let H, C, D be bialgebras. Let $\alpha : C \longrightarrow H \otimes H$, $\beta : D \longrightarrow H \otimes H$ be linear maps such that $\beta(1_D) = 1_H \otimes 1_H$. Then $C \times {}^\alpha H^\beta \times D$ equipped with two-sided tensor product algebra $C \otimes H \otimes D$ becomes a bialgebra if and only if the following conditions hold:

$$\alpha(1_C) = 1_H \otimes 1_H, \varphi(1_C) = 1_H \otimes 1_C, \psi(1_D) = 1_D \otimes 1_H; \tag{2.22}$$
$$(cc')_{-1}xx' \otimes (cc')_0 = c_{-1}xc'_{-1}x' \otimes c_0c'_0; \tag{2.23}$$
$$(dd')_{(0)} \otimes xx'(dd')_{(1)} = d_{(0)}d'_{(0)} \otimes xd_{(1)}x'd'_{(1)}; \tag{2.24}$$
$$\alpha(cc')^1 \otimes \alpha(cc')^2 = \alpha(c)^1\alpha(c)^1 \otimes \alpha(c)^2\alpha(c')^2; \tag{2.25}$$
$$\beta(dd')^1 \otimes \beta(dd')^2 = \beta(d)^1\beta(d')^1 \otimes \beta(d)^2\beta(d')^2; \tag{2.26}$$
$$\beta(d)^1x_1 \otimes \beta(d)^2x_2 = x_1\beta(d)^1 \otimes x_2\beta(d)^2; \tag{2.27}$$
$$\alpha(c)^1x_1 \otimes \alpha(c)^2x_2 = x_1\alpha(c)^1 \otimes x_2\alpha(c)^2; \tag{2.28}$$
$$\beta(d)^1\alpha(c)^1 \otimes \beta(d)^2\alpha(c)^2 = \alpha(c)^1\beta(d)^1 \otimes \alpha(c)^2\beta(d)^2, \tag{2.29}$$

where $c, c' \in C,\ x, x' \in H,\ d, d' \in D$. In this case, we call this bialgebra two-sided twisted crossed coproduct bialgebra and denoted by $C \diamond {}^\alpha H^\beta \diamond D$. Furthermore, if C, D are Hopf algebras with antipodes S_C and S_D, $S_H : H \longrightarrow H$ is a linear map, then $C \diamond {}^\alpha H^\beta \diamond D$ is a Hopf algebra with antipode $\overline{S}$ defined by

$$\overline{S}(c \otimes x \otimes d) = S_C(c_0) \otimes S_H(c_{-1}xd_{(1)}) \otimes S_D(d_{(0)})$$

if and only if H is a (α, β)-Hopf algebra.

Proof. Let $T(c \otimes x) = c_{-1}x \otimes c_0, G(c \otimes x) = \alpha(c)^1x_1 \otimes \alpha(c)^2x_2, R(x \otimes d) = d_{(0)} \otimes xd_{(1)}$ in Theorem 2.9. Then we obtain Eqs.(2.22)-(2.24) and

$$\alpha(cc')^1(xx')_1\beta(dd')^1 \otimes \alpha(cc')^2(xx')_2\beta(dd')^2 \tag{2.30}$$

$$= \alpha(c)^1 x_1 \beta(d)^1 \alpha(c')^1 x'_1 \beta(d')^1 \otimes \alpha(c)^2 x_2 \beta(d)^2 \alpha(c')^2 x'_2 \beta(d')^2.$$

In fact, Eq.(2.30) is equivalent to Eqs.(2.25)-(2.29). One can calculate as follows.

$$\begin{aligned}
&\alpha(c)^1 x_1 \beta(d)^1 \alpha(c')^1 x'_1 \beta(d')^1 \otimes \alpha(c)^2 x_2 \beta(d)^2 \alpha(c')^2 x'_2 \beta(d')^2\\
&\overset{(2.29)}{=} \quad \alpha(c)^1 x_1 \alpha(c')^1 \beta(d)^1 x'_1 \beta(d')^1 \otimes \alpha(c)^2 x_2 \alpha(c')^2 \beta(d)^2 x'_2 \beta(d')^2\\
&\overset{(2.28)}{=} \quad \alpha(c)^1 \alpha(c')^1 x_1 \beta(d)^1 x'_1 \beta(d')^1 \otimes \alpha(c)^2 \alpha(c')^2 x_2 \beta(d)^2 x'_2 \beta(d')^2\\
&\overset{(2.27)}{=} \quad \alpha(c)^1 \alpha(c')^1 x_1 x'_1 \beta(d)^1 \beta(d')^1 \otimes \alpha(c)^2 \alpha(c')^2 x_2 x'_2 \beta(d)^2 \beta(d')^2\\
&\overset{(2.26)(2.25)}{=} \quad \alpha(cc')^1 (xx')_1 \beta(dd')^1 \otimes \alpha(cc')^2 (xx')_2 \beta(dd')^2.
\end{aligned}$$

And the inverse is direct. Thus we can finish the proof by setting $T(c \otimes x) = c_{-1}x \otimes c_0, G(c \otimes x) = \alpha(c)^1 x_1 \otimes \alpha(c)^2 x_2, R(x \otimes d) = d_{(0)} \otimes x d_{(1)}$ in Proposition 2.12. □

Remark 2.14. If H is commutative in Example 2.13, then Eqs.(2.27)-(2.29) hold automatically and Eqs.(2.22)-(2.26) just show that $\alpha, \beta, \varphi, \psi$ are algebra maps.

Example 2.15. Let H, C, D be bialgebras. Let $T : C \otimes H \to H \otimes C$, $R : H \otimes D \to D \otimes H$ be linear maps. Then $C \times {}_T H_R \times D$ equipped with the two-sided tensor product algebra $C \otimes H \otimes D$ becomes a bialgebra if and only if T and R are algebra maps. In this case, we call this bialgebra two-sided twisted tensor coproduct bialgebra and denoted by $C \diamond_T H_R \diamond D$. Furthermore, if H, C, D are Hopf algebras with antipodes S_H, S_C and S_D, then $C \diamond_T H_R \diamond D$ is a Hopf algebra with antipode $\overline{S}$ defined in Eq.(2.21).

Proof. Let G and β be trivial in Theorem 2.9 and Proposition 2.12. □

3. An extended version of Majid's double biproduct

In this section, we investigate the necessary and sufficient conditions for two-sided crossed coproduct coalgebra and two-sided smash product algebra to be a bialgebra, which improve the main result [8, Theorem 2.2.] by simplifying the conditions, especially the condition Eq.(3.13).

Firstly, let us recall [8, Theorem 2.2.].

Lemma 3.1. *Let H be a bialgebra. Let $\alpha : C \to H \otimes H$ be a linear map, where C a left H-module algebra such that $\varepsilon_C(1_C) = 1$, $\alpha(1_C) = 1_H \otimes 1_H$, and a coalgebra with a left H-weak coaction. Let $\beta : D \to H \otimes H$ be a linear map, where D a right H-module algebra such that $\varepsilon_D(1_D) = 1$, $\beta(1_D) = 1_H \otimes 1_H$, and a coalgebra with a right H-weak coaction. Then the two-sided crossed coproduct coalgebra $C \times {}^{\alpha} H^{\beta} \times D$ eqquipped with the two-sided smash product algebra $C \# H \# D$ becomes a bialgebra if and only if the following conditions hold:*

$$\Delta_C(1_C) = 1_C \otimes 1_C, \Delta_D(1_D) = 1_D \otimes 1_D; \tag{3.1}$$

$$\varepsilon_C, \varepsilon_D \textit{ are algebra maps}; \tag{3.2}$$

$$1_{C_{-1}} \otimes 1_{C_0} = 1_H \otimes 1_C, 1_{D(0)} \otimes 1_{D(1)} = 1_D \otimes 1_H; \tag{3.3}$$

$$\varepsilon_C(x \triangleright c) = \varepsilon(x)\varepsilon(c), \varepsilon_D(d \triangleleft y) = \varepsilon(d)\varepsilon(y); \tag{3.4}$$

$$c(x_1 \triangleright c') \otimes x_2 = c_1(\alpha(c_2)^1 x_1 \triangleright c') \otimes \alpha(c_2)^2 x_2; \tag{3.5}$$

$$(d \triangleleft x_1)d' \otimes x_2 = (d \triangleleft x_2 \beta(d'_1)^2) d'_2 \otimes x_1 \beta(d'_1)^1; \tag{3.6}$$

$$(c(x \triangleright c'))_1 \otimes (c(x \triangleright c'))_2 = c_1(c_{2-1}\alpha(c_3)^1 x_1 \triangleright c'_1) \otimes c_{20}(\alpha(c_3)^2 x_2 \triangleright c'_2); \tag{3.7}$$

$$((d \triangleleft x)d')_1 \otimes ((d \triangleleft x)d')_2 = (d_1 \triangleleft x_1 \beta(d'_1)^1) d'_{2(0)} \otimes (d_2 \triangleleft x_2 \beta(d'_1)^2 d'_{2(1)}) d'_3; \tag{3.8}$$

$$\alpha(c(x_1 \triangleright c'))^1 x_2 \otimes \alpha(c(x_1 \triangleright c'))^2 x_3 = \alpha(c)^1 x_1 \alpha(c')^1 \otimes \alpha(c)^2 x_2 \alpha(c')^2; \tag{3.9}$$

$$x_1 \beta((d \triangleleft x_3)d')^1 \otimes x_2 \beta((d \triangleleft x_3)d')^2 = \beta(d)^1 x_1 \beta(d')^1 \otimes \beta(d)^2 x_2 \beta(d')^2; \tag{3.10}$$

$$(c(x_1 \triangleright c'))_{-1} x_2 \otimes (c(x_1 \triangleright c'))_0 = c_{1-1}\alpha(c_2)^1 x_1 c'_{-1} \otimes c_{10}\alpha(c_2)^2 x_2 c'_0; \tag{3.11}$$

$$((d\triangleleft x_2)d')_{(0)}\otimes x_1((d\triangleleft x_2)d')_{(1)}=(d_{(0)}\triangleleft x_1\beta(d'_1)^1)d'_{2(0)}\otimes d_{(1)}x_2\beta(d'_1)^2d'_{2(1)};\qquad(3.12)$$

$$(x_1\triangleright c_1)\otimes x_2c_{2-1}\alpha(c_3)^1\beta(d)^1x'_1\otimes(d_{2(0)}\triangleleft x'_2)\otimes(x_3\triangleright c_{20})\otimes x_4\alpha(c_3)^2\beta(d)^2d_{2(1)}x'_3\qquad(3.13)$$
$$\otimes(d_3\triangleleft x'_4)=((x_1\beta(d_1)^1{}_1)\triangleright c_1)\otimes x_2\beta(d_1)^1{}_2c_{2-11}\alpha(c_3)^1{}_1x'_1\otimes(d_{2(0)}\triangleleft c_{2-12}\alpha(c_3)^1{}_2x'_2)$$
$$\otimes x_3\beta(d_1)^2{}_1d_{2(1)1}\triangleright c_{20}\otimes x_4\beta(d_1)^2{}_2d_{2(1)2}\alpha(c_3)^2{}_1x'_3\otimes d_3\triangleleft\alpha(c_3)^2{}_2x'_4.$$

In this case, we call this bialgebra the double crossed biproduct denoted by $C\star{}^{\alpha}H^{\beta}\star D$.

Next we give an improved version of the above result.

Theorem 3.2. *Under the assumption of Lemma 3.1. The two-sided crossed coproduct* $C\times^{\alpha}H^{\beta}\times D$ *equipped with the two-sided smash product* $C\#H\#D$ *becomes a bialgebra if and only if Eqs.(3.1)-(3.6) and the following conditions hold* ($x\in H, c,c'\in C, d,d'\in D$):

$$(cc')_1\otimes(cc')_2=c_1(c_{2-1}\alpha(c_3)^1\triangleright c'_1)\otimes c_{20}(\alpha(c_3)^2\triangleright c'_2);\qquad(3.14)$$
$$(x\triangleright c)_1\otimes(x\triangleright c)_2=(x_1\triangleright c_1)\otimes(x_2\triangleright c_2);\qquad(3.15)$$
$$(dd')_1\otimes(dd')_2=(d_1\triangleleft\beta(d'_1)^1)d'_{2(0)}\otimes(d_2\triangleleft\beta(d'_1)^2)d'_{2(1)})d'_3;\qquad(3.16)$$
$$(d\triangleleft x)_1\otimes(d\triangleleft x)_2=(d_1\triangleleft x_1)\otimes(d_2\triangleleft x_2);\qquad(3.17)$$
$$\alpha(x_1\triangleright c)^1x_2\otimes\alpha(x_1\triangleright c)^2x_3=x_1\alpha(c)^1\otimes x_2\alpha(c)^2;\qquad(3.18)$$
$$\alpha(cc')^1\otimes\alpha(cc')^2=\alpha(c)^1\alpha(c')^1\otimes\alpha(c)^2\alpha(c')^2;\qquad(3.19)$$
$$x_1\beta(d\triangleleft x_3)^1\otimes x_2\beta(d\triangleleft x_3)^2=\beta(d)^1x_1\otimes\beta(d)^2x_2;\qquad(3.20)$$
$$\beta(dd')^1\otimes\beta(dd')^2=\beta(d)^1\beta(d')^1\otimes\beta(d)^2\beta(d')^2;\qquad(3.21)$$
$$(cc')_{-1}\otimes(cc')_0=c_{1-1}\alpha(c_2)^1c'_{-1}\otimes c_{10}(\alpha(c_2)^2\triangleright c'_0);\qquad(3.22)$$
$$(x_1\triangleright c)_{-1}x_2\otimes(x_1\triangleright c)_0=x_1c_{-1}\otimes x_2\triangleright c_0;\qquad(3.23)$$
$$(d\triangleleft x_2)_{(0)}\otimes x_1(d\triangleleft x_2)_{(1)}=(d_{(0)}\triangleleft x_1)\otimes d_{(1)}x_2;\qquad(3.24)$$
$$(dd')_{(0)}\otimes(dd')_{(1)}=(d_{(0)}\triangleleft\beta(d'_1)^1)d'_{2(0)}\otimes d_{(1)}\beta(d'_1)^2d'_{2(1)};\qquad(3.25)$$
$$c\otimes 1_H\varepsilon(d)=(\beta(d)^1\triangleright c)\otimes\beta(d)^2;\qquad(3.26)$$
$$\varepsilon(c)1_H\otimes d=\alpha(c)^1\otimes(d\triangleleft\alpha(c)^2);\qquad(3.27)$$
$$d_{(1)}\triangleright c_0\otimes d_{(0)}\triangleleft c_{-1}=c\otimes d;\qquad(3.28)$$
$$c_1\otimes c_{2-1}\alpha(c_3)^1\beta(d_1)^1\otimes d_{2(0)}\otimes c_{20}\otimes\alpha(c_3)^2\beta(d_1)^2d_{2(1)}\otimes d_3=(\beta(d_1)^1{}_1\triangleright c_1)\qquad(3.29)$$
$$\otimes\beta(d_1)^1{}_2c_{2-1}\alpha(c_3)^1{}_1\otimes(d_{2(0)}\triangleleft\alpha(c_3)^1{}_2)\otimes(\beta(d_1)^2{}_1\triangleright c_{20})\otimes\beta(d_1)^2{}_2d_{2(1)}\alpha(c_3)^2{}_1$$
$$\otimes(d_3\triangleleft\alpha(c_3)^2{}_2).$$

In this case, we also call this bialgebra the double crossed biproduct still denoted by $C\star{}^{\alpha}H^{\beta}\star D$.

Proof. By Lemma 3.1, we only need to prove that Eqs.(3.7)-(3.13) are equivalent to Eqs.(3.14)-(3.29). In what follows, we take two steps to prove the result.

Step 1. Let $x=1_H$ and $c=1_C$ in Eq.(3.7), one gets Eq.(3.14) and Eq.(3.15), respectively.

Conversely, we have

$$\begin{aligned}(c(x\triangleright c'))_1\otimes(c(x\triangleright c'))_2&\overset{(3.14)}{=}c_1(c_{2-1}\alpha(c_3)^1\triangleright(x\triangleright c')_1)\otimes c_{20}(\alpha(c_3)^2\triangleright(x\triangleright c')_2)\\&\overset{(3.15)}{=}c_1(c_{2-1}\alpha(c_3)^1\triangleright(x_1\triangleright c'_1))\otimes c_{20}(\alpha(c_3)^2\triangleright(x_2\triangleright c'_2))\\&=c_1(c_{2-1}\alpha(c_3)^1x_1\triangleright c'_1)\otimes(c_{20}(\alpha(c_3)^2x_2\triangleright c'_2)),\end{aligned}$$

i.e., Eq.(3.7) holds. Thus Eq.(3.7)⇔ Eq.(3.14) and (3.15).

Likewise, one can obtain that Eq.(3.8)⇔ Eq.(3.16) and (3.17); Eq.(3.9)⇔ Eqs.(3.18) and (3.19); Eq.(3.10)⇔ Eqs.(3.20) and (3.21); Eq.(3.11)⇔ Eqs.(3.22) and (3.23); Eq.(3.12)⇔ Eqs.(3.24) and (3.25).

Step 2. This step is technical, we will show that Eq.(3.13) $\Leftrightarrow$ Eqs.(3.26)-(3.29).

Firstly, by applying $\mathrm{id}_C\otimes\varepsilon_H\otimes\varepsilon_D\otimes\varepsilon_C\otimes\mathrm{id}_H\otimes\varepsilon_D$ to Eq.(3.13), we have

$$(x_1\triangleright c)\otimes x_2\varepsilon(d)\varepsilon(x')=x_1\beta(d)^1\triangleright c_1\otimes x_2\beta(d)^2\varepsilon(x').$$

Let $x=x'=1_H$ in the above equation, one gets Eq.(3.26). Applying $\varepsilon_C\otimes\mathrm{id}_H\otimes\varepsilon_D\otimes\varepsilon_C\otimes\varepsilon_H\otimes\mathrm{id}_D$ to Eq.(3.13), we have

$$\varepsilon(c)xx'_1\otimes d\triangleleft x'_2=\varepsilon(c)x\alpha(c_3)^1x'_1\otimes d\triangleleft\alpha(c_3)^2x'_2. \tag{3.30}$$

Let $x=x'=1_H$ in the above equation, we have Eq.(3.27). Setting $x=x'=1_H$ in Eq.(3.13), we have

$$\begin{aligned}&c_1\otimes c_{2-1}\alpha(c_3)^1\beta(d)^1\otimes d_{2(0)}\otimes c_{20}\otimes\alpha(c_3)^2\beta(d)^2d_{2(1)}\otimes d_3\\&=((\beta(d_1)^1{}_1)\triangleright c_1)\otimes\beta(d_1)^1{}_2c_{2-11}\alpha(c_3)^1{}_1\otimes(d_{2(0)}\triangleleft c_{2-12}\alpha(c_3)^1{}_2)\\&\otimes(\beta(d_1)^2{}_1d_{2(1)1}\triangleright c_{20}\otimes\beta(d_1)^2{}_2d_{2(1)2}\alpha(c_3)^2{}_1\otimes(d_3\triangleleft\alpha(c_3)^2{}_2).\end{aligned}\tag{3.31}$$

Applying $\varepsilon_C\otimes\varepsilon_H\otimes\mathrm{id}_D\otimes\mathrm{id}_C\otimes\varepsilon_H\otimes\varepsilon_D$ to Eq.(3.31), we can get Eq.(3.28). And

$$\begin{aligned}&\text{RHS of Eq.(3.31)}\\&\overset{(3.26)(3.27)}{=}(\beta(d_1)^1{}_1\triangleright c_1)\otimes\beta(d_1)^1{}_2\alpha(c_2)^1c_{3-11}\alpha(c_4)^1{}_1\otimes(d_{2(0)}\triangleleft\alpha(c_2)^2c_{3-12}\alpha(c_4)^1{}_2)\\&\qquad\otimes(\beta(d_1)^2{}_1d_{2(1)1}\beta(d_3)^1\triangleright c_{30})\otimes\beta(d_1)^2{}_2d_{2(1)2}\beta(d_3)^2\alpha(c_4)^2{}_1\otimes(d_4\triangleleft\alpha(c_4)^2{}_2)\\&\overset{(1.9)(2.9)}{=}(\beta(d_1)^1{}_1\triangleright c_1)\otimes\beta(d_1)^1{}_2c_{2-1}\alpha(c_3)^1\alpha(c_4)^1{}_1\otimes(d_{3(0)(0)}\triangleleft c_{20-1}\alpha(c_3)^2\alpha(c_4)^1{}_2)\\&\qquad\otimes(\beta(d_1)^2{}_1\beta(d_2)^1d_{3(0)1}\triangleright c_{200})\otimes\beta(d_1)^2{}_2\beta(d_2)^2d_{3(1)}\alpha(c_4)^2{}_1\otimes(d_4\triangleleft\alpha(c_4)^2{}_2)\\&\overset{(3.28)}{=}(\beta(d_1)^1{}_1\triangleright c_1)\otimes\beta(d_1)^1{}_2c_{2-1}\alpha(c_3)^1\alpha(c_4)^1{}_1\otimes(d_{3(0)}\triangleleft\alpha(c_3)^2\alpha(c_4)^1{}_2)\\&\qquad\otimes(\beta(d_1)^2{}_1\beta(d_2)^1\triangleright c_{20})\otimes\beta(d_1)^2{}_2\beta(d_2)^2d_{3(1)}\alpha(c_4)^2{}_1\otimes(d_4\triangleleft\alpha(c_4)^2{}_2)\\&\overset{(3.26)(3.27)}{=}(\beta(d_1)^1{}_1\triangleright c_1)\otimes\beta(d_1)^1{}_2c_{2-1}\alpha(c_3)^1{}_1\otimes(d_{2(0)}\triangleleft\alpha(c_3)^1{}_2)\\&\qquad\otimes(\beta(d_1)^2{}_1\triangleright c_{20})\otimes\beta(d_1)^2{}_2\beta(d_2)^2d_{3(1)}\alpha(c_3)^2{}_1\otimes(d_3\triangleleft\alpha(c_3)^2{}_2)\\&=\ \text{RHS of Eq.(3.29)}.\end{aligned}$$

And LHS of Eq.(3.31) is exactly LHS of Eq.(3.29), so we obtain Eq.(3.29). Inversely, we have

$$\begin{aligned}&\text{RHS of Eq.(3.13)}\\&\overset{(3.26)(3.27)}{=}((x_1\beta(d_1)^1{}_1)\triangleright c_1)\otimes x_2\beta(d_1)^1{}_2\alpha(c_2)^1c_{3-11}\alpha(c_4)^1{}_1x'_1\otimes(d_{2(0)}\triangleleft\alpha(c_2)^2c_{3-12}\\&\qquad\times\alpha(c_4)^1{}_2x'_2)\otimes(x_3\beta(d_1)^2{}_1d_{2(1)1}\beta(d_3)^1\triangleright c_{30})\otimes x_4\beta(d_1)^2{}_2d_{2(1)2}\beta(d_3)^2\alpha(c_4)^2{}_1x'_3\\&\qquad\otimes(d_3\triangleleft\alpha(c_4)^2{}_2x'_4)\\&\overset{(1.9)}{=}((x_1\beta(d_1)^1{}_1)\triangleright c_1)\otimes x_2\beta(d_1)^1{}_2c_{2-1}\alpha(c_3)^1\alpha(c_4)^1{}_1x'_1\otimes(d_{2(0)}\triangleleft c_{20-1}\alpha(c_3)^2\\&\qquad\times\alpha(c_4)^1{}_2x'_2)\otimes(x_3\beta(d_1)^2{}_1d_{2(1)1}\beta(d_3)^1\triangleright c_{200})\otimes x_4\beta(d_1)^2{}_2d_{2(1)2}\beta(d_3)^2\alpha(c_4)^2{}_1x'_3\\&\qquad\otimes(d_3\triangleleft\alpha(c_4)^2{}_2x'_4)\\&\overset{(2.9)}{=}((x_1\beta(d_1)^1{}_1)\triangleright c_1)\otimes x_2\beta(d_1)^1{}_2c_{2-1}\alpha(c_3)^1\alpha(c_4)^1{}_1x'_1\otimes(d_{3(0)(0)}\triangleleft c_{20-1}\alpha(c_3)^2\\&\qquad\times\alpha(c_4)^1{}_2x'_2)\otimes(x_3\beta^2{}_1\beta(d_2)^1d_{3(0)1}\triangleright c_{200})\otimes x_4\beta(d_1)^2{}_2\beta(d_2)^2d_{3(1)}\alpha(c_4)^2{}_1x'_3\\&\qquad\otimes(d_4\triangleleft\alpha(c_4)^2{}_2x'_4)\\&\overset{(3.28)}{=}((x_1\triangleright\beta(d_1)^1{}_1)\triangleright c_1)\otimes x_2\beta(d_1)^1{}_2c_{2-1}\alpha(c_3)^1\alpha(c_4)^1{}_1x'_1\otimes(d_{3(0)}\triangleleft\alpha(c_3)^2\\&\qquad\times\alpha(c_4)^1{}_2x'_2)\otimes(x_3\beta(d_1)^2{}_1\beta(d_2)^1\triangleright c_{20})\otimes x_4\beta(d_1)^2{}_2\beta(d_2)^2d_{3(1)}\alpha(c_4)^2{}_1x'_3\\&\qquad\otimes(d_4\triangleleft\alpha(c_4)^2{}_2\triangleleft x'_4)\end{aligned}$$

$$\overset{(3.26)(3.27)}{=} ((x_1 \triangleright \beta(d_1)^1{}_1) \triangleright c_1) \otimes x_2\beta(d_1)^1{}_2 c_{2-1}\alpha(c_3)^1{}_1 x'_1 \otimes (d_{2(0)} \triangleleft \alpha(c_3)^1{}_2 x'_2)$$
$$\otimes (x_3\beta(d_1)^2{}_1 \triangleright c_{20}) \otimes x_4\beta(d_1)^2{}_2 d_{2(1)}\alpha(c_4)^2{}_1 x'_3 \otimes (d_3 \triangleleft \alpha(c_3)^2{}_2 \triangleleft x'_4)$$
$$\overset{(3.29)}{=} (x_1 \triangleright c_1) \otimes x_2 c_{2-1}\alpha(c_3)^1\beta(d_1)^1 x'_1 \otimes (d_{2(0)} \triangleleft x'_2) \otimes (x_3 \triangleright c_{20})$$
$$\otimes x_4\alpha(c_3)^2\beta(d_1)^2 d_{2(1)} x'_3 \otimes (d_3 \triangleleft x'_4)$$
$$= \text{LHS of Eq.(3.13)}.$$

Thus Eq.(3.13)$\Leftrightarrow$ Eqs.(3.26)-(3.29). These finish the proof. □

Corollary 3.3. *Let H be a bialgebra. Let $\alpha : C \longrightarrow H \otimes H$ be a linear map, where C a left H-module algebra, and a coalgebra with a left H-weak coaction. Then the left crossed coproduct $C \times^\alpha H$ equipped with the left smash product $C\#H$ becomes a bialgebra if and only if the parts about C of Eqs.(3.1)-(3.4), $\alpha(1_C) = 1_H \otimes 1_H$, Eqs.(3.5), (3.14),(3.15),(3.18),(3.19),(3.22) and (3.23) hold. In this case, we call this bialgebra* **left crossed biprodcut** *and denote it by $C \star^\alpha H$.*

Proof. Let $D = K$ in Theorem 3.2. □

Remark 3.4. In [14, Theorem 1.1], Wang-Wang-Yao provided a bialgebra construction for the left crossed coproduct $C \times^\alpha H$ and the left smash product $C\#H$ as follows: Let H be a bialgebra. Let $\alpha : C \longrightarrow H \otimes H$ be a linear map, where C a left H-module algebra, and a coalgebra with a left H-weak coaction. Assume that α is convolution invertible and for all $c, c' \in C$,

$$(\alpha(c)^1 \triangleright c'_1) \otimes \alpha(\alpha(c)^2 \triangleright c'_2)^1 \otimes \alpha(\alpha(c)^2 \triangleright c'_2)^2$$
$$= (\alpha(c)^1 \triangleright c'_1) \otimes \alpha(c)^2{}_1\alpha(c'_2)^1 \otimes \alpha(c)^2{}_2\alpha(c'_2)^2. \tag{3.32}$$

Then the left crossed coproduct $C \times^\alpha H$ equipped with the left smash product $C\#H$ becomes a bialgebra if and only if the parts about C of Eqs.(3.1)-(3.4), $\alpha(1_C) = 1_H \otimes 1_H$, Eqs.(3.14), (3.15), (3.18), (3.19), (3.22), (3.23) and

$$c_{10}(\alpha(c_2)^2\alpha(c_3)^1{}_2 \triangleright c') \otimes c_{1-1}\alpha(c_2)^1\alpha(c_3)^1{}_1 \otimes \alpha(c_3)^2$$
$$= c_{10}(\alpha(c_2)^2{}_1 \triangleright c') \otimes c_{1-1}\alpha(c_2)^1 \otimes \alpha(c_2)^2{}_2 \tag{3.33}$$

hold.

By comparing the necessary conditions of Wang-Wang-Yao's above and ours in Corollary 3.3, we can get that the conditions imposed on α are redundant, i.e., the conditions that α is convolution invertible and Eq.(3.32) should be omitted.

Proof. We only need to prove that Eq.(3.5) is equivalent to Eq.(3.33). Applying $\mathrm{id}_C \otimes \varepsilon_H \otimes \mathrm{id}_H$ to Eq.(3.33), we have

$$c_1(\alpha(c_2)^1 \triangleright c') \otimes \alpha(c_2)^2 = cc' \otimes 1_H. \tag{3.34}$$

Then

$$c(x_1 \triangleright c') \otimes x_2 \overset{(3.34)}{=} c_1(\alpha(c_2)^1 \triangleright (x_1 \triangleright c')) \otimes \alpha(c_2)^2 x_2$$
$$= c_1(\alpha(c_2)^1 x_1 \triangleright c') \otimes \alpha(c_2)^2 x_2.$$

Thus Eq.(3.5) holds. The inverse can be checked as follows.

$$c_{10}((\alpha(c_2)^2\alpha(c_3)^1_2) \triangleright c') \otimes c_{1-1}\alpha(c_2)^1\alpha(c_3)^1_1 \otimes \alpha(c_3)^2$$
$$\overset{(1.8)}{=} c_{10}((\alpha(c_{20})^1\alpha(c_3)^2_1) \triangleright c') \otimes c_{1-1}c_{2-1}\alpha(c_3)^1 \otimes \alpha(c_{20})^2\alpha(c_3)^2_2$$
$$\overset{(1.6)}{=} c_{101}((\alpha(c_{102})^1\alpha(c_2)^2_1) \triangleright c') \otimes c_{1-1}\alpha(c_2)^1 \otimes \alpha(c_{102})^2\alpha(c_2)^2_2$$
$$\overset{(3.5)}{=} c_{10}(\alpha(c_2)^2_1 \triangleright c') \otimes c_{1-1}\alpha(c_2)^1 \otimes \alpha(c_2)^2_2,$$

finishing the proof. □

Corollary 3.5. *Let H be a bialgebra. Let $\beta : D \longrightarrow H \otimes H$ be a linear map, where D a right H-module algebra, and a coalgebra with a right H-weak coaction. Then the right crossed coproduct $H^{\beta} \times D$ equipped with the right smash product $H\#D$ becomes a bialgebra if and only if the parts about D of Eqs.(3.1)-(3.4), $\beta(1_D) = 1_H \otimes 1_H$, Eqs.(3.6), (3.16),(3.17),(3.20),(3.21),(3.24) and (3.25) hold. In this case, we call this bialgebra* **right crossed biprodcut** *and denote it by $H^{\beta} \star D$.*

Proof. Let $C = K$ in Theorem 3.2. □

Based on Theorem 3.2 and Corollaries 3.3, 3.5, we have

Proposition 3.6. *Under the assumption of Lemma 3.1. Suppose that $C \star^{\alpha} H$ is a left crossed biprodcut and $H^{\beta} \star D$ is a right crossed biproduct. Then the two-sided crossed coproduct $C \times^{\alpha} H^{\beta} \times D$ equipped with the two-sided smash product $C\#H\#D$ becomes a bialgebra if and only if (3.26)-(3.29) hold.*

Corollary 3.7. ([9,10]) *Let H be a Hopf algebra, C a bialgebra in ${}^{H}_{H}\mathbb{YD}$, and D a bialgebra in $\mathbb{YD}^{H}_{H}$. Then the two-sided smash product algebra $C\#H\#D$ and the two-sided smash coproduct coalgebra $C \times H \times D$ form a bialgebra, named the Majid's double biproduct (or double-bosonization) and denoted by $C \star H \star D$ if and only if Eq.(3.28) holds.*

Question. Let H be a Hopf algebra. C is a bialgebra in ${}^{H}_{H}\mathbb{YD}$ if and only if $C \star H$ is a left Radford biproduct, D is a bialgebra in $\mathbb{YD}^{H}_{H}$ if and only if $H \star D$ is a right Radford biproduct. Is there a tensor category ${}^{H}_{H}\mathbb{YD}^{\alpha}$ (or ${}^{\beta}\mathbb{YD}^{H}_{H}$) such that C is a bialgebra in ${}^{H}_{H}\mathbb{YD}^{\alpha}$ (or ${}^{\beta}\mathbb{YD}^{H}_{H}$) if and only if $C \star^{\alpha} H$ (or $H^{\beta} \star D$) is a left (or right) crossed biproduct?

At last, we provide three specific examples for double crossed biproducts, of which the first one is the case of α, β trivial and the last two are the case of α, β non-trivial.

Example 3.8. Assume that char$k \neq 2$. Let $H = K\{1, g\}$ be the group bialgebra. Let $C = sp\{1, x\}$, its algebra structure is defined by $x^2 = 0$; its coalgebra structure is defined by $\Delta(1) = 1 \otimes 1,\ \varepsilon(1) = 1,\ \Delta(x) = x \otimes 1 + 1 \otimes x$, and $\varepsilon(x) = 0$. Let $D = sp\{1, y\}$, its algebra structure is defined by $y^2 = 0$; its coalgebra structure is defined by $\Delta(1) = 1 \otimes 1,\ \varepsilon(1) = 1,\ \Delta(y) = y \otimes 1 + 1 \otimes y$, and $\varepsilon(y) = 0$.

Define the linear maps:

$$\triangleright : H \otimes C \longrightarrow C \text{ such that } g \triangleright x := -x, g \triangleright 1 := 1;$$
$$\rho : C \longrightarrow H \otimes C \text{ such that } \rho(x) := g \otimes x, \rho(1) := 1 \otimes 1;$$
$$\triangleleft : D \otimes H \longrightarrow D \text{ such that } y \triangleleft g := -y, 1 \triangleleft g := 1;$$
$$\gamma : D \longrightarrow D \otimes H \text{ such that } \gamma(y) := y \otimes g, \gamma(1) := 1 \otimes 1.$$

Then $C \otimes H \otimes D = K\{a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8\}$, where $a_1 = 1_C \otimes 1_H \otimes 1_D,\ a_2 = 1_C \otimes 1_H \otimes y,\ a_3 = 1_C \otimes g \otimes 1_D,\ a_4 = 1_C \otimes g \otimes y,\ a_5 = x \otimes 1_H \otimes 1_D,\ a_6 = x \otimes 1_H \otimes y,\ a_7 = x \otimes g \otimes 1_D,\ a_8 = x \otimes g \otimes y$, is a double biproduct with the multiplication

$\cdot$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_2	a_2	0	$-a_4$	0	a_6	0	$-a_8$	0
a_3	a_3	a_4	$-a_1$	$-a_2$	$-a_7$	$-a_8$	$-a_5$	$-a_6$
a_4	a_4	0	a_2	0	$-a_8$	0	a_6	0
a_5	a_5	a_6	a_7	a_8	0	0	0	0
a_6	a_6	0	$-a_8$	0	0	0	0	0
a_7	a_7	a_8	a_5	a_6	0	0	0	0
a_8	a_8	0	$-a_6$	0	0	0	0	0

and comultiplication

$$\Delta(a_1) \quad = \quad a_1 \otimes a_1;$$

$$
\begin{aligned}
\Delta(a_2) &= a_2 \otimes a_3 + a_1 \otimes a_2;\\
\Delta(a_3) &= a_3 \otimes a_3;\\
\Delta(a_4) &= a_4 \otimes a_1 + a_3 \otimes a_4;\\
\Delta(a_5) &= a_5 \otimes a_1 + a_3 \otimes a_5;\\
\Delta(a_6) &= a_6 \otimes a_3 + a_5 \otimes a_2 + a_4 \otimes a_7 + a_3 \otimes a_6;\\
\Delta(a_7) &= a_7 \otimes a_3 + a_1 \otimes a_7;\\
\Delta(a_8) &= a_8 \otimes a_1 + a_7 \otimes a_4 + a_2 \otimes a_5 + a_1 \otimes a_8.
\end{aligned}
$$

Example 3.9. Let $H = K\{1_H, g, x, gx\}$ be a bialgebra, its algebra structure is defined by $g^2 = 1_H,\ x^2 = 0,\ gx = -xg$; its coalgebra structure is defined by $\Delta(1_H) = 1_H \otimes 1_H,\ \varepsilon(1_H) = 1,\ \Delta(g) = g \otimes g,\ \varepsilon(g) = 1,\ \Delta(x) = 1_H \otimes x + x \otimes g,\ \varepsilon(x) = 0,\ \Delta(gx) = g \otimes gx + gx \otimes 1_H,\ \varepsilon(gx) = 0$. Let $C = sp\{1_C, y\}$, its algebra structure is defined by $y^2 = 0$; its coalgebra structure is defined by $\Delta(1_C) = 1_C \otimes 1_C,\ \varepsilon(1_C) = 1,\ \Delta(y) = y \otimes 1_C + 1_C \otimes y,\ \varepsilon(y) = 0$. Let $D = sp\{1_D, z\}$, its algebra structure is defined by $z^2 = 0$; its coalgebra structure is defined by $\Delta(1_D) = 1_D \otimes 1_D,\ \varepsilon(1_D) = 1,\ \Delta(z) = z \otimes 1_D + 1_D \otimes z$ and $\varepsilon(z) = 0$.

Define linear maps $\triangleright : H \otimes C \longrightarrow C,\ \rho : C \longrightarrow H \otimes C,\ \alpha : C \longrightarrow H \otimes H,\ \triangleleft : D \otimes H \longrightarrow D,\ \gamma : D \longrightarrow D \otimes H,\ \beta : D \longrightarrow H \otimes H$ by

$$1_H \triangleright 1_C := 1_C,\ g \triangleright 1_C := 1_C,\ 1_H \triangleright y := y,\ g \triangleright y := y;$$

$$\rho(1_C) := 1_H \otimes 1_C,\ \rho(y) := g \otimes y;$$

$$\alpha(1_C) := 1_H \otimes 1_H,\ \rho(y) := k1_H \otimes x - kg \otimes x - kx \otimes 1_H + kx \otimes g;$$

$$1_D \triangleleft 1_H := 1_D,\ 1_D \triangleleft g := 1_D,\ z \triangleleft 1_H := z, z \triangleleft g := z;$$

$$\gamma(1_D) := 1_D \otimes 1_H,\ \gamma(z) := z \otimes g;$$

$$\beta(1_D) := 1_H \otimes 1_H,\ \beta(y) := \ell 1_H \otimes gx - \ell g \otimes gx - \ell gx \otimes 1_H + \ell gx \otimes g,$$

where $\forall\ \ k, \ell \in \mathbb{Z}$ (the set of integers).

Then $C \otimes H \otimes D = K\{b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6,\ b_7,\ b_8,\ b_9,\ b_{10},\ b_{11},\ b_{12},\ b_{13},\ b_{14},\ b_{15},\ b_{16}\}$, where $b_1 = 1_C \otimes 1_H \otimes 1_D,\ b_2 = 1_C \otimes 1_H \otimes z,\ b_3 = 1_C \otimes g \otimes 1_D,\ b_4 = 1_C \otimes g \otimes z,\ b_5 = 1_C \otimes x \otimes 1_D,\ b_6 = 1_C \otimes x \otimes z,\ b_7 = 1_C \otimes gx \otimes 1_D,\ b_8 = 1_C \otimes gx \otimes z,\ b_9 = y \otimes 1_H \otimes 1_D,\ b_{10} = y \otimes 1_H \otimes z,\ b_{11} = y \otimes g \otimes 1_D,\ b_{12} = y \otimes g \otimes z,\ b_{13} = y \otimes x \otimes 1_D,\ b_{14} = y \otimes x \otimes z,\ b_{15} = y \otimes gx \otimes 1_D,\ b_{16} = y \otimes gx \otimes z$, is a double crossed biproduct with the multiplication

$\cdot$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{12}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
b_1	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{12}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
b_2	b_2	0	b_4	0	b_6	0	b_8	0	b_{10}	0	b_{12}	0	b_{14}	0	b_{16}	0
b_3	b_3	b_4	b_1	b_2	b_7	b_8	b_5	b_6	b_{11}	b_{12}	b_9	b_{10}	b_{15}	b_{16}	b_{13}	b_{14}
b_4	b_4	0	b_2	0	b_8	0	b_6	0	b_{12}	0	b_{10}	0	b_{16}	0	b_{14}	0
b_5	b_5	b_6	$-b_7$	$-b_8$	0	0	0	0	b_{13}	b_{14}	$-b_{15}$	$-b_{16}$	0	0	0	0
b_6	b_6	0	$-b_8$	0	0	0	0	0	b_{14}	0	$-b_{16}$	0	0	0	0	0
b_7	b_7	b_8	$-b_5$	$-b_6$	0	0	0	0	b_{15}	b_{16}	$-b_{13}$	$-b_{14}$	0	0	0	0
b_8	b_8	0	$-b_6$	0	0	0	0	0	0	b_{16}	0	$-b_{14}$	0	0	0	0
b_9	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	0	0	0	0	0	0	0	0
b_{10}	b_{10}	0	b_{12}	0	b_{14}	0	b_{16}	0	0	0	0	0	0	0	0	0
b_{11}	b_{11}	b_{12}	b_9	b_{10}	b_{15}	b_{16}	b_{13}	b_{14}	0	0	0	0	0	0	0	0
b_{12}	b_{12}	0	b_{10}	0	b_{16}	0	b_{14}	0	0	0	0	0	0	0	0	0
b_{13}	b_{13}	b_{14}	$-b_{15}$	$-b_{16}$	0	0	0	0	0	0	0	0	0	0	0	0
b_{14}	b_{14}	0	$-b_{16}$	0	0	0	0	0	0	0	0	0	0	0	0	0
b_{15}	b_{15}	b_{16}	$-b_{13}$	$-b_{14}$	0	0	0	0	0	0	0	0	0	0	0	0
b_{16}	b_{16}	0	$-b_{14}$	0	0	0	0	0	0	0	0	0	0	0	0	0

and comultiplication

$$\Delta(b_1) = b_1 \otimes b_1;$$

$$\Delta(b_2) = b_1 \otimes b_2 + b_2 \otimes b_4;$$

$$\begin{aligned}
\Delta(b_3) &= b_3 \otimes b_3;\\
\Delta(b_4) &= b_3 \otimes b_4 + b_4 \otimes b_2;\\
\Delta(b_5) &= b_1 \otimes b_5 + b_5 \otimes b_1;\\
\Delta(b_6) &= -b_2 \otimes b_7 + b_1 \otimes b_6 + b_6 \otimes b_3 + b_5 \otimes b_2;\\
\Delta(b_7) &= b_7 \otimes b_1 + b_3 \otimes b_7;\\
\Delta(b_8) &= -b_4 \otimes b_5 + b_7 \otimes b_2 + b_8 \otimes b_3 + b_3 \otimes b_8;\\
\Delta(b_9) &= b_9 \otimes b_1 + b_3 \otimes b_9;\\
\Delta(b_{10}) &= b_{10} \otimes b_3 + b_9 \otimes b_2 + b_4 \otimes b_{11} + b_3 \otimes b_{10};\\
\Delta(b_{11}) &= b_{11} \otimes b_3 + b_1 \otimes b_{11};\\
\Delta(b_{12}) &= b_{12} \otimes b_1 + b_{11} \otimes b_4 + b_2 \otimes b_{10} + b_1 \otimes b_{12};\\
\Delta(b_{13}) &= b_3 \otimes b_{13} + b_9 \otimes b_5 + b_7 \otimes b_{11} + b_{13} \otimes b_3;\\
\Delta(b_{14}) &= b_9 \otimes b_6 - b_{10} \otimes b_7 + b_3 \otimes b_{14} - b_4 \otimes b_{15} + b_{13} \otimes b_4 + b_{14} \otimes b_1 + b_8 \otimes b_9 + b_7 \otimes b_{12};\\
\Delta(b_{15}) &= b_{11} \otimes b_7 + b_1 \otimes b_{15} + b_{15} \otimes b_1 + b_5 \otimes b_9;\\
\Delta(b_{16}) &= b_{11} \otimes b_8 - b_{12} \otimes b_5 + b_1 \otimes b_{16} - b_2 \otimes b_{13} + b_{16} \otimes b_3 + b_{15} \otimes b_2 + b_6 \otimes b_{11} + b_5 \otimes b_{10}.
\end{aligned}$$

Example 3.10. Let $H = K\{\lambda, \varsigma\}$ be a bialgebra, its algebra structure is defined by $\lambda^2 = \lambda,\ \varsigma^2 = \varsigma,\ \lambda \cdot \varsigma = \varsigma \cdot \lambda = 0$; its coalgebra structure is defined by $\Delta(\lambda) = \lambda \otimes \lambda + \varsigma \otimes \varsigma,\ \varepsilon(\lambda) = 1,\ \Delta(\varsigma) = \lambda \otimes \varsigma + \varsigma \otimes \lambda,\ \varepsilon(\varsigma) = 0$. Let $C = sp\{u^{(0)}, u^{(1)}, u^{(2)}, u^{(3)}\}$, its algebra structure is defined by

$\cdot$	$u^{(0)}$	$u^{(1)}$	$u^{(2)}$	$u^{(3)}$
$u^{(0)}$	$u^{(0)}$	0	0	0
$u^{(1)}$	0	$u^{(1)}$	0	0
$u^{(2)}$	0	0	$u^{(2)}$	0
$u^{(3)}$	0	0	0	$u^{(3)}$

,

its coalgebra structure is defined by

$$\begin{aligned}
\Delta(u^{(0)}) &= u^{(0)} \otimes u^{(0)} + u^{(1)} \otimes u^{(3)} + u^{(3)} \otimes u^{(1)} + u^{(2)} \otimes u^{(2)};\\
\Delta(u^{(1)}) &= u^{(0)} \otimes u^{(1)} + u^{(1)} \otimes u^{(0)} + u^{(2)} \otimes u^{(3)} + u^{(3)} \otimes u^{(2)};\\
\Delta(u^{(2)}) &= u^{(0)} \otimes u^{(2)} + u^{(2)} \otimes u^{(0)} + u^{(1)} \otimes u^{(1)} + u^{(3)} \otimes u^{(3)};\\
\Delta(u^{(3)}) &= u^{(0)} \otimes u^{(3)} + u^{(3)} \otimes u^{(0)} + u^{(1)} \otimes u^{(2)} + u^{(2)} \otimes u^{(1)}.
\end{aligned}$$

Let $D = sp\{v^{(0)}, v^{(1)}, v^{(2)}, v^{(3)}\}$, its algebra structure is defined by

$\cdot$	$v^{(0)}$	$v^{(1)}$	$v^{(2)}$	$v^{(3)}$
$v^{(0)}$	$v^{(0)}$	0	0	0
$v^{(1)}$	0	$v^{(1)}$	0	0
$v^{(2)}$	0	0	$v^{(2)}$	0
$v^{(3)}$	0	0	0	$v^{(3)}$

,

its coalgebra structure is defined by

$$\begin{aligned}
\Delta(v^{(0)}) &= v^{(0)} \otimes v^{(0)} + v^{(1)} \otimes v^{(3)} + v^{(3)} \otimes v^{(1)} + v^{(2)} \otimes v^{(2)};\\
\Delta(v^{(1)}) &= v^{(0)} \otimes v^{(1)} + v^{(1)} \otimes v^{(0)} + v^{(2)} \otimes v^{(3)} + v^{(3)} \otimes v^{(2)};\\
\Delta(v^{(2)}) &= v^{(0)} \otimes v^{(2)} + v^{(2)} \otimes v^{(0)} + v^{(1)} \otimes v^{(1)} + v^{(3)} \otimes v^{(3)};\\
\Delta(v^{(3)}) &= v^{(0)} \otimes v^{(3)} + v^{(3)} \otimes v^{(0)} + v^{(1)} \otimes v^{(2)} + v^{(2)} \otimes v^{(1)}.
\end{aligned}$$

Define linear maps $\triangleright : H \otimes C \longrightarrow C,\ \rho : C \longrightarrow H \otimes C,\ \alpha : C \longrightarrow H \otimes H,\ \triangleleft : D \otimes H \longrightarrow D,\ \gamma : D \longrightarrow D \otimes H,\ \beta : D \longrightarrow H \otimes H$ by

$$\rho(u^{(0)}) := \lambda \otimes u^{(0)} + \varsigma \otimes u^{(0)},\ \rho(u^{(1)}) := \lambda \otimes u^{(1)} + \varsigma \otimes u^{(3)},$$
$$\rho(u^{(2)}) := \lambda \otimes u^{(2)} + \varsigma \otimes u^{(2)},\ \rho(u^{(3)}) := \lambda \otimes u^{(3)} + \varsigma \otimes u^{(1)};$$

$$\alpha(u^{(0)}) := \lambda\otimes\lambda+\lambda\otimes\varsigma+\varsigma\otimes\lambda,\ \alpha(u^{(1)}) := \varsigma\otimes\varsigma,\ \alpha(u^{(2)}) := 0,\ \alpha(u^{(3)}) := 0;$$
$$\gamma(v^{(0)}) := v^{(0)}\otimes\lambda+v^{(0)}\otimes\varsigma,\ \gamma(v^{(1)}) := v^{(1)}\otimes\lambda+v^{(3)}\otimes\varsigma,$$
$$\gamma(v^{(2)}) := v^{(2)}\otimes\lambda+v^{(2)}\otimes\varsigma,\ \gamma(v^{(3)}) := v^{(3)}\otimes\lambda+v^{(1)}\otimes\varsigma;$$
$$\beta(v^{(0)}) := \lambda\otimes\lambda+\lambda\otimes\varsigma+\varsigma\otimes\lambda,\ \beta(v^{(1)}) := \varsigma\otimes\varsigma,\ \beta(v^{(2)}) := 0,\ \beta(v^{(3)}) := 0.$$

Then $C\otimes H\otimes D = K\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17}, c_{18}, c_{19}, c_{20}, c_{21}, c_{22}, c_{23}, c_{24}, c_{25}, c_{26}, c_{27}, c_{28}, c_{29}, c_{30}, c_{31}, c_{32}\}$, where $c_1 = u^{(0)}\otimes\lambda\otimes v^{(0)}$, $c_2 = u^{(0)}\otimes\lambda\otimes v^{(1)}$, $c_3 = u^{(0)}\otimes\lambda\otimes v^{(2)}$, $c_4 = u^{(0)}\otimes\lambda\otimes v^{(3)}$, $c_5 = u^{(1)}\otimes\lambda\otimes v^{(0)}$, $c_6 = u^{(1)}\otimes\lambda\otimes v^{(1)}$, $c_7 = u^{(1)}\otimes\lambda\otimes v^{(2)}$, $c_8 = u^{(1)}\otimes\lambda\otimes v^{(3)}$, $c_9 = u^{(2)}\otimes\lambda\otimes v^{(0)}$, $c_{10} = u^{(2)}\otimes\lambda\otimes v^{(1)}$, $c_{11} = u^{(2)}\otimes\lambda\otimes v^{(2)}$, $c_{12} = u^{(2)}\otimes\lambda\otimes v^{(3)}$, $c_{13} = u^{(3)}\otimes\lambda\otimes v^{(0)}$, $c_{14} = u^{(3)}\otimes\lambda\otimes v^{(1)}$, $c_{15} = u^{(3)}\otimes\lambda\otimes v^{(2)}$, $c_{16} = u^{(3)}\otimes\lambda\otimes v^{(3)}$, $c_{17} = u^{(0)}\otimes\varsigma\otimes v^{(0)}$, $c_{18} = u^{(0)}\otimes\varsigma\otimes v^{(1)}$, $c_{19} = u^{(0)}\otimes\varsigma\otimes v^{(2)}$, $c_{20} = u^{(0)}\otimes\varsigma\otimes v^{(3)}$, $c_{21} = u^{(1)}\otimes\varsigma\otimes v^{(0)}$, $c_{22} = u^{(1)}\otimes\varsigma\otimes v^{(1)}$, $c_{23} = u^{(1)}\otimes\varsigma\otimes v^{(2)}$, $c_{24} = u^{(1)}\otimes\varsigma\otimes v^{(3)}$, $c_{25} = u^{(2)}\otimes\varsigma\otimes v^{(0)}$, $c_{26} = u^{(2)}\otimes\varsigma\otimes v^{(1)}$, $c_{27} = u^{(2)}\otimes\varsigma\otimes v^{(2)}$, $c_{28} = u^{(2)}\otimes\varsigma\otimes v^{(3)}$, $c_{29} = u^{(3)}\otimes\varsigma\otimes v^{(0)}$, $c_{30} = u^{(3)}\otimes\varsigma\otimes v^{(1)}$, $c_{31} = u^{(3)}\otimes\varsigma\otimes v^{(2)}$, $c_{32} = u^{(3)}\otimes\varsigma\otimes v^{(3)}$, is a double crossed biproduct with the multiplication of the two-sided tensor product algebra and the comultiplication defined by

$$\begin{aligned}
\Delta(c_1) = {} & c_1\otimes c_1+c_2\otimes c_4+c_3\otimes c_3+c_4\otimes c_2+c_5\otimes c_5+c_8\otimes c_6+c_7\otimes c_7+c_6\otimes c_8\\
&+c_9\otimes c_9+c_{10}\otimes c_{12}+c_{11}\otimes c_{11}+c_{12}\otimes c_{10}+c_{13}\otimes c_{13}+c_{14}\otimes c_{16}+c_{15}\otimes c_{15}\\
&+c_{16}\otimes c_{14}+c_{17}\otimes c_{17}+c_{20}\otimes c_{20}+c_{19}\otimes c_{19}+c_{18}\otimes c_{18}+c_{21}\otimes c_{29}+c_{24}\otimes c_{32}\\
&+c_{23}\otimes c_{31}+c_{22}\otimes c_{30}+c_{25}\otimes c_{25}+c_{28}\otimes c_{28}+c_{27}\otimes c_{27}+c_{26}\otimes c_{26}+c_{29}\otimes c_{21}\\
&+c_{32}\otimes c_{24}+c_{31}\otimes c_{23}+c_{30}\otimes c_{22};\\
\Delta(c_2) = {} & c_1\otimes c_5+c_2\otimes c_8+c_3\otimes c_7+c_4\otimes c_6+c_5\otimes c_1+c_8\otimes c_2+c_7\otimes c_3+c_6\otimes c_4\\
&+c_9\otimes c_{13}+c_{10}\otimes c_{16}+c_{11}\otimes c_{15}+c_{12}\otimes c_{14}+c_{13}\otimes c_{19}+c_{16}\otimes c_{10}+c_{15}\otimes c_{11}\\
&+c_{14}\otimes c_{12}+c_{17}\otimes c_{21}+c_{18}\otimes c_{24}+c_{19}\otimes c_{23}+c_{20}\otimes c_{22}+c_{21}\otimes c_{17}+c_{24}\otimes c_{18}\\
&+c_{19}\otimes c_{23}+c_{22}\otimes c_{20}+c_{25}\otimes c_{29}+c_{26}\otimes c_{32}+c_{27}\otimes c_{31}+c_{28}\otimes c_{30}+c_{29}\otimes c_{25}\\
&+c_{32}\otimes c_{26}+c_{31}\otimes c_{27}+c_{30}\otimes c_{28};\\
\Delta(c_3) = {} & c_1\otimes c_3+c_3\otimes c_1+c_2\otimes c_2+c_4\otimes c_4+c_5\otimes c_{15}+c_7\otimes c_{13}+c_6\otimes c_{14}+c_8\otimes c_{16}\\
&+c_9\otimes c_{11}+c_{11}\otimes c_9+c_{10}\otimes c_{10}+c_{12}\otimes c_{12}+c_{13}\otimes c_7+c_{15}\otimes c_5+c_{14}\otimes c_6\\
&+c_{16}\otimes c_8+c_{17}\otimes c_{19}+c_{19}\otimes c_{17}+c_{20}\otimes c_{18}+c_{18}\otimes c_{20}+c_{21}\otimes c_{31}+c_{23}\otimes c_{29}\\
&+c_{22}\otimes c_{30}+c_{24}\otimes c_{32}+c_{25}\otimes c_{27}+c_{27}\otimes c_{25}+c_{26}\otimes c_{26}+c_{28}\otimes c_{28}+c_{29}\otimes c_{23}\\
&+c_{31}\otimes c_{21}+c_{30}\otimes c_{22}+c_{32}\otimes c_{24};\\
\Delta(c_4) = {} & c_1\otimes c_4+c_4\otimes c_1+c_2\otimes c_3+c_3\otimes c_2+c_5\otimes c_{16}+c_8\otimes c_{13}+c_6\otimes c_{15}+c_7\otimes c_{14}\\
&+c_9\otimes c_{12}+c_{12}\otimes c_9+c_{10}\otimes c_{11}+c_{11}\otimes c_{10}+c_{13}\otimes c_8+c_{16}\otimes c_5+c_{14}\otimes c_7\\
&+c_{15}\otimes c_6+c_{17}\otimes c_{20}+c_{20}\otimes c_{17}+c_{18}\otimes c_{19}+c_{19}\otimes c_{18}+c_{21}\otimes c_{32}+c_{24}\otimes c_{29}\\
&+c_{22}\otimes c_{31}+c_{23}\otimes c_{30}+c_{25}\otimes c_{28}+c_{28}\otimes c_{25}+c_{26}\otimes c_{27}+c_{27}\otimes c_{26}+c_{29}\otimes c_{24}\\
&+c_{32}\otimes c_{21}+c_{30}\otimes c_{23}+c_{31}\otimes c_{22};\\
\Delta(c_5) = {} & c_1\otimes c_5+c_2\otimes c_8+c_3\otimes c_7+c_4\otimes c_6+c_5\otimes c_1+c_8\otimes c_2+c_7\otimes c_3+c_6\otimes c_4\\
&+c_9\otimes c_{13}+c_{10}\otimes c_{16}+c_{11}\otimes c_{15}+c_{12}\otimes c_{14}+c_{13}\otimes c_9+c_{16}\otimes c_{10}+c_{15}\otimes c_{11}\\
&+c_{14}\otimes c_{12}+c_{17}\otimes c_{21}+c_{20}\otimes c_{24}+c_{18}\otimes c_{22}+c_{19}\otimes c_{23}+c_{21}\otimes c_{17}+c_{24}\otimes c_{20}\\
&+c_{22}\otimes c_{28}+c_{23}\otimes c_{19}+c_{25}\otimes c_{29}+c_{26}\otimes c_{32}+c_{27}\otimes c_{31}+c_{28}\otimes c_{30}+c_{29}\otimes c_{25}\\
&+c_{32}\otimes c_{26}+c_{31}\otimes c_{27}+c_{30}\otimes c_{28};\\
\Delta(c_6) = {} & c_1\otimes c_6+c_2\otimes c_5+c_3\otimes c_8+c_4\otimes c_7+c_6\otimes c_1+c_5\otimes c_2+c_8\otimes c_3+c_7\otimes c_4\\
&+c_9\otimes c_{14}+c_{10}\otimes c_{13}+c_{11}\otimes c_{16}+c_{12}\otimes c_{15}+c_{14}\otimes c_9+c_{13}\otimes c_{10}+c_{16}\otimes c_{11}
\end{aligned}$$

$$
\begin{aligned}
&+c_{15}\otimes c_{12}+c_{17}\otimes c_{22}+c_{20}\otimes c_{21}+c_{19}\otimes c_{24}+c_{18}\otimes c_{23}+c_{22}\otimes c_{17}+c_{21}\otimes c_{20}\\
&+c_{24}\otimes c_{19}+c_{23}\otimes c_{18}+c_{25}\otimes c_{30}+c_{28}\otimes c_{29}+c_{27}\otimes c_{32}+c_{26}\otimes c_{31}+c_{30}\otimes c_{25}\\
&+c_{29}\otimes c_{28}+c_{32}\otimes c_{27}+c_{31}\otimes c_{26};
\end{aligned}
$$

$$
\begin{aligned}
\Delta(c_7)=\,&c_1\otimes c_7+c_3\otimes c_5+c_2\otimes c_6+c_4\otimes c_8+c_5\otimes c_3+c_7\otimes c_1+c_8\otimes c_4+c_6\otimes c_2\\
&+c_9\otimes c_{15}+c_{11}\otimes c_{13}+c_{10}\otimes c_{14}+c_{12}\otimes c_{16}+c_{13}\otimes c_{11}+c_{15}\otimes c_9+c_{16}\otimes c_{12}\\
&+c_{14}\otimes c_{10}+c_{17}\otimes c_{23}+c_{19}\otimes c_{21}+c_{18}\otimes c_{22}+c_{20}\otimes c_{24}+c_{21}\otimes c_{19}+c_{23}\otimes c_{17}\\
&+c_{24}\otimes c_{20}+c_{22}\otimes c_{18}+c_{25}\otimes c_{31}+c_{27}\otimes c_{29}+c_{26}\otimes c_{30}+c_{28}\otimes c_{32}+c_{29}\otimes c_{27}\\
&+c_{31}\otimes c_{25}+c_{32}\otimes c_{28}+c_{30}\otimes c_{26};
\end{aligned}
$$

$$
\begin{aligned}
\Delta(c_8)=\,&c_1\otimes c_8+c_3\otimes c_5+c_2\otimes c_7+c_4\otimes c_6+c_5\otimes c_3+c_8\otimes c_1+c_4\otimes c_6+c_7\otimes c_2\\
&+c_9\otimes c_{16}+c_{11}\otimes c_{13}+c_{10}\otimes c_{15}+c_{12}\otimes c_{14}+c_{13}\otimes c_{11}+c_{16}\otimes c_9+c_{14}\otimes c_{12}\\
&+c_{15}\otimes c_{10}+c_{17}\otimes c_{24}+c_{18}\otimes c_{21}+c_{20}\otimes c_{23}+c_{19}\otimes c_{22}+c_{21}\otimes c_{18}+c_{24}\otimes c_{17}\\
&+c_{22}\otimes c_{19}+c_{23}\otimes c_{20}+c_{25}\otimes c_{32}+c_{26}\otimes c_{29}+c_{28}\otimes c_{31}+c_{27}\otimes c_{30}+c_{29}\otimes c_{26}\\
&+c_{32}\otimes c_{25}+c_{31}\otimes c_{28}+c_{30}\otimes c_{27};
\end{aligned}
$$

$$
\begin{aligned}
\Delta(c_9)=\,&c_1\otimes c_9+c_2\otimes c_{12}+c_3\otimes c_{11}+c_4\otimes c_{10}+c_9\otimes c_1+c_{10}\otimes c_4+c_{11}\otimes c_3+c_{12}\otimes c_2\\
&+c_5\otimes c_5+c_6\otimes c_8+c_7\otimes c_7+c_8\otimes c_6+c_{13}\otimes c_{13}+c_{14}\otimes c_{16}+c_{15}\otimes c_{15}\\
&+c_{16}\otimes c_{14}+c_{17}\otimes c_{25}+c_{20}\otimes c_{28}+c_{19}\otimes c_{27}+c_{18}\otimes c_{26}+c_{25}\otimes c_{17}+c_{28}\otimes c_{20}\\
&+c_{27}\otimes c_{19}+c_{26}\otimes c_{18}+c_{21}\otimes c_{21}+c_{24}\otimes c_{24}+c_{23}\otimes c_{23}+c_{22}\otimes c_{22}+c_{29}\otimes c_{29}\\
&+c_{32}\otimes c_{32}+c_{31}\otimes c_{31}+c_{30}\otimes c_{30};
\end{aligned}
$$

$$
\begin{aligned}
\Delta(c_{10})=\,&c_1\otimes c_{10}+c_2\otimes c_9+c_3\otimes c_{12}+c_4\otimes c_{11}+c_9\otimes c_2+c_{10}\otimes c_1+c_{11}\otimes c_4+c_{12}\otimes c_3\\
&+c_5\otimes c_6+c_6\otimes c_5+c_7\otimes c_8+c_8\otimes c_7+c_{13}\otimes c_{14}+c_{14}\otimes c_{13}+c_{15}\otimes c_{16}\\
&+c_{16}\otimes c_{15}+c_{17}\otimes c_{26}+c_{20}\otimes c_{25}+c_{19}\otimes c_{28}+c_{18}\otimes c_{27}+c_{25}\otimes c_{18}+c_{28}\otimes c_{17}\\
&+c_{27}\otimes c_{20}+c_{26}\otimes c_{19}+c_{21}\otimes c_{22}+c_{24}\otimes c_{21}+c_{23}\otimes c_{24}+c_{22}\otimes c_{23}+c_{29}\otimes c_{30}\\
&+c_{32}\otimes c_{29}+c_{31}\otimes c_{32}+c_{30}\otimes c_{31};
\end{aligned}
$$

$$
\begin{aligned}
\Delta(c_{11})=\,&c_1\otimes c_{11}+c_3\otimes c_9+c_2\otimes c_{10}+c_4\otimes c_{12}+c_9\otimes c_3+c_{11}\otimes c_1+c_{12}\otimes c_4+c_{10}\otimes c_2\\
&+c_5\otimes c_7+c_7\otimes c_5+c_6\otimes c_6+c_9\otimes c_8+c_{13}\otimes c_{15}+c_{15}\otimes c_{13}+c_{14}\otimes c_{14}\\
&+c_{16}\otimes c_{16}+c_{17}\otimes c_{27}+c_{19}\otimes c_{25}+c_{20}\otimes c_{26}+c_{18}\otimes c_{28}+c_{25}\otimes c_{19}+c_{27}\otimes c_{17}\\
&+c_{26}\otimes c_{20}+c_{28}\otimes c_{18}+c_{21}\otimes c_{23}+c_{23}\otimes c_{21}+c_{22}\otimes c_{22}+c_{24}\otimes c_{24}+c_{29}\otimes c_{31}\\
&+c_{31}\otimes c_{29}+c_{30}\otimes c_{30}+c_{32}\otimes c_{32};
\end{aligned}
$$

$$
\begin{aligned}
\Delta(c_{12})=\,&c_1\otimes c_{12}+c_4\otimes c_9+c_2\otimes c_{11}+c_3\otimes c_{10}+c_9\otimes c_4+c_{12}\otimes c_1+c_{10}\otimes c_3+c_{11}\otimes c_2\\
&+c_5\otimes c_8+c_8\otimes c_5+c_6\otimes c_7+c_7\otimes c_6+c_{13}\otimes c_{16}+c_{16}\otimes c_{13}+c_{14}\otimes c_{15}\\
&+c_{15}\otimes c_{14}+c_{17}\otimes c_{28}+c_{18}\otimes c_{25}+c_{20}\otimes c_{27}+c_{19}\otimes c_{26}+c_{25}\otimes c_{20}+c_{26}\otimes c_{17}\\
&+c_{28}\otimes c_{19}+c_{27}\otimes c_{18}+c_{21}\otimes c_{24}+c_{22}\otimes c_{21}+c_{24}\otimes c_{23}+c_{23}\otimes c_{22}+c_{29}\otimes c_{32}\\
&+c_{30}\otimes c_{29}+c_{32}\otimes c_{31}+c_{31}\otimes c_{30};
\end{aligned}
$$

$$
\begin{aligned}
\Delta(c_{13})=\,&c_1\otimes c_{13}+c_2\otimes c_{16}+c_3\otimes c_{15}+c_4\otimes c_{14}+c_{13}\otimes c_1+c_{14}\otimes c_4+c_{15}\otimes c_3\\
&+c_{16}\otimes c_2+c_5\otimes c_9+c_6\otimes c_{12}+c_7\otimes c_{11}+c_8\otimes c_{10}+c_9\otimes c_5+c_{12}\otimes c_6\\
&+c_{11}\otimes c_7+c_{10}\otimes c_8+c_{17}\otimes c_{29}+c_{20}\otimes c_{32}+c_{19}\otimes c_{31}+c_{18}\otimes c_{30}+c_{29}\otimes c_{17}\\
&+c_{32}\otimes c_{20}+c_{31}\otimes c_{19}+c_{30}\otimes c_{18}+c_{21}\otimes c_{25}+c_{24}\otimes c_{28}+c_{23}\otimes c_{27}+c_{22}\otimes c_{26}\\
&+c_{25}\otimes c_{21}+c_{28}\otimes c_{24}+c_{27}\otimes c_{23}+c_{26}\otimes c_{22};
\end{aligned}
$$

$$
\begin{aligned}
\Delta(c_{14})=\,&c_1\otimes c_{14}+c_2\otimes c_{13}+c_3\otimes c_{16}+c_4\otimes c_{15}+c_{13}\otimes c_2+c_{14}\otimes c_1+c_{15}\otimes c_4\\
&+c_{16}\otimes c_3+c_5\otimes c_{10}+c_6\otimes c_9+c_7\otimes c_{12}+c_8\otimes c_{11}+c_9\otimes c_6+c_{10}\otimes c_5\\
&+c_{11}\otimes c_8+c_{12}\otimes c_7+c_{17}\otimes c_{30}+c_{20}\otimes c_{29}+c_{18}\otimes c_{32}+c_{19}\otimes c_{31}+c_{29}\otimes c_{18}
\end{aligned}
$$

$$\begin{aligned}
&+c_{32}\otimes c_{17}+c_{31}\otimes c_{20}+c_{30}\otimes c_{19}+c_{21}\otimes c_{26}+c_{24}\otimes c_{25}+c_{23}\otimes c_{28}+c_{22}\otimes c_{27}\\
&+c_{25}\otimes c_{22}+c_{28}\otimes c_{21}+c_{27}\otimes c_{24}+c_{26}\otimes c_{23};\\
\Delta(c_{15})=&\,c_{1}\otimes c_{15}+c_{3}\otimes c_{13}+c_{2}\otimes c_{14}+c_{4}\otimes c_{16}+c_{13}\otimes c_{3}+c_{15}\otimes c_{1}+c_{16}\otimes c_{4}\\
&+c_{14}\otimes c_{2}+c_{5}\otimes c_{11}+c_{7}\otimes c_{9}+c_{6}\otimes c_{10}+c_{8}\otimes c_{12}+c_{9}\otimes c_{7}+c_{11}\otimes c_{5}\\
&+c_{12}\otimes c_{8}+c_{10}\otimes c_{6}+c_{17}\otimes c_{31}+c_{19}\otimes c_{29}+c_{20}\otimes c_{30}+c_{18}\otimes c_{32}+c_{29}\otimes c_{19}\\
&+c_{31}\otimes c_{17}+c_{32}\otimes c_{18}+c_{30}\otimes c_{20}+c_{21}\otimes c_{27}+c_{23}\otimes c_{25}+c_{24}\otimes c_{26}+c_{22}\otimes c_{28}\\
&+c_{25}\otimes c_{23}+c_{27}\otimes c_{21}+c_{28}\otimes c_{22}+c_{26}\otimes c_{24};\\
\Delta(c_{16})=&\,c_{1}\otimes c_{16}+c_{4}\otimes c_{13}+c_{2}\otimes c_{15}+c_{3}\otimes c_{14}+c_{13}\otimes c_{4}+c_{16}\otimes c_{1}+c_{14}\otimes c_{3}\\
&+c_{15}\otimes c_{2}+c_{5}\otimes c_{12}+c_{8}\otimes c_{9}+c_{6}\otimes c_{11}+c_{7}\otimes c_{10}+c_{9}\otimes c_{8}+c_{12}\otimes c_{5}\\
&+c_{10}\otimes c_{7}+c_{11}\otimes c_{6}+c_{17}\otimes c_{32}+c_{18}\otimes c_{29}+c_{20}\otimes c_{31}+c_{19}\otimes c_{30}+c_{29}\otimes c_{20}\\
&+c_{30}\otimes c_{17}+c_{32}\otimes c_{19}+c_{31}\otimes c_{18}+c_{21}\otimes c_{28}+c_{22}\otimes c_{25}+c_{24}\otimes c_{27}+c_{23}\otimes c_{26}\\
&+c_{25}\otimes c_{24}+c_{26}\otimes c_{21}+c_{28}\otimes c_{23}+c_{27}\otimes c_{22};\\
\Delta(c_{17})=&\,c_{1}\otimes c_{17}+c_{4}\otimes c_{20}+c_{3}\otimes c_{19}+c_{2}\otimes c_{18}+c_{5}\otimes c_{29}+c_{8}\otimes c_{32}+c_{7}\otimes c_{31}\\
&+c_{6}\otimes c_{30}+c_{9}\otimes c_{25}+c_{12}\otimes c_{28}+c_{11}\otimes c_{27}+c_{10}\otimes c_{26}+c_{13}\otimes c_{21}+c_{16}\otimes c_{24}\\
&+c_{15}\otimes c_{23}+c_{14}\otimes c_{22}+c_{17}\otimes c_{1}+c_{18}\otimes c_{4}+c_{19}\otimes c_{3}+c_{20}\otimes c_{2}+c_{21}\otimes c_{13}\\
&+c_{22}\otimes c_{16}+c_{23}\otimes c_{15}+c_{24}\otimes c_{14}+c_{25}\otimes c_{9}+c_{26}\otimes c_{12}+c_{27}\otimes c_{11}+c_{28}\otimes c_{10}\\
&+c_{29}\otimes c_{5}+c_{30}\otimes c_{8}+c_{31}\otimes c_{7}+c_{32}\otimes c_{6};\\
\Delta(c_{18})=&\,c_{1}\otimes c_{18}+c_{4}\otimes c_{17}+c_{3}\otimes c_{20}+c_{2}\otimes c_{19}+c_{5}\otimes c_{30}+c_{8}\otimes c_{29}+c_{7}\otimes c_{32}\\
&+c_{6}\otimes c_{31}+c_{9}\otimes c_{26}+c_{12}\otimes c_{25}+c_{11}\otimes c_{28}+c_{10}\otimes c_{27}+c_{13}\otimes c_{22}+c_{16}\otimes c_{21}\\
&+c_{15}\otimes c_{24}+c_{14}\otimes c_{23}+c_{17}\otimes c_{2}+c_{18}\otimes c_{1}+c_{19}\otimes c_{4}+c_{20}\otimes c_{3}+c_{21}\otimes c_{14}\\
&+c_{22}\otimes c_{13}+c_{23}\otimes c_{16}+c_{24}\otimes c_{15}+c_{25}\otimes c_{10}+c_{26}\otimes c_{9}+c_{27}\otimes c_{12}+c_{28}\otimes c_{11}\\
&+c_{29}\otimes c_{6}+c_{30}\otimes c_{5}+c_{31}\otimes c_{8}+c_{32}\otimes c_{7};\\
\Delta(c_{19})=&\,c_{1}\otimes c_{19}+c_{3}\otimes c_{17}+c_{4}\otimes c_{18}+c_{2}\otimes c_{20}+c_{5}\otimes c_{31}+c_{7}\otimes c_{29}+c_{8}\otimes c_{30}\\
&+c_{6}\otimes c_{32}+c_{9}\otimes c_{27}+c_{11}\otimes c_{25}+c_{12}\otimes c_{26}+c_{10}\otimes c_{28}+c_{13}\otimes c_{23}+c_{15}\otimes c_{21}\\
&+c_{16}\otimes c_{22}+c_{14}\otimes c_{24}+c_{17}\otimes c_{3}+c_{19}\otimes c_{1}+c_{18}\otimes c_{2}+c_{20}\otimes c_{4}+c_{21}\otimes c_{15}\\
&+c_{23}\otimes c_{12}+c_{22}\otimes c_{14}+c_{24}\otimes c_{16}+c_{25}\otimes c_{11}+c_{27}\otimes c_{9}+c_{26}\otimes c_{10}+c_{28}\otimes c_{12}\\
&+c_{29}\otimes c_{7}+c_{31}\otimes c_{5}+c_{30}\otimes c_{6}+c_{32}\otimes c_{8};\\
\Delta(c_{20})=&\,c_{1}\otimes c_{20}+c_{2}\otimes c_{17}+c_{4}\otimes c_{19}+c_{3}\otimes c_{18}+c_{5}\otimes c_{32}+c_{6}\otimes c_{29}+c_{8}\otimes c_{31}\\
&+c_{7}\otimes c_{30}+c_{9}\otimes c_{28}+c_{10}\otimes c_{25}+c_{12}\otimes c_{27}+c_{11}\otimes c_{26}+c_{13}\otimes c_{24}+c_{14}\otimes c_{21}\\
&+c_{16}\otimes c_{23}+c_{15}\otimes c_{22}+c_{17}\otimes c_{4}+c_{20}\otimes c_{1}+c_{18}\otimes c_{3}+c_{19}\otimes c_{2}+c_{21}\otimes c_{16}\\
&+c_{24}\otimes c_{13}+c_{22}\otimes c_{15}+c_{23}\otimes c_{14}+c_{25}\otimes c_{12}+c_{28}\otimes c_{9}+c_{26}\otimes c_{11}+c_{27}\otimes c_{10}\\
&+c_{29}\otimes c_{8}+c_{32}\otimes c_{5}+c_{30}\otimes c_{7}+c_{31}\otimes c_{6};\\
\Delta(c_{21})=&\,c_{1}\otimes c_{21}+c_{4}\otimes c_{24}+c_{3}\otimes c_{23}+c_{2}\otimes c_{22}+c_{5}\otimes c_{17}+c_{8}\otimes c_{20}+c_{7}\otimes c_{19}\\
&+c_{6}\otimes c_{18}+c_{9}\otimes c_{29}+c_{12}\otimes c_{32}+c_{11}\otimes c_{31}+c_{10}\otimes c_{30}+c_{13}\otimes c_{25}+c_{16}\otimes c_{28}\\
&+c_{15}\otimes c_{27}+c_{14}\otimes c_{26}+c_{17}\otimes c_{13}+c_{18}\otimes c_{16}+c_{19}\otimes c_{15}+c_{20}\otimes c_{14}+c_{21}\otimes c_{1}\\
&+c_{22}\otimes c_{4}+c_{23}\otimes c_{3}+c_{24}\otimes c_{2}+c_{25}\otimes c_{5}+c_{26}\otimes c_{8}+c_{27}\otimes c_{7}+c_{28}\otimes c_{6}\\
&+c_{29}\otimes c_{9}+c_{30}\otimes c_{12}+c_{31}\otimes c_{11}+c_{32}\otimes c_{10};\\
\Delta(c_{22})=&\,c_{1}\otimes c_{22}+c_{4}\otimes c_{21}+c_{3}\otimes c_{24}+c_{2}\otimes c_{23}+c_{5}\otimes c_{18}+c_{8}\otimes c_{17}+c_{7}\otimes c_{20}\\
&+c_{6}\otimes c_{19}+c_{9}\otimes c_{30}+c_{12}\otimes c_{29}+c_{11}\otimes c_{32}+c_{10}\otimes c_{31}+c_{13}\otimes c_{26}+c_{16}\otimes c_{25}\\
&+c_{15}\otimes c_{28}+c_{14}\otimes c_{27}+c_{17}\otimes c_{14}+c_{18}\otimes c_{13}+c_{19}\otimes c_{16}+c_{20}\otimes c_{15}+c_{21}\otimes c_{2}\\
&+c_{22}\otimes c_{1}+c_{23}\otimes c_{4}+c_{24}\otimes c_{3}+c_{25}\otimes c_{6}+c_{26}\otimes c_{5}+c_{27}\otimes c_{8}+c_{28}\otimes c_{7}
\end{aligned}$$

$$+c_{29}\otimes c_{10}+c_{30}\otimes c_{9}+c_{31}\otimes c_{12}+c_{32}\otimes c_{11};$$

$$\begin{aligned}\Delta(c_{23})=&c_1\otimes c_{23}+c_3\otimes c_{21}+c_4\otimes c_{22}+c_2\otimes c_{24}+c_5\otimes c_{19}+c_7\otimes c_{17}+c_8\otimes c_{18}\\&+c_6\otimes c_{20}+c_9\otimes c_{31}+c_{11}\otimes c_{29}+c_{12}\otimes c_{30}+c_{10}\otimes c_{32}+c_{13}\otimes c_{27}+c_{15}\otimes c_{25}\\&+c_{16}\otimes c_{26}+c_{14}\otimes c_{28}+c_{17}\otimes c_{15}+c_{19}\otimes c_{13}+c_{18}\otimes c_{14}+c_{20}\otimes c_{16}+c_{21}\otimes c_{3}\\&+c_{23}\otimes c_{1}+c_{22}\otimes c_{2}+c_{24}\otimes c_{4}+c_{25}\otimes c_{7}+c_{27}\otimes c_{5}+c_{26}\otimes c_{6}+c_{28}\otimes c_{8}\\&+c_{29}\otimes c_{11}+c_{31}\otimes c_{9}+c_{30}\otimes c_{10}+c_{32}\otimes c_{12};\end{aligned}$$

$$\begin{aligned}\Delta(c_{24})=&c_1\otimes c_{24}+c_2\otimes c_{21}+c_4\otimes c_{23}+c_3\otimes c_{22}+c_5\otimes c_{20}+c_6\otimes c_{17}+c_8\otimes c_{19}\\&+c_7\otimes c_{18}+c_9\otimes c_{32}+c_{10}\otimes c_{29}+c_{12}\otimes c_{31}+c_{11}\otimes c_{30}+c_{13}\otimes c_{28}+c_{14}\otimes c_{25}\\&+c_{16}\otimes c_{27}+c_{15}\otimes c_{26}+c_{17}\otimes c_{16}+c_{20}\otimes c_{13}+c_{18}\otimes c_{15}+c_{19}\otimes c_{14}+c_{21}\otimes c_{4}\\&+c_{24}\otimes c_{1}+c_{22}\otimes c_{3}+c_{23}\otimes c_{2}+c_{25}\otimes c_{8}+c_{28}\otimes c_{5}+c_{26}\otimes c_{7}+c_{27}\otimes c_{6}\\&+c_{29}\otimes c_{12}+c_{32}\otimes c_{9}+c_{30}\otimes c_{11}+c_{31}\otimes c_{10};\end{aligned}$$

$$\begin{aligned}\Delta(c_{25})=&c_1\otimes c_{25}+c_4\otimes c_{28}+c_3\otimes c_{27}+c_2\otimes c_{26}+c_9\otimes c_{17}+c_{12}\otimes c_{20}+c_{11}\otimes c_{19}\\&+c_{10}\otimes c_{18}+c_5\otimes c_{21}+c_8\otimes c_{24}+c_7\otimes c_{23}+c_6\otimes c_{22}+c_{13}\otimes c_{29}+c_{16}\otimes c_{32}\\&+c_{15}\otimes c_{31}+c_{14}\otimes c_{30}+c_{17}\otimes c_{9}+c_{18}\otimes c_{12}+c_{19}\otimes c_{11}+c_{20}\otimes c_{10}+c_{25}\otimes c_{1}\\&+c_{26}\otimes c_{4}+c_{27}\otimes c_{3}+c_{28}\otimes c_{2}+c_{21}\otimes c_{13}+c_{22}\otimes c_{16}+c_{23}\otimes c_{15}+c_{24}\otimes c_{14}\\&+c_{29}\otimes c_{5}+c_{30}\otimes c_{8}+c_{31}\otimes c_{7}+c_{32}\otimes c_{6};\end{aligned}$$

$$\begin{aligned}\Delta(c_{26})=&c_1\otimes c_{26}+c_4\otimes c_{25}+c_3\otimes c_{28}+c_2\otimes c_{27}+c_9\otimes c_{18}+c_{12}\otimes c_{17}+c_{11}\otimes c_{20}\\&+c_{10}\otimes c_{19}+c_5\otimes c_{22}+c_8\otimes c_{21}+c_7\otimes c_{24}+c_6\otimes c_{23}+c_{13}\otimes c_{30}+c_{16}\otimes c_{29}\\&+c_{15}\otimes c_{32}+c_{14}\otimes c_{31}+c_{17}\otimes c_{10}+c_{18}\otimes c_{9}+c_{19}\otimes c_{12}+c_{20}\otimes c_{11}+c_{25}\otimes c_{2}\\&+c_{26}\otimes c_{1}+c_{27}\otimes c_{4}+c_{28}\otimes c_{3}+c_{21}\otimes c_{14}+c_{22}\otimes c_{13}+c_{23}\otimes c_{16}+c_{24}\otimes c_{15}\\&+c_{29}\otimes c_{6}+c_{30}\otimes c_{5}+c_{31}\otimes c_{8}+c_{32}\otimes c_{7};\end{aligned}$$

$$\begin{aligned}\Delta(c_{27})=&c_1\otimes c_{27}+c_3\otimes c_{25}+c_4\otimes c_{26}+c_2\otimes c_{28}+c_9\otimes c_{19}+c_{11}\otimes c_{17}+c_{12}\otimes c_{18}\\&+c_{10}\otimes c_{20}+c_5\otimes c_{23}+c_7\otimes c_{21}+c_8\otimes c_{22}+c_6\otimes c_{24}+c_{13}\otimes c_{31}+c_{15}\otimes c_{29}\\&+c_{16}\otimes c_{30}+c_{14}\otimes c_{32}+c_{17}\otimes c_{11}+c_{19}\otimes c_{9}+c_{18}\otimes c_{10}+c_{20}\otimes c_{12}+c_{25}\otimes c_{3}\\&+c_{27}\otimes c_{1}+c_{26}\otimes c_{2}+c_{28}\otimes c_{4}+c_{21}\otimes c_{15}+c_{23}\otimes c_{13}+c_{22}\otimes c_{14}+c_{24}\otimes c_{16}\\&+c_{29}\otimes c_{7}+c_{31}\otimes c_{5}+c_{30}\otimes c_{6}+c_{32}\otimes c_{8};\end{aligned}$$

$$\begin{aligned}\Delta(c_{28})=&c_1\otimes c_{28}+c_2\otimes c_{25}+c_4\otimes c_{27}+c_3\otimes c_{26}+c_9\otimes c_{20}+c_{10}\otimes c_{17}+c_{12}\otimes c_{19}\\&+c_{11}\otimes c_{18}+c_5\otimes c_{24}+c_6\otimes c_{21}+c_8\otimes c_{23}+c_7\otimes c_{22}+c_{13}\otimes c_{32}+c_{14}\otimes c_{29}\\&+c_{16}\otimes c_{31}+c_{15}\otimes c_{30}+c_{17}\otimes c_{12}+c_{20}\otimes c_{9}+c_{18}\otimes c_{11}+c_{19}\otimes c_{10}+c_{25}\otimes c_{4}\\&+c_{28}\otimes c_{1}+c_{26}\otimes c_{3}+c_{27}\otimes c_{2}+c_{21}\otimes c_{16}+c_{24}\otimes c_{13}+c_{22}\otimes c_{15}+c_{23}\otimes c_{14}\\&+c_{29}\otimes c_{8}+c_{32}\otimes c_{5}+c_{30}\otimes c_{7}+c_{31}\otimes c_{6};\end{aligned}$$

$$\begin{aligned}\Delta(c_{29})=&c_1\otimes c_{29}+c_4\otimes c_{32}+c_3\otimes c_{31}+c_2\otimes c_{30}+c_{13}\otimes c_{17}+c_{16}\otimes c_{20}+c_{15}\otimes c_{19}\\&+c_{14}\otimes c_{18}+c_5\otimes c_{25}+c_8\otimes c_{28}+c_7\otimes c_{27}+c_6\otimes c_{26}+c_9\otimes c_{21}+c_{12}\otimes c_{24}\\&+c_{11}\otimes c_{23}+c_{10}\otimes c_{22}+c_{17}\otimes c_{5}+c_{18}\otimes c_{8}+c_{19}\otimes c_{7}+c_{20}\otimes c_{6}+c_{29}\otimes c_{1}\\&+c_{30}\otimes c_{4}+c_{31}\otimes c_{3}+c_{32}\otimes c_{2}+c_{21}\otimes c_{9}+c_{22}\otimes c_{12}+c_{23}\otimes c_{11}+c_{24}\otimes c_{10}\\&+c_{25}\otimes c_{13}+c_{26}\otimes c_{16}+c_{27}\otimes c_{15}+c_{28}\otimes c_{14};\end{aligned}$$

$$\begin{aligned}\Delta(c_{30})=&c_1\otimes c_{30}+c_4\otimes c_{29}+c_3\otimes c_{32}+c_2\otimes c_{31}+c_{13}\otimes c_{18}+c_{16}\otimes c_{17}+c_{15}\otimes c_{20}\\&+c_{14}\otimes c_{19}+c_5\otimes c_{26}+c_8\otimes c_{25}+c_7\otimes c_{28}+c_6\otimes c_{27}+c_9\otimes c_{22}+c_{12}\otimes c_{21}\\&+c_{11}\otimes c_{24}+c_{10}\otimes c_{23}+c_{17}\otimes c_{6}+c_{18}\otimes c_{5}+c_{19}\otimes c_{8}+c_{20}\otimes c_{7}+c_{29}\otimes c_{2}\\&+c_{30}\otimes c_{1}+c_{31}\otimes c_{4}+c_{32}\otimes c_{3}+c_{21}\otimes c_{10}+c_{22}\otimes c_{9}+c_{23}\otimes c_{12}+c_{24}\otimes c_{11}\\&+c_{25}\otimes c_{14}+c_{26}\otimes c_{13}+c_{27}\otimes c_{16}+c_{28}\otimes c_{15};\end{aligned}$$

$$\begin{aligned}\Delta(c_{31})=&\,c_1\otimes c_{31}+c_3\otimes c_{29}+c_4\otimes c_{30}+c_2\otimes c_{32}+c_{13}\otimes c_{19}+c_{15}\otimes c_{17}+c_{16}\otimes c_{18}\\&+c_{14}\otimes c_{20}+c_5\otimes c_{27}+c_7\otimes c_{25}+c_8\otimes c_{26}+c_6\otimes c_{28}+c_9\otimes c_{23}+c_{11}\otimes c_{21}\\&+c_{12}\otimes c_{22}+c_{10}\otimes c_{24}+c_{17}\otimes c_7+c_{19}\otimes c_5+c_{18}\otimes c_6+c_{20}\otimes c_8+c_{29}\otimes c_3\\&+c_{31}\otimes c_1+c_{30}\otimes c_2+c_{32}\otimes c_4+c_{21}\otimes c_{11}+c_{23}\otimes c_9+c_{22}\otimes c_{10}+c_{24}\otimes c_{12}\\&+c_{25}\otimes c_{15}+c_{27}\otimes c_{13}+c_{26}\otimes c_{14}+c_{28}\otimes c_{16};\\ \Delta(c_{32})=&\,c_1\otimes c_{32}+c_2\otimes c_{29}+c_4\otimes c_{31}+c_3\otimes c_{30}+c_{13}\otimes c_{20}+c_{14}\otimes c_{17}+c_{16}\otimes c_{19}\\&+c_{15}\otimes c_{18}+c_5\otimes c_{28}+c_6\otimes c_{25}+c_8\otimes c_{27}+c_7\otimes c_{26}+c_9\otimes c_{24}+c_{10}\otimes c_{21}\\&+c_{12}\otimes c_{23}+c_{11}\otimes c_{22}+c_{17}\otimes c_8+c_{20}\otimes c_5+c_{18}\otimes c_7+c_{19}\otimes c_6+c_{29}\otimes c_4\\&+c_{32}\otimes c_1+c_{30}\otimes c_3+c_{31}\otimes c_2+c_{21}\otimes c_{12}+c_{24}\otimes c_9+c_{22}\otimes c_{11}+c_{23}\otimes c_{10}\\&+c_{25}\otimes c_{16}+c_{28}\otimes c_{13}+c_{26}\otimes c_{15}+c_{27}\otimes c_{14}.\end{aligned}$$

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